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Subordination chains and univalence criteria

Nicolae N. Pascu, Dorina Răducanu and Shigeyoshi Owa

Abstract

The object of the present paper is to give an univalence condition for analytic functions in the open unit disk U by using the properties for subordination chain.

1 Introduction

Let U be the open unit disk in the complex plane $\mathbf{C}$, i.e. $U = \{z \in \mathbf{C} : |z| < 1\}$. We denote by A the class of functions $f(z)$ which are analytic in U with $f(0) = 0$, $f'(0) = 1$ and by S the subclass of the class A consisting of univalent functions.

If $f(z) \in A$ and $g(z) \in S$, then $f(z)$ is said to be *subordinate* to $g(z)$(written by $f(z) \prec g(z)$)in U if $f(U) \subset g(U)$.

A function $L : U \times [0, \infty) \to \mathbf{C}$ is said to be a *subordination chain* if $L(\cdot, t)$ is analytic and univalent in U, for all $t \in [0, \infty)$ and $L(z, s) \prec L(z, t)$, whenever $o \leq s \leq t < \infty$.

The following result concerning subordination chains is due to Ch. Pommerenke [3].

Theorem 1 *Let* $L(z, t) = a_1(t) z + \ldots.$*be a function from* $U \times [0, \infty)$ *into* $\mathbf{C}$, *such that:*

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(i) $L(\cdot,t)$ *is analytic in* U*, for all* $t\in[0,\infty)$.

(ii) $L(z,t)$ *is a locally absolutely continuous function of* t*, locally uniformly with respect to* $z\in U$.

(iii) $a_1(t)\neq 0$*, for all* $t\in[0,\infty)$ *and* $\lim_{t\to\infty}|a_1(t)|=\infty$.

(iv) the family of function $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ *forms a normal family in* U.

Let $p: U\times[0,\infty)\to\mathbf{C}$ *be an analytic function in* U *with* $\operatorname{Re}p(z,t)>0$ *for all* $(z,t)\in U\times[0,\infty)$ *and such that:*

$$\frac{\partial L(z,t)}{\partial t}=zp(z,t)\frac{\partial L(z,t)}{\partial z}, \tag{1}$$

a.e. $t\in[0,\infty)$, *for all* $z\in U$.

Then the function $L(z,t)$ *is a subordination chain in* U.

2 Sufficient conditions for univalence

By using Theorem 1 we obtain an univalence condition which generalize some known univalence criteria for analytic functions in the open unit disk U.

Let $a(t)$ be a complex valued function on $[0,\infty)$ satisfying:

$$\bullet a\in C^1[0,\infty),\ a(0)=1,\ a(t)\neq 0 \tag{2}$$

and

$$a(t)+a'(t)\neq 0,\ t\in[0,\infty),$$

$$\bullet \text{the modulus of } a(t) \text{ is increasing to } \infty. \tag{3}$$

Definition 1 *Let $F = F(u,v)$ be a function from $U \times \mathbf{C}$ into $\mathbf{C}$ and let $L(z,t) = F(e^{-t}z, a(t)z)$, for all $(z,t) \in U \times [0,\infty)$. We say that the function F satisfies (PA) conditions if:*

(i) $L(\cdot,t)$ is analytic in U, for all $t \in [0,\infty)$.

(ii) $L(z,t)$ is a locally absolutely continuous function of t, locally uniformly with respect to $z \in U$.

(iii) the function $\frac{\partial L(z,t)}{\partial t} / z\frac{\partial L(z,t)}{\partial t}$ is analytic in $\bar{U}$, for all $t > 0$ and is analytic in U for $t = 0$.

(iv) $\frac{\partial F(0,0)}{\partial \nu} \neq 0$ and $\frac{\partial F(0,0)}{\partial u} / \frac{\partial F(0,0)}{\partial \nu} \notin (-\infty,-1]$.

(v) the family of functions

$$\left\{ F\left(e^{-t}z, a(t)z\right) / \left[e^{-t}\frac{\partial F(0,0)}{\partial u} + a(t)\frac{\partial F(0,0)}{\partial v} \right] \right\}_{t \geq 0}$$

is a normal family in U.

Theorem 2 *Let $a : [0,\infty) \to \mathbf{C}$ be a function satisfying (2) and (3). Further, suppose $F : U \times \mathbf{C}$ is a function which satisfies (PA) conditions. If*

$$\left| G(z,z) + \frac{a(t) - a'(t)}{2a(t)} \right| < \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in U, \quad t \geq 0 \tag{4}$$

and

$$\max_{|z|=e^{-t}} \left| G\left(z, a(t)\frac{z}{|z|}\right) + \frac{a(t) - a'(t)}{2a(t)} \right| \tag{5}$$

$$\leq \frac{|a(t) + a'(t)|}{2|a(t)|}, \quad z \in U \setminus \{0\}, \quad t \geq 0,$$

where

$$G(u,v) = \frac{u}{v} \cdot \frac{\partial F(u,v)}{\partial u} / \frac{\partial F(u,v)}{\partial v}, \tag{6}$$

then $F(z,z)$ is an univalent function in U.

Proof. We wish to show that the function $L(z,t) = F(e^{-t}z, a(t)z)$ satisfies the conditions of Theorem 1 and hence $L(\cdot,t)$ is univalent in U, for all $t \in [0,\infty)$.

If $F(e^{-t}z, a(t)z) = a_1(t)z + \ldots$, then

$$a_1(t) = e^{-t}\frac{\partial F(0,0)}{\partial u} + a(t)\frac{\partial F(0,0)}{\partial v}.$$

By using the conditions (iv) and (v) of the Definition 1 we have $a_1(t) \neq 0$ for all $t \geq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and the family of functions $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ is a normal family in U. Let

$$p(z,t) = \frac{\partial L(z,t)}{\partial t} / z\frac{\partial L(z,t)}{\partial z}, \quad (z,t) \in U \times [0,\infty). \tag{7}$$

Then the condition (1) of Theorem 1 is satisfied for all $z \in U$ and $t \in [0,\infty)$. It remains to prove that the function $p(z,t)$ has a positive real part in U, for all $t \in [0,\infty)$. If

$$w(z,t) = \frac{1 - p(z,t)}{1 + p(z,t)}, \quad (z,t) \in U \times [0,\infty), \tag{8}$$

then $\operatorname{Re} p(z,t) > 0$ if and only if $|w(z,t)| < 1$. According with (6), (7), (8) we have

$$w(z,t) = \frac{2a(t)}{a(t) + a'(t)} G\left(e^{-t}z, a(t)z\right) \tag{9}$$

$$+ \frac{a(t) - a'(t)}{a(t) + a'(t)}, \quad (z,t) \in U \times [0,\infty).$$

By using the inequality (4) we obtain $|w(z,0)| < 1$, for all $z \in U$. For $t > 0$ the function $p(z,t)$ is analytic in $\bar{U}$ and it follows

$$|w(z,t)| < \max_{|\zeta|=1} |w(\varsigma,t)| = \max_{|\zeta|=1} \left| \frac{2a(t)}{a(t) + a'(t)} G\left(e^{-t}\varsigma, a(t)\varsigma\right) + \frac{a(t) - a'(t)}{a(t) + a'(t)} \right|.$$

If we let $z = e^{-t}\varsigma$ with $|\zeta| = 1$, then $|z| = e^{-t}$ and by using (5) we have

$$|w(z,t)| < \max_{|z|=e^{-t}} \left| \frac{2a(t)}{a(t)+a'(t)} G\left(z, a(t)\frac{z}{|z|}\right) + \frac{a(t)-a'(t)}{a(t)+a'(t)} \right| \leq 1.$$

Since $L(z,t)$ satisfies all the conditions of Theorem 1, it follows that $L(z,t)$ is a subordination chain in U and $F(z,z) = L(z,0)$ is an univalent function in U.

Remark 1

1. If $a(t) = e^t$ we obtain the univalence condition due to N.N. Pascu [1].

2. If
$$F(u,v) = f(u) + \frac{(v-u)R(u)}{1-(v-u)Q(u)},$$
where $u = e^{-t}z, v = a(t)z$ and $R(z), Q(z)$are analytic functions in U, we obtain the results concerning univalence criteria due to J.A. Pfaltzgraff [2].

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Nicolae N. Pascu , Dorina Răducanu:
Department of Mathematics
Faculty of Sciences, Transilvania University
Iuliu Maniu, 50
Braşov 2200
România

Shigeyoshi Owa:
Department of Mathematics
Kinki University
Higashi - Osaka, Osaka 577-8502
Japan