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# THE NOTION OF COMMENSURABILITY IN GROUP THEORY AND GEOMETRY

LUISA PAOLUZZI

## 1. INTRODUCTION

This work is based on a talk given by the author at the RIMS seminar "Representation spaces, twisted topological invariants and geometric structures of 3-manifolds", held in Hakone from May 28th to June 1st, 2012, and is intended to provide a digest of known results concerning the problems of classifying topological or geometric spaces up to commensurability, and of understanding their commensurators. Given the extent of the matter at hand, it is obviously unreasonable to try and cover all the possible domains in which these concepts appear, so choices had to be made which only reflect the personal taste of the author, and certainly not the interest of the subject. Similarly, even in those areas that will be touched upon here, only a limited selection of results will be mentioned. Hopefully, the reader will find something to stimulate his or her curiosity, and will be induced to investigate further the references listed in the bibliography.

Originally, the concept of commensurability comes from elementary group theory: Two subgroups $H_1$ and $H_2$ of the same group $G$ are said to be *commensurable* if their intersection $H_1 \cap H_2$ is of finite index in both of them. A basic exercise in group theory shows that the intersection of two finite-index subgroups is again a finite index subgroup. It follows that *being commensurable* is an equivalence relation on the set of subgroups of a given group. For instance, if $G$ is a finite group, all its subgroups are commensurable, while if $G = \mathbb{Z}$ its subgroups belong to two commensurability classes according to whether they are trivial or infinite cyclic. Note also that, if $G = \mathbb{R}$, two infinite cyclic subgroups $H_1$ and $H_2$, generated by positive elements $x_1$ and $x_2$ respectively, are commensurable if and only if the lengths $x_1$ and $x_2$ are commensurable in the sense of Euclid.

As it is, this notion does not seem particularly interesting, but translating it into more topological terms suggests ways to generalise the initial restrictive definition in order to make it more exploitable in a topological or geometric setting. Roughly speaking, the algebraic notion of "being commensurable" can be translated into the geometric notion of "having a common finite-sheeted cover". One way to see this is the following. Assume $G$ is the fundamental group of some connected space $X$ admitting a universal cover. In this context, $H_1$ and $H_2$ are the fundamental groups of two spaces, $Y_1$ and $Y_2$, which cover $X$, and there is a space $Y$ with fundamental group $H_1 \cap H_2$ which is a finite-sheeted cover of both $Y_1$ and $Y_2$.

This will be made more precise in the next section, where the definitions of *weak commensurability* and *abstract commensurability* will be given and motivations as why one would introduce these new definitions will also be provided. The main inspiration here will come from hyperbolic geometry in dimensions 2 and 3, although we shall try to stress that these notions apply in much larger settings.

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In Section 3, we will introduce the definition of *weak commensurators* and *abstract commensurators* and discuss some of their straightforward properties. Some results concerning the structure of (abstract) commensurators will also be discussed here.

In the following sections we will move into a more geometric world. We shall start by recalling some classical results about the classification of finite-volume hyperbolic manifolds (or orbifolds) up to commensurability (Section 4) with a special focus on the case of hyperbolic knots (Section 5) which have recently received a great deal of attention giving rise to several new results.

The last part of this work will be devoted to geometric group theory. The relationship between abstract commensurability and quasi-isometry will be discussed here (Section 6) before some classification results will be recalled (Section 7).

## 2. Weak and abstract commensurability

As we have already seen in Section 1, the algebraic notion of commensurability translates geometrically into having a common finite-sheeted cover. A slightly different way to transpose commensurability into geometry is by assuming that the group $G$ is itself endowed with some kind of geometric structure or, at least, acts geometrically on some space. Assume, for instance, that $G$ is a Lie group. In this case it is convenient to choose $H_1$ and $H_2$ to be discrete subgroups of $G$, or even lattices. In what follows we will mainly be interested in the case where $G$ is $PSL_2(\mathbb{C})$ or $PSL_2(\mathbb{R})$ acting on the hyperbolic 3-space and plane respectively, while $H_1$ and $H_2$ are fundamental groups of oriented finite-volume hyperbolic orbifolds of dimension 3 or 2. It is again clear that commensurable subgroups of $G$ are (orbifold) fundamental groups of orbifolds admitting a common finite-sheeted orbifold cover. Readers who are not at ease with orbifolds can think of manifolds instead: we will see in a little while that this is not a major assumption.

If one wishes to go back from geometry to algebra, though, there are some ambiguities that have to be taken into account. Indeed, for a fixed hyperbolic manifold or orbifold there are several ways to represent its fundamental group as a discrete subgroup of the isometry group of the hyperbolic space. These depend basically on the choice of a basepoint and developing map, and differ by conjugacy in $G$, the isometry group. The natural definition of commensurability in a geometric setting is thus a weaker one, namely commensurability up to conjugacy:

**Definition 2.1.** Let $H_1$ and $H_2$ be two subgroups of a group $G$. We say that $H_1$ and $H_2$ are *weakly commensurable* if there is an element $g \in G$ such that $H_1$ and $gH_2g^{-1}$ are commensurable in the strict sense.

**Remark 2.2.** It is straightforward to see that being weakly commensurable is again an equivalence relation on the subgroups of a given group $G$. This equivalence relation also makes sense in a geometric setting. Indeed, consider two hyperbolic orbifolds $Y_1$ and $Y_2$ of the same dimension and of finite volume. By the very definition of weak commensurability, $Y_1$ and $Y_2$ have a common finite orbifold cover if and only if the images of their fundamental groups in any holonomy representation are weakly commensurable in the isometry group of the hyperbolic space. As a consequence, weak commensurability is an equivalence relation on the hyperbolic orbifolds of given dimension and finite volume.

Before exploring further the notion of commensurability from a geometric and topological viewpoint, let us observe that, as a consequence of Selberg's lemma (see, for instance, [27]), fundamental groups of finite volume hyperbolic orbifolds are virtually torsion free.

It follows that each hyperbolic orbifold is finitely covered by a hyperbolic manifold, so when considering commensurability it is not restrictive to consider only manifolds. Indeed, each commensurability class of orbifolds contains a manifold, and two manifolds are obviously commensurable as such if and only if they are commensurable as orbifolds. On the other hand note that, although each manifold is finitely covered by an oriented one, so that we can only take into account orientable manifolds, commensurability classes can change if we consider orientable manifolds or oriented ones. This comes from the fact that subgroups of the isometry group of the hyperbolic space can be weakly commensurable in the full group of isometries but not in the group of orientation preserving isometries.

Restricting now our attention to the cases of dimension 2 and 3 we will see that this notion of commensurability is well-adapted to geometry, but is not supple enough for topology. Assume $H_1$ and $H_2$ are fundamental groups of two 3-manifolds. Assume, moreover, that the first manifold admits a hyperbolic structure and the two manifolds admit a common finite-sheeted cover. In this case, it follows from Mostow's rigidity theorem (see [27]) that the second manifold is also hyperbolic and that $H_1$ and $H_2$ are weakly commensurable when viewed as discrete subgroups of the isometry group of hyperbolic 3-space, regardless of the (faithful and discrete) representation chosen.

In dimension 2, though, one topological surface supports several metric structures. So, although any two (topological) closed surfaces of negative Euler characteristic have a common finite-sheeted cover, once they are equipped with hyperbolic structures this may no longer be true. In other words, two uniform lattices of $PSL_2(\mathbb{R})$, which are virtual surface groups, are not weakly commensurable in general. This is easily seen by noticing that a hyperbolic surface admits uncountably many hyperbolic structures, while there are only countably many surfaces in a commensurability class.

This suggests an even weaker and more flexible definition which seems better adapted to topology or, rather, large-scale geometry:

**Definition 2.3.** Two groups $H_1$ and $H_2$ are *abstractly commensurable* if there are finite-index subgroups $K_i \subset H_i$, $i = 1, 2$, which are isomorphic.

For instance, all finite groups are abstractly commensurable, as are all (hyperbolic) surface groups, and all free groups of finite rank $r$, with $r \geq 2$. On the other hand, two free abelian groups are abstractly commensurable if and only if they are isomorphic. Note also that two closed surface groups are abstractly commensurable if and only if the two surfaces admit the same type of geometry, that is spherical, Euclidean or hyperbolic.

Remark that on any given set of groups the relation "being abstractly commensurable" is reflexive, symmetric, and transitive.

## 3. Commensurators

This section will be devoted to discuss some properties of *commensurators* and *abstract commensurators*. These can be seen as invariants of a weak or abstract commensurability class respectively. We will start by giving their definitions.

**Definition 3.1.** Let $H$ a subgroup of a group $G$. The *commensurator of $H$ in $G$*, noted $Comm_G(H)$, is the set of elements $g \in G$ such that $H$ and $gHg^{-1}$ are commensurable.

It is not very difficult to see that $Comm_G(H)$ is a subgroup of $G$ which can be seen as a sort of generalised normaliser of $H$. Indeed, $Comm_G(H)$ can be thought of as the stabiliser of the commensurability class of $H$ for the conjugacy action of $G$ on the set of commensurability classes of its subgroups. The following inclusions are easy to verify:

$$\mathcal{Z}(G) \subset \mathcal{C}_G(H) \subset \mathcal{N}_G(H) \subset Comm_G(H)$$

where $\mathcal{Z}(G)$ denotes the centre of $G$, while $\mathcal{C}_G(H)$ and $\mathcal{N}_G(H)$ denote the centraliser and normaliser of $H$ in $G$ respectively. Of course one has

$$H \subset \mathcal{N}_G(H) \subset Comm_G(H).$$

Note that commensurable subgroups in the strong sense have the same commensurator while weakly commensurable ones have conjugate commensurators.

If commensurators can be seen as generalised normalisers, abstract commensurators generalise the notion of group of automorphism. The definition is a bit involved. Let $H$ be a group. Consider the set of all isomorphisms $f : K_1 \longrightarrow K_2$ between any two finite-index subgroups $K_1$ and $K_2$ of $H$. Define an equivalence relation on this set by imposing that two isomorphisms $f$ and $f'$ are equivalent if there exist a finite-index subgroup of $H$ on which $f$ and $f'$ coincide. Let $Comm(H)$ be the set of equivalence classes with respect to this relation.

**Definition 3.2.** The set $Comm(H)$ is called the *abstract commensurator of $H$*. It is endowed with a group structure induced by partial composition.

Observe that abstractly commensurable groups have isomorphic abstract commensurators. To see this, it suffices to observe that if $K$ is a finite-index subgroup of $H$ then $Comm(K) \cong Comm(H)$.

**Remark 3.3.** Given a group $H$, there is a natural group morphism from its automorphism group $Aut(H)$ to its commensurator group $Comm(H)$. Similarly, if $H$ is the subgroup of some group $G$, there is a natural morphism from $Comm_G(H)$ to $Comm(H)$.

Clearly, these natural group homomorphisms need not be injective. For instance, the latter morphism always factorises through $Comm_G(H)/\mathcal{C}_G(H)$. The following result can be found in [3] and gives a sufficient condition for the former natural morphism to be injective.

**Proposition 3.4.** *Let $H$ be a group with the* unique root property. *We have that $Aut(H)$ injects into $Comm(H)$.*

Before passing to the elementary proof of this result, we recall the definition of unique root property:

**Definition 3.5.** A group $H$ has the *unique root property* if the equality $h_1^n = h_2^n$, for elements $h_1$ and $h_2$ in $H$ and $n \in \mathbb{N}^*$, implies $h_1 = h_2$.

*Proof.* Let $f \in Aut(H)$ be an automorphism which is in the kernel of the natural morphism between $Aut(H)$ and $Comm(H)$. This means that there is a finite-index subgroup $K$ of $H$ on which $f$ acts as the identity. Since every finite-index subgroup contains a normal finite-index subgroup, without loss of generality, we can assume that $K$ is normal in $H$. The group $H/K$ is finite and, by Lagrange's theorem, there is positive integer $n$ such that, for each $h \in H$, $h^n \in K$. We have $h^n = f(h^n)$, because $f$ acts as the identity on the elements of $K$, and thus $h^n = f(h)^n$. The unique root property assures that $h = f(h)$ which implies that $f$ is the identity of $H$. $\square$

The above result applies, for instance, to free groups and free abelian groups. For free abelian groups it is not difficult to determine the commensurators groups:

$$GL_n(\mathbb{Z}) \cong Aut(\mathbb{Z}^n) \subset Comm(\mathbb{Z}^n) \cong GL_n(\mathbb{G}).$$

In general, though, abstract commensurators are not easily computed (some more examples will be provided later in this section). In their paper [3], Bartholdi and Bogopolski opt for a different approach, that is to provide information on the structure of these groups under the unique root property assumption. They establish a criterion for abstract commensurators to be non-finitely generated:

**Theorem 3.6** (Bartholdi-Bogopolski). *Let $H$ be a group with the unique root property. Assume that for infinitely many primes $p$ there are a normal subgroup $K$ of $H$ of index $p$, and an automorphism $f_K$ of $K$ which is not the restriction of some automorphism of $H$. Under these hypotheses $Comm(H)$ is not finitely generated.*

The above result applies in particular to free groups, showing that $Comm(F_n)$ is not finitely generated. It also applies to free abelian groups and indeed $Comm(\mathbb{Z}^n)$ is not finitely generated. For the interested reader, we mention that, in the same paper, Bartholdi and Bogopolski use this main result to deduce criteria for abstract commensurators not to be finitely generated in a variety of situations (free products of certain groups, as well as free products with amalgamation or HNN-extensions, both over $\mathbb{Z}$).

Among the groups whose abstract commensurators are known are the braid groups [25]:

**Theorem 3.7** (Leininger-Margalit). *For $n \geq 4$ one has*

$$Comm(B_n) \cong Mod(S_{n+1}) \ltimes (\mathbb{Q}^* \ltimes \mathbb{Q}^\infty),$$

*where $Mod(S_{n+1}) = \pi_0(Homeo(S_{n+1}))$ is the extended mapping class group of the sphere with $n+1$ punctures, $\mathbb{Q}^*$ is the multiplicative group of non-zero rationals, while $\mathbb{Q}^\infty$ is the product of countably many copies of the additive rationals.*

For braid groups, commensurators of $B_n$ seen as the natural subgroup of $B_m$, $n \leq m$, of the braids acting trivially on the last $m - n$ strands are also known [36]:

**Theorem 3.8** (Rolfsen). *For $1 \leq n \leq m \leq +\infty$ one has*

$$Comm_{B_m}(B_n) = \mathcal{N}_{B_m}(B_n) = \langle B_n, C_{B_m}(B_n)\rangle \cong B_n \times (B_{m-n+1})_1$$

*where $B_n$ acts on the first $n$ strands while $B_{m-n+1}$ acts on the last $m-n+1$ strands, and $(B_{m-n+1})_1$ denotes the stabiliser of the first of the $m - n + 1$ strands (corresponding to the $n$th strand of the $m$).*

Although the abstract commensurators we have met so far are quite large since they are all non finitely generated, they are not necessarily this way all the time, even for groups which are closely related to free groups and braid groups [18]:

**Theorem 3.9** (Farb-Handel). *For $n \geq 4$ one has*

$$Comm(Out(F_n)) \cong Out(F_n),$$

*where $Out(F_n)$ denotes the outer automorphism group of the free group of rank $n$.*

Note that the braid group $B_n$ injects into $Aut(F_n)$ and $B_n/\mathcal{Z}(B_n)$ injects into $Out(F_n)$. Remark, moreover, that for a whole large class of Artin groups Crisp proved that the abstract commensurators coincide with the automorphism groups [14].

We end this section by mentioning a result which provides some insight on the algorithmic complexity of abstract commensurators [2]:

**Theorem 3.10** (Arzhantseva-Lafont-Minasyan). *There are*

(i) *a class of finite presentations of groups in which the isomorphism problem is solvable but the abstract commensurability problem is not*
*and, conversely,*
(ii) *a class of finite presentation of groups in which the abstract commensurability problem is solvable but the isomorphism problem is not.*

## 4. Commensurability of hyperbolic 3-manifolds

We will now turn our attention to the study of commensurability of hyperbolic 3-manifolds. As we have seen, from an algebraic point of view this boils down to studying commensurability of lattices in $PSL_2(\mathbb{C})$. The contents of this section and the next one are mainly based on Walsh's survey on the commensurability of hyperbolic 3-orbifolds [38], although her approach is perhaps more geometric than the one proposed here. Some considerations on the commensurability of arbitrary 3-manifolds will also be found in this section.

The first remarkable result on the structure of the commensurability classes of lattices in $PSL_2(\mathbb{C})$ was obtained by Margulis in [28]. It establishes a dichotomy in behaviour for the classes of irreducible lattices $H$ in connected semi-simple Lie groups $G$ with trivial centre. When $G = PSL_2(\mathbb{C})$ Margulis results can be stated as follows:

**Theorem 4.1** (Margulis). *Let $H$ be a discrete subgroup of $G = PSL_2(\mathbb{C})$ of finite covolume. Then either $H$ is of finite index in $Comm_G(H)$ or $Comm_G(H)$ is dense in $G$. This second situation happens if and only if $H$ is* arithmetic.

Recall that a lattice in $PSL_2(\mathbb{C})$ is *arithmetic* if the trace field of $H$ is a number field with exactly one complex place. For the definition of *arithmetic lattice* in a generic connected semi-simple Lie group the reader is referred to [27]. A fundamental class of examples of arithmetic subgroups of $PSL_2(\mathbb{C})$ is that of *Bianchi groups*, i.e. subgroups of the form $PSL_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of the number field $\mathbb{Q}(\sqrt{-d})$, with $d \in \mathbb{N}$ square-free. In fact, these are basically the only examples of non-uniform arithmetic lattices, for any non-uniform arithmetic lattice is weakly commensurable to a Bianchi group.

A straightforward consequence of Margulis's result is that the commensurability class of a non-arithmetic manifold, that is a manifold whose fundamental group is a non-arithmetic lattice, contains a *minimal element*. In other words, there is an orbifold which is finitely covered by any other manifold and orbifold in the class. The fundamental group of such orbifold is precisely the commensurator of any group of the class. This suggests that commensurability classes of non-arithmetic manifolds are relatively easy to understand. On the other hand, for arithmetic manifolds one has [8]:

**Theorem 4.2** (Borel). *The commensurability class of an arithmetic 3-orbifold contains infinitely many minimal elements.*

Here a *minimal element* is an orbifold which does not cover any other orbifold in the commensurability class, apart from itself.

The above results provide a description of the structure of the commensurability classes of hyperbolic manifolds, but it is of course interesting to investigate other aspects of commensurability. For instance, one might want to classify manifolds up to commensurability or understand how certain particular classes of manifolds are distributed in the commensurability classes.

A possible way to answer the first question is to produce commensurability invariants. As we have seen, the arithmeticity (or non-arithmeticity) of a hyperbolic manifold is a very rough commensurability invariant. Of course, hyperbolic volume can also be used as a commensurability invariant: two commensurable manifolds have necessarily commensurable volumes. The converse is not true in general, though. For instance, the complements of two hyperbolic mutant links have the same volume (as well as same trace field and Bloch invariant) but need not be isometric; moreover, in the case where they are not isometric, the two complements may or may not be commensurable (see [13]).

The commensurator of the fundamental group is yet another invariant, albeit hard to handle. Other algebraic invariants are also known, like the *invariant trace field*, [27, 33], that is the smallest field containing the traces of the elements of the group which are products of squares, and, in the case of cusped manifolds, the *cusp field* generated by the *cusp parameter* or *cusp shape*, see for instance [31], where other commensurability invariants are also discussed.

The interest of the second question lies in the fact that this could have been a way to address Thurston's virtual fibering conjecture, now proved in the positive by Agol. The strategy consists in establishing whether each commensurability class contains a fibred manifold. This is, for instance, the spirit of a paper by Calegari and Dunfield [11], where they show that there are non-fibred knot complements in rational homology spheres which are not commensurable to any fibred knot complement in a rational homology sphere. More specifically, Hoste and Shanahan proved that any non-fibred twist knot (in the 3-sphere) is not commensurable to a fibred knot in a $\mathbb{Z}/2\mathbb{Z}$-homology sphere [23]. Of course these results underline some difficulties that one may encounter when trying to prove the virtual fibring conjecture this way.

In a similar vein, the problem of understanding the distribution of knots complements in the commensurability classes has been considered initially by Reid and Walsh and has since received a good deal of attention. The state of the art in the subject will be discussed in detail in the next section.

We end this section with some comments about the commensurability classification of other 3-manifolds admitting a geometric structure. This is well-known and can be found in [30] where the case of graph manifolds is also studied:

**Proposition 4.3.** *All compact orientable manifolds admitting a fixed geometry which is neither hyperbolic nor Sol, i.e. Seifert manifolds, belong to one commensurability class, with respect to the global group of isometries of the geometry.*

*The commensurability classes of compact Sol manifolds are in one-to-one correspondence with real quadratic number fields, generated by the eigenvalues of the monodromy of a torus bundle in the class.*

*All manifolds which are cusped but not hyperbolic belong to a unique commensurability class.*

The reader interested in the case of graph manifolds can also check [5].

## 5. The case of knots

A consequence of Proposition 4.3 is that all torus-knot complements are commensurable. According to what is known so far, the situation for hyperbolic knots seem to be completely different. The first result concerning the commensurability of hyperbolic knot complements can be deduced from a theorem obtained by Reid in [34]:

**Theorem 5.1** (Reid). *There is a unique arithmetic knot (in $\mathbf{S}^3$), that is the figure-eight.*

It follows that no other knot is commensurable to the figure-eight. Note that, by abuse of language, we say that *two knots are commensurable* to mean that so are their complements. This does not pose a problem, for knots are determined by their complements (see [21]). Ried's result is extremely striking if one considers that there are lots of arithmetic links. Indeed, it is trivial to observe that there are infinitely many links with two components whose complements are arithmetic, simply because there are infinitely many links whose complements are homeomorphic to that of the Whitehead link. More to the point, though, Reid observes that there are infinitely many two-component links whose complements are arithmetic and pairwise non-homeomorphic [34].

Since Ried's result other knots have been proved to be alone in their commensurability class. This is the case of all 2-bridge knots [35] and of all hyperbolic $(-2, 3, n)$-pretzel knots, with $n \in \mathbb{Z} \setminus \{7\}$, [26]. On the other hand, Ried and Walsh determined that there are precisely three knots in the commensurability class of the $(-2, 3, 7)$-pretzel knot [35]. Besides the $(-2, 3, 7)$-pretzel knot, infinitely many hyperbolic knots are known to be commensurable to two other knots [22].

Again in [35], Ried and Walsh conjectured that a commensurability class of hyperbolic manifolds contains at most three knot complements. The conjecture has been recently proved for a large class of knots [7]:

**Theorem 5.2** (Boileau-Boyer-Cebanu-Walsh). *If a hyperbolic knot has* no hidden symmetries *then its commensurability class contains at most two other knots.*

There are several equivalent ways to express the condition on the knot that appears in the statement of the theorem: A hyperbolic knot has *no hidden symmetries* if its fundamental group is normal in its commensurator. Equivalently, the knot complement is a finite regular cover of the minimal element in its commensurability class. Different known facts about knots with hidden symmetries seem to provide evidence that knots of this type must be rare. Indeed, besides the figure-eight knot, only two (non-arithmetic) knots with hidden symmetries are known so far. These are the two dodecahedral knots constructed by Aitchison and Rubinstein in [1]. Moreover, if a non-arithmetic knot has hidden symmetries, the minimal element in its commensurability class has a rigid cusp, i.e. a Euclidean turnover, and in particular its cusp shape belongs either to $\mathbb{Q}[\sqrt{-3}]$, like the figure-eight's, or to $\mathbb{Q}[i]$ (see [31]). Note that the cusp shape of the two dodecahedral knots belongs to $\mathbb{Q}[\sqrt{-3}]$, while just one knot with cusp shape in $\mathbb{Q}[i]$ is known and it has no hidden symmetries (see [20]). Conjecturally, besides the dodecahedral knots there are no other non-arithmetic knots with hidden symmetries (see [7, 31]).

From Boileau, Boyer, Cebanu, and Walsh's work it also emerges that a knot must satisfy several properties in order to be *cyclically commensurable* to another knot. By *cyclically commensurable* one means that the complements of the two knots have a common finite-sheeted cyclic cover or, equivalently, a common cyclic finite quotient.

**Theorem 5.3** (Boileau-Boyer-Cebanu-Walsh). *Let $K$ be a hyperbolic knot cyclically commensurable to another knot $K'$. Then*

(i) *$K$ and $K'$ are fibred;*
(ii) *$K$ and $K'$ have the same genus;*
(iii) *$K$ and $K'$ have different volume, in particular $K$ and $K'$ are not mutants;*
(iv) *$K$ and $K'$ are chiral and not commensurable with their mirror images.*

Note that two commensurable knots with no hidden symmetries are in fact cyclically commensurable.

**Remark 5.4** (Boileau-Boyer-Cebanu-Walsh). In contrast to what happens to cyclically commensurable knots according to Theorem 5.3, the two dodecahedral knots, which belong to the same commensurability class but are not cyclically commensurable,

(i) are one fibred and the other not;
(ii) have different genera;
(iii) have the same volume;
(iv) are both amphicheiral.

To provide some insight on Boileau, Boyer, Cebanu, and Walsh's result as well as to stress its connection with the cyclic surgery theorem [16], which also explains the expected bound on the number of knots in a commensurability class, we will prove a much weaker version of their result under several simplifying assumptions (see also [35]):

**Proposition 5.5.** *Let $K$ be a hyperbolic knot with no hidden symmetries and with no symmetries. Assume that all knots in its commensurability class have no hidden symmetries. The commensurability class of $K$ contains at most three knots.*

*Proof.* The fact that $K$ has no hidden symmetries and no symmetries implies that its complement is the minimal element in its commensurability class. Let $K'$ be another knot whose complement is commensurable to $K$. Since $K'$ has no hidden symmetries, its complement must be a finite regular cover of $\mathbf{S}^3 \setminus K$. Of course, since the covering induces a covering of the peripheral tori, it must be cyclic. The covering transformation group extends to the whole 3-sphere containing $K'$. The extended action is free, according to Smith's conjecture [29]. The quotient by this action is a lens space obtained by Dehn surgery on $K$. The cyclic surgery theorem implies that, besides the meridian, at most two other slopes on the boundary of the exterior of $K$ can produce a lens space. As a consequence, this limits the number of possible covers to two. Notice that, for a given slope, the order of the covering is determined by the type of lens space obtained. □

## 6. Commensurability and quasi-isometry

The considerations made in Section 2 suggested that abstractly commensurable manifolds shared similar large-scale geometry properties. Indeed, abstractly commensurable manifolds are *quasi-isometric.* We recall the definition [10]:

**Definition 6.1.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A map $\phi : X \longrightarrow Y$ is a *quasi-isometry* if there exist constants $L \geq 1$ and $C \geq 0$ such that

$$L^{-1} d_X(x, x') - C \leq d_Y(\phi(x), \phi(x')) \leq L d_X(x, x') + C$$

for all $x, x' \in X$, and for all $y \in Y$ there is $x \in X$ such that

$$d_Y(y, \phi(x)) \leq C.$$

Recall that given a finitely generated group $H$ and a finite generating set it is possible to define a natural metric structure, called *word metric*, on $H$ induced by the natural embedding of $H$ as the set of vertices of its Cayley graph. One can prove that the identity map on $H$ is a quasi-isometry with respect to any two metric structures associated to two finite sets of generators for $H$. Similarly, the inclusion map of a finite index subgroup into a finitely-generated group is a quasi-isometry (with respect to any word metric). It follows that two finitely-generated, abstractly-commensurable groups are quasi-isometric.

The converse is not true in general. Indeed, recall that, if $H$ is a finitely generated group of isometries of a simply-connected metric space $X$, such that the action of $H$ is properly discontinuous and cocompact, then $X$ and $H$ (with any fixed word metric) are quasi-isometric. As a consequence, all uniform lattices in $PSL_2(\mathbb{C})$ are quasi-isometric, for they are quasi-isometric to hyperbolic 3-space, although they are not all commensurable. Note also that, as a consequence of Mostow's rigidity, two uniform lattices of $PSL_2(\mathbb{C})$ are weakly commensurable if and only if they are abstractly commensurable.

On the other hand, for non uniform lattices, which are not quasi-isometric to $\mathbf{H}^3$, the two notions coincide [37]:

**Theorem 6.2** (Schwartz). *Non uniform lattices of $PSL_2(\mathbb{C})$ are commensurable if and only if they are quasi-isometric.*

As we have already seen, the two notions coincide also for lattices of $PSL_2(\mathbb{R})$. Indeed, all uniform lattices are virtual hyperbolic surface groups and all hyperbolic surface groups are abstractly commensurable while all non uniform lattices are virtually non-abelian free groups of finite rank which are again all commensurable.

More surprisingly, if $H_1$ is any finitely generated group which is quasi-isometric to a uniform lattice $H_2$ of $PSL_2(\mathbb{R})$, then $H_1$ and $H_2$ are abstractly commensurable. This is a consequence of two very deep results. The first one is due to Bowditch and gives a topological characterisation of hyperbolic groups as groups that act as *uniform convergence groups* on a compactum, which is their boundary [9]. The second result shows that a group of homeomorphisms of the circle is conjugate to a lattice in $PSL_2(\mathbb{R})$ if and only if the group is a *convergence group* and is due to Casson-Jungreis [12] and Gabai [19].

For the readers convenience, we recall one of the possible equivalent definitions of convergence groups:

**Definition 6.3.** Let $H$ be a group acting on a space $X$. $H$ is a *convergence group* if the induced diagonal action on the set of triples of pairwise distinct points of $X$ is properly discontinuous. If moreover the induce action is cocompact, $H$ is a *uniform convergence group.*

Cannon's conjecture claims that the above result holds true in dimension 3 as well, that is, a group which is quasi-isometric to a uniform lattice of $PSL_2(\mathbb{C})$ should be also commensurable to it. Cannon's conjecture is definitely one of the paramount open problems relating geometric group theory and low dimensional topology and geometry.

We end this section by observing that if $H$ is any group which is abstractly commensurable to a uniform lattice in $PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{C})$ then $H$ is an extension of a uniform lattice by a finite group. To see this in the case of dimension 2, it is enough to consider the convergence action on the boundary: $H$ is the extension of its image in $PSL_2(\mathbb{R})$ by the kernel of the action. In dimension 3, one reasons similarly by considering a uniform lattice $K$ of $PSL_2(\mathbb{C})$ which is a normal subgroup of finite index inside $H$ and mapping $H$ to $Aut(K) \subset PSL_2(\mathbb{C})$.

## 7. Some classification results in geometric group theory

In this section we will present some results of classification up to abstract commensurability of some families of groups. When possible, a comparison with the classification up to quasi-isometry for the same family will be given. We will mainly be interested in Artin and Coxeter groups. Artin groups are defined by the specific form of their presentations which are encoded by non-oriented labelled graphs. The vertices of the graph correspond to the generators of the group, and there is precisely one relation for each edge in the graph: If $v_1$ and $v_2$ are generators corresponding to two vertices connected by an edge labelled $m \in \mathbb{N} \setminus \{0,1\}$, the relation associated to the edge is $v_1v_2 \cdots = v_2v_1 \ldots$, where the two alternating words in the letters $v_1$ and $v_2$ on both sides of the equality have length $m$[1]. To each Artin group one can associate a Coxeter group whose presentation is obtained by the one for the Artin group by imposing that all generators have order 2. As a consequence, Coxeter groups can, too, be encoded by a labelled graph.

Artin groups constitute a large class of groups and some of the groups we have already encountered are in fact Artin groups. Among these, braid groups are probably the most well-known family of Artin groups. Other Artin groups we have already seen are free groups, corresponding to graphs without edges, and free abelian groups, corresponding to complete graphs all of whose edges are labelled with a 2. These are moreover examples of *right-angled Artin groups*, that is Artin groups associated to graphs all of whose labels are equal to 2 (and, as a consequence, all labels can be omitted). Free products of free abelian groups provide another example of right-angled Artin groups. These groups correspond to graphs which are disjoint unions of complete graphs. The classification of (non trivial) free products of free abelian groups up to quasi-isometry was obtained by Papasoglu and Whyte [32] and that up to commensurability by Behrstock, Januszkiewicz, and Neumann [4]. It turns out the two classifications coincide:

**Theorem 7.1** (Behrstock-Januszkiewicz-Neumann+Papasoglu-Whyte). *Let $H = H_1 * \cdots * H_n$ and $H' = H_1' * \cdots * H_m'$ be two non-trivial (i.e. $n, m > 1$) free products of finitely generated non-trivial abelian groups. Assume $H, H' \neq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. The following are equivalent:*

(i) *$H$ and $H'$ are commensurable;*
(ii) *$H$ and $H'$ are quasi-isometric;*
(iii) $\{\mathrm{rk}(H_1), \ldots, \mathrm{rk}(H_n)\} \setminus \{0,1\} = \{\mathrm{rk}(H_1'), \ldots, \mathrm{rk}(H_m')\} \setminus \{0,1\}$.

The classification up to quasi-isometry is known for other classes of right-angled Artin groups, while the classification up to commensurability seems still relatively elusive. Besides those discussed above, other right-angled Artin groups that are classified up to quasi-isometry are:

- Groups whose presentation graphs are trees [5, 24]: there are four quasi-isometry classes according to whether the diameter of the tree is 0, 1, 2, or $\geq 3$;
- Groups whose presentation graphs are *atomic* [6] which are classified by their graphs; by definition a graph is *atomic* if it is connected, has no valence-one vertices, no cycles of length less than five, and no separating closed vertex stars;
- Certain right-angled Artin groups of higher dimension [4].

[1] Note that in the literature one can find other conventions for encoding the presentation of an Artin group. We are not going to discuss them here.

For arbitrary Artin groups a classification result up to abstract commensurability of a specific class was obtained by Crisp in [14]. Crisp considered the class of 2-dimensional Artin groups with connected, large type, triangle-free defining graphs with no separating edge or vertex, and proved that two groups of the class are abstractly commensurable if and only if they have label-isomorphic presentation graphs.

The following result by Davis and Januszkiewicz [17] shows that the classification of right-angled Coxeter groups up to abstract commensurability would imply the same type of classification for right-angled Artin groups. More important still, an immediate consequence of their result is that right-angled Artin groups are linear, since so are right-angled Coxeter ones.

**Theorem 7.2** (Davis-Januszkiewicz). *For each right-angled Artin group there is a right-angled Coxeter group that contains it of finite index.*

So far very little is known on the classification up to abstract commensurability of Coxeter groups and even of right-angled ones. The simplest examples of Coxeter groups are obtained by considering discrete groups of isometries of spherical, Euclidean or hyperbolic space generated by reflections in the faces of spherical, Euclidean or hyperbolic polyhedra respectively. When the faces of the polyhedra always meet at right angles, the associated Coxeter groups are right-angled.

In dimension 2, that is in the case of polygons, and assuming the polygons are compact, Coxeter groups are virtual surface groups, and so they belong to three abstract commensurability classes (or, equivalently, quasi-isometry classes), according to the geometry of the polygon. Note that the graphs associated to these Coxeter groups are just circles with a finite number of vertices marked. The number of vertices and edges is precisely that of the reflection polygon. However, the graph is somehow dual to the perimeter of the reflection polygon with edges of the graph corresponding to vertices of the polygon and a label $m$ on an edge if the angle at the corresponding vertex of the polygon is $\pi/m$. Note that the discrete groups of reflections in the faces of a non-compact polygon are virtually-free Coxeter groups.

The first example of classification up to abstract commensurability of a non-trivial class of right-angled Coxeter groups (of dimension 2) was obtained in [15]. The classification up to quasi-isometry of the family considered in [15] is still unknown as is the existence of embeddings of the groups of the class into the group of isometries of hyperbolic 3-space.

The graphs associated to the groups of the family are theta-curves with one arc containing precisely one vertex in its interior, and the other two arcs containing at least two vertices (and, of course, all edges with labels equal to 2). These can also be seen as two circles with at least five vertices each identified along two closed sub-paths of length two. The identification is reflected in the structure of these groups which are obtained as free products of two groups of reflections in the faces of two right-angled hyperbolic polygons amalgamated along an infinite dihedral group times a cyclic group of order 2. Geometrically, these groups are just orbifold fundamental groups of 2-complexes obtained by gluing together along an edge a right-angled hyperbolic $(n+4)$-gon and a right-angled hyperbolic $(m+4)$-gon whose edges are all silvered. Note that the common edge is also a silvered edge. In other terms the two reflection polygons are not "contained in the same plane".

The classification result in [15] is as follows:

**Theorem 7.3** (Crisp-Paoluzzi). *Let $W_{m,n}$ the right-angled Coxeter group associated to a theta-curve with arcs of lengths 2, $n+2$ and $m+2$ respectively, where $1 \leq m \leq n$. The groups $W_{m,n}$ and $W_{k,l}$ are abstractly commensurable if and only if $\frac{m}{n} = \frac{k}{l}$.*

We end by observing that the sufficiency of the condition is easy to establish. Indeed, consider the orbifold 2-complex obtained by gluing together a right-angled hyperbolic $(n+4)$-gon and a right-angled hyperbolic $(m+4)$-gon. By reflecting the complex $s$ times with respect to its sides which are orthogonal to the "double edge" one obtains another orbifold 2-complex of the same type but obtained by gluing together a right-angled hyperbolic $(sn+4)$-gon and a right-angled hyperbolic $(sm+4)$-gon.

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LATP, Aix-Marseille Univ.

*E-mail address*: `paoluzzi@cmi.univ-mrs.fr`