On inducing degenerate sums through 2-labellings

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Labellings

Definition

A *k-labelling* of a graph G is a function $\ell : E(G) \rightarrow \{1, \dots, k\}$.

Definition

We call *resulting sum* (relative to a labelling ℓ) of a vertex u the sum of labels on edges incident to u. We denote it $\sigma_\ell(u)$.

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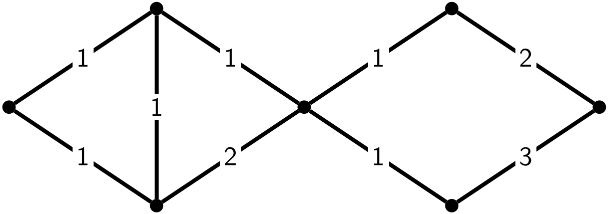
We say a labelling ℓ is *distinguishing* if for every two adjacent vertices u and v of G, $\sigma_\ell(u) \neq \sigma_\ell(v)$.

Small example



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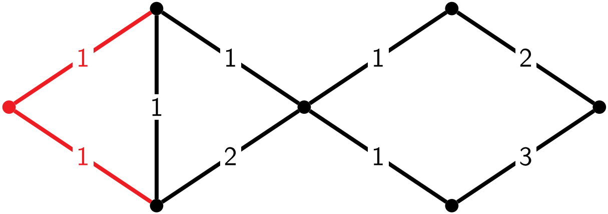

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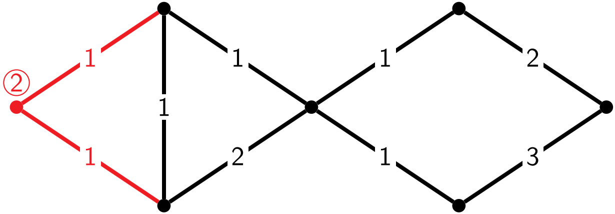

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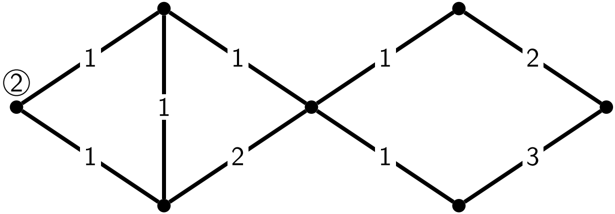

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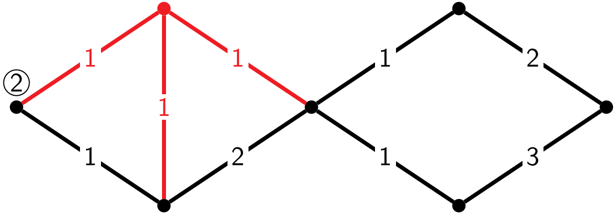

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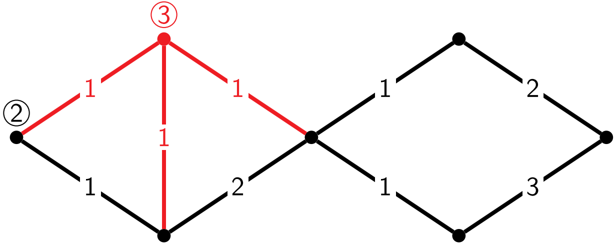

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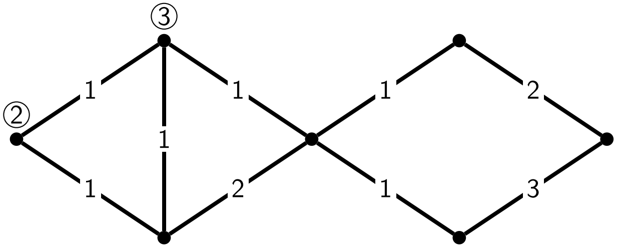

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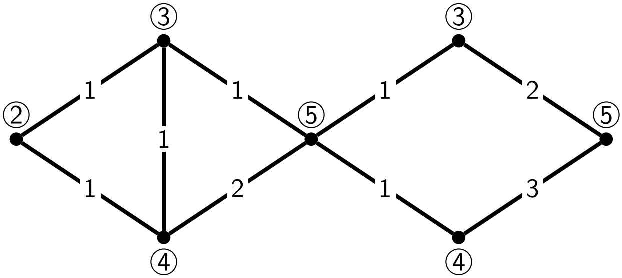

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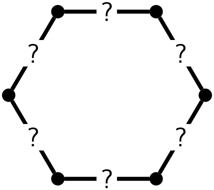

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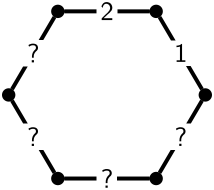

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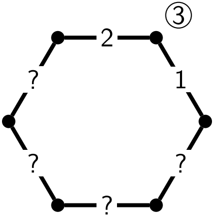

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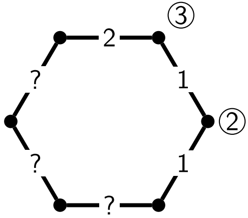

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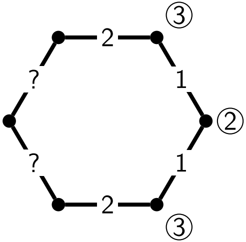

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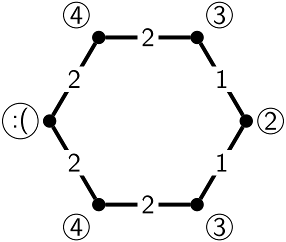

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These are the main conjectures and theorems in the field:

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All graphs admit a distinguishing 3-labelling.

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And some specific cases. For instance, the 1-2-3 Conjecture is known to hold for 3-colorable graphs.

Labellings (again)

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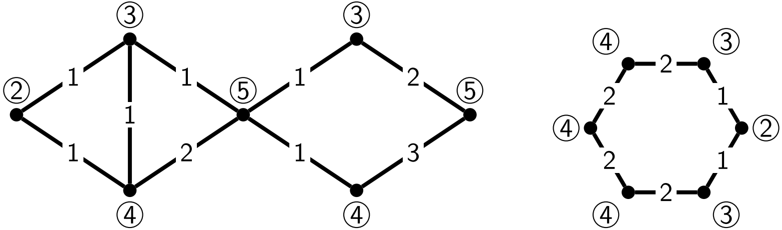

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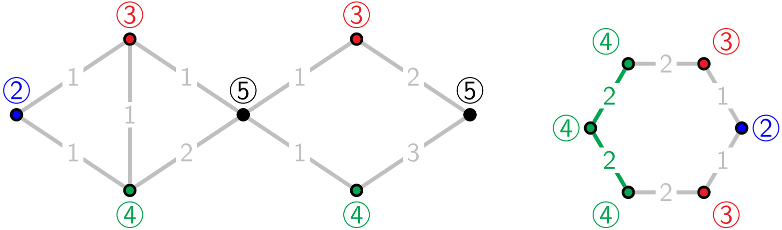

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Degenerate 1-2 Conjecture (Gao *et al., 2015*)

All graphs admit a degenerate 2-labelling.

They proved the conjecture for:

- Graphs with $mad \leq 3$.
- Series-parallel graphs.
- Complete bipartite graphs.
- Cycles.
- Complete graphs.

Our contribution

From this exploratory work, our contribution consists in multiple things:

- Re-defining the problem in usual terms;
- Link it to several well-known graph notions;
- Improve the result on complete bipartite graphs to all bipartite graphs;
- Improve the results on series-parallel graphs to all 2-degenerate graphs;
- Improve the *mad* bound to $\frac{10}{3}$.

Theorem (Bensmail *et al.*, 2023+)

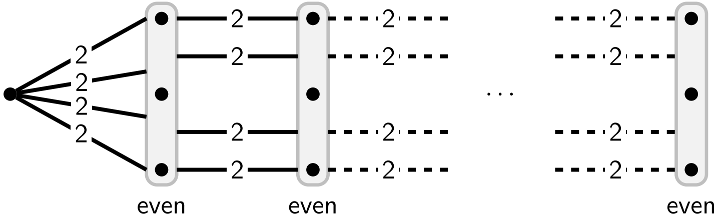
Let G be a graph. If G is bipartite, 2-degenerate or of $mad < \frac{10}{3}$, then G admits a degenerate 2-labelling.

Bipartite graphs

Theorem

Let G be a bipartite graph. Then G admits a degenerate 2-labelling.

We use a classical technique of path swapping from the field.



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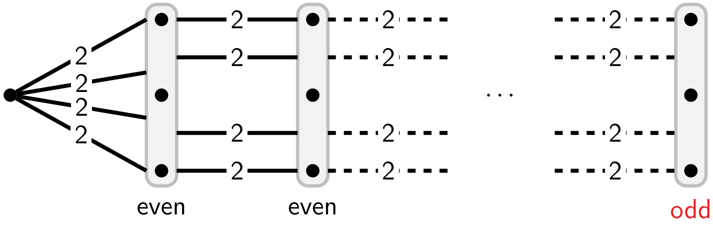

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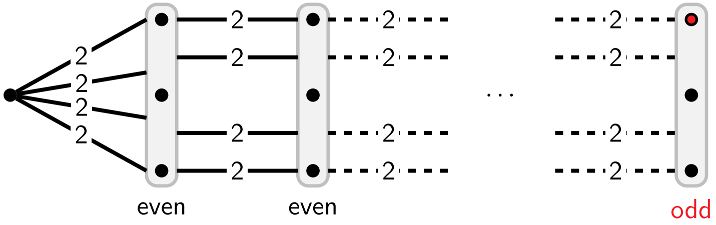

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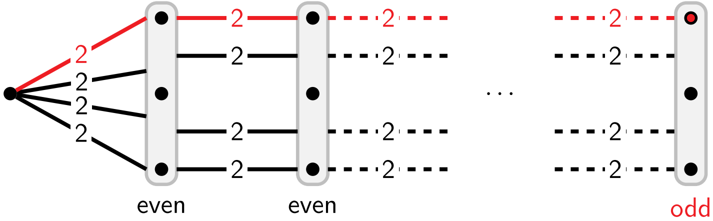

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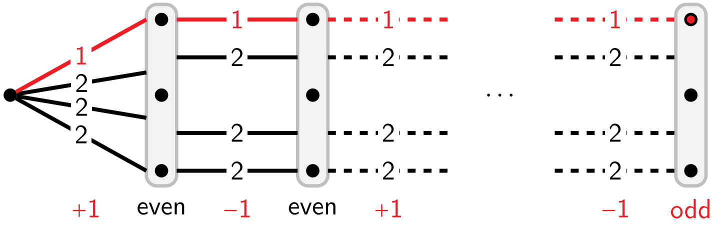

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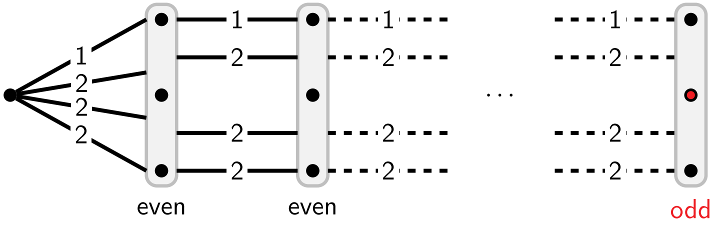

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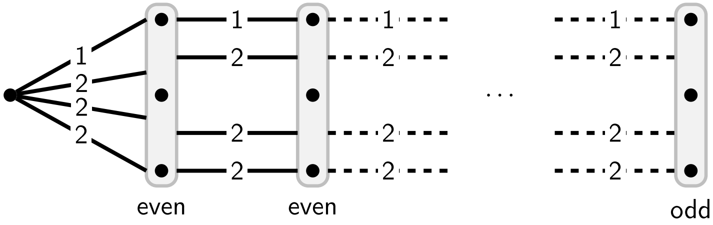

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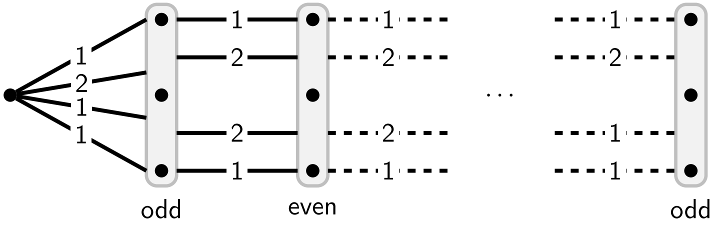

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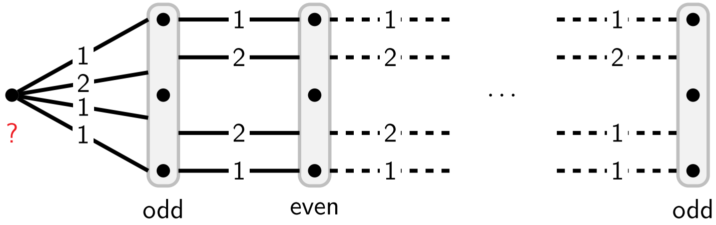

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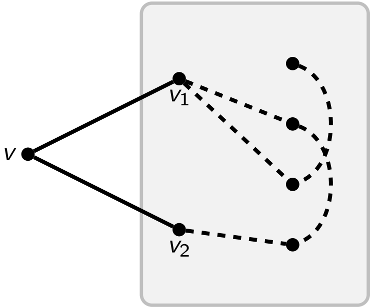
We swap paths according to a degenerate 2-coloring layout.



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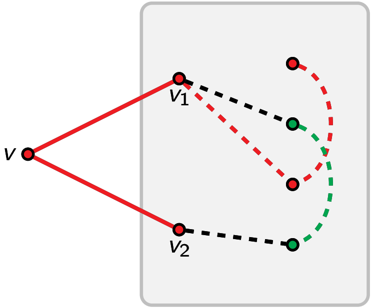

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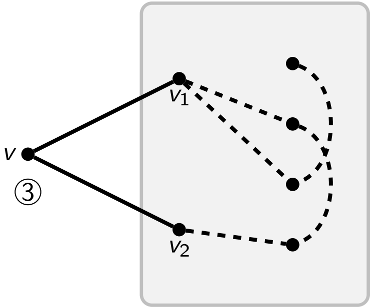

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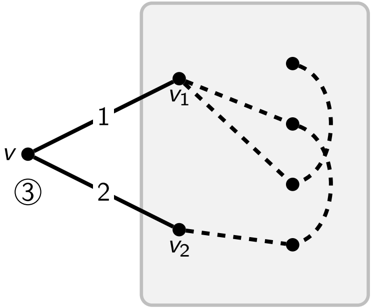

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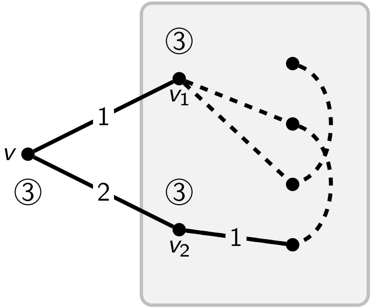

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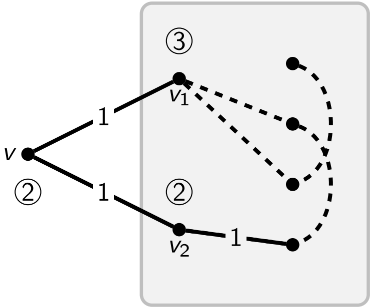

Main result

$mad(G)$ is the maximum density of an induced subgraph of G.

Theorem (Gao *et al.*, 2015)

Let G be a graph of $mad \leq 3$. Then G admits a degenerate 2-labelling.

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Corollary

Let G be a planar graph of girth 6. Then G admits a degenerate 2-labelling.

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Theorem (Bensmail *et al.*, 2023+)

Let G be a graph of $mad < \frac{10}{3}$. Then G admits a degenerate 2-labelling.

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Let G be a planar graph of girth 5. Then G admits a degenerate 2-labelling.

Main ideas

Definition

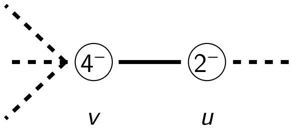
We say a graph G with $mad(G) < \frac{10}{3}$ is a *minimal counter example* (relative to a theorem) if there is no H with $mad(H) < \frac{10}{3}$ such that $|E(H)| + |V(H)| < |E(G)| + |V(G)|$.

- Suppose we have a minimal counterexample (minimal CE) to the theorem;
- Prove that it cannot contain some sparse structures;
- Put charge $d(v) - \frac{10}{3}$ on every vertex v;
- Move charges between vertices;
- Prove that the *mad* is too big.

An example of a reducible configuration

Theorem

Let G be a minimal CE. Then G does not contain a 4^{-}-vertex adjacent to a 2^{-}-vertex.



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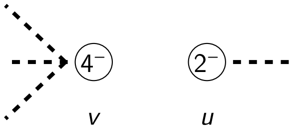
By minimality of G, we can compute ℓ' a degenerate 2-labelling of $G - \{uv\}$.



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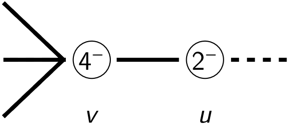
Suppose $d(v) = 4$.



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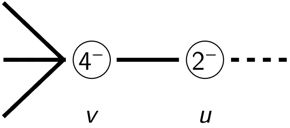
Suppose $d(v) = 4$.
Then $\sigma_\ell(v) > \sigma_\ell(u)$.



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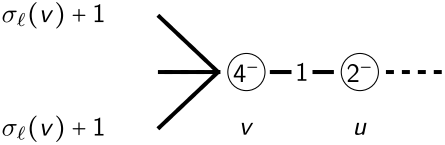
We can pick a fitting label!



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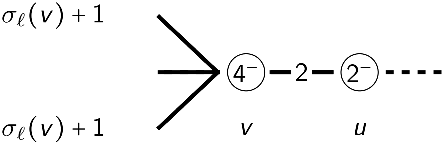

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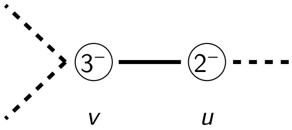

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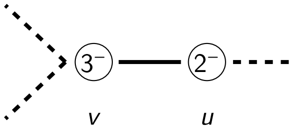

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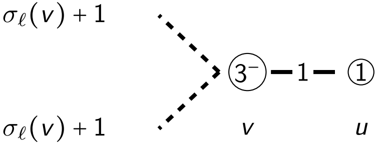
By arguments very similar.



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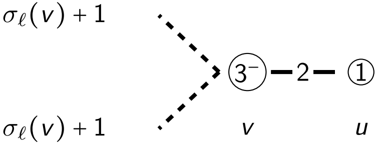

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Suppose $d(v) = 2$.
We can choose a label such that $\sigma_{\ell}(v) + \ell'(uv)$ is not equal to the resulting sum of the only neighbour of v other than u.

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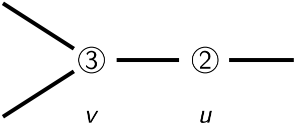
Let G be a minimal CE. Then G does not contain a 4^{-}-vertex adjacent to a 2^{-}-vertex.

We can now assume $d(v) = 3$.



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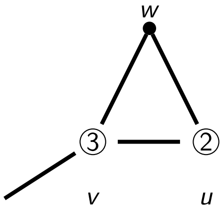
Let G be a minimal CE. Then G does not contain a 4^{-}-vertex adjacent to a 2^{-}-vertex.

We can now assume $d(v) = 3$. We just pick a label by ℓ' for uv such that $\sigma_{\ell}(v) + \ell'(uv) \neq \sigma_{\ell}(w)$.



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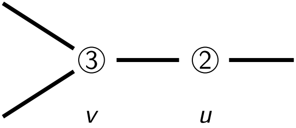
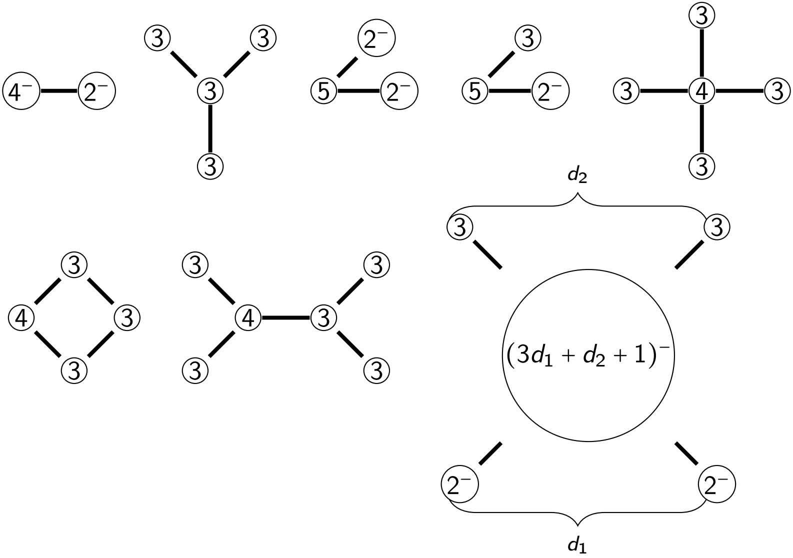
We generalize the idea in a lemma.



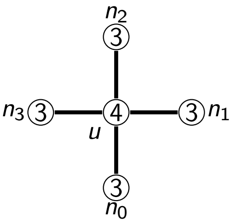
Reducible configurations



Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

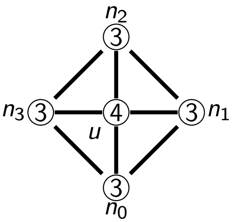


We need to consider possible inner edges, and complete a labelling of $G - \{u\}$ and without the inner edges.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

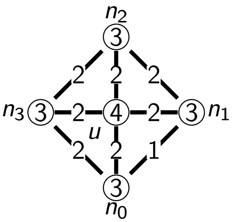


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 4$.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

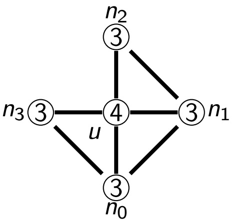


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 4$.
Then G is a wheel of order 5.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

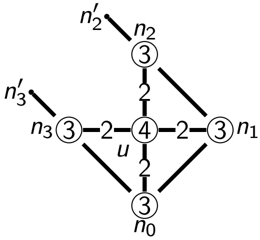


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 3$.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

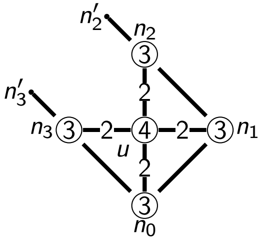


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 3$. We ensure a large sum for u.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

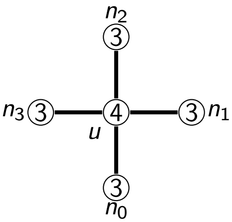


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 3$. We ensure a large sum for u. We pick remaining labels according to the resulting sums of n_2' and n_3'.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

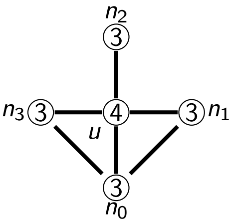


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$. There are two subcases.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

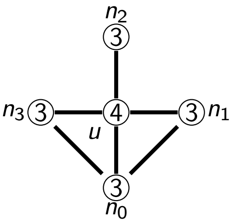


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
Assume inner edges are incident to a common vertex.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

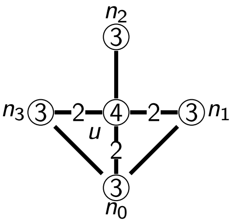


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$. Assume inner edges are incident to a common vertex. We have no control over the label of un_2.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

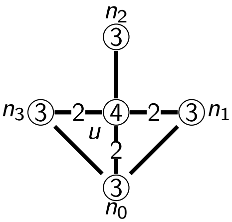


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$. Assume inner edges are incident to a common vertex. We can still ensure that the resulting sum of u is large.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

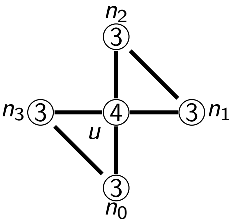


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$. Assume inner edges are incident to a common vertex. We can still ensure that the resulting sum of u is large. We conclude with Lemma.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

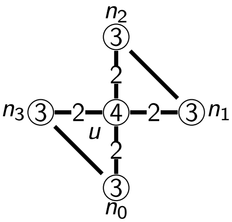


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$.
Now assume this configuration.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

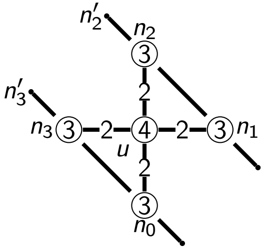


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$. Now assume this configuration. The resulting sum of u is large.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

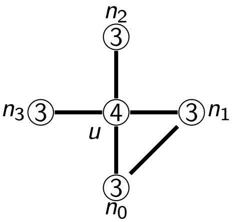


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 2$. Now assume this configuration. The resulting sum of u is large. We pick labels for $n_1 n_2$ and $n_0 n_3$ to differentiate them from their neighbours.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

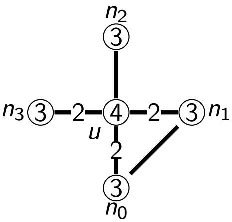


Assume
$|E(G[\{n_0, n_1, n_2, n_3\}])| = 1.$

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

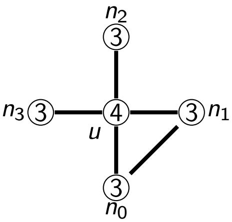


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 1$. If we have the choice for at least one of un_2 or un_3, we can ensure a large resulting sum for u, and pick a label for n_0n_1 like in last case.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

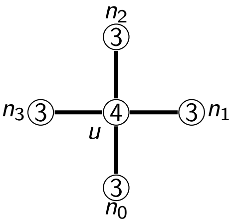


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 1$.
Otherwise, we pick fitting labels for un_2, un_3, and we can make the resulting sum of one of n_0 or n_1 strictly bigger than the other.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

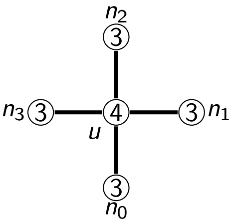


Assume
$|E(G[\{n_0, n_1, n_2, n_3\}])| = 0$.

Some more Reductions

Configuration 5

Let G be a minimal CE. Then G does not contain a 4-vertex adjacent to four 3-vertices.

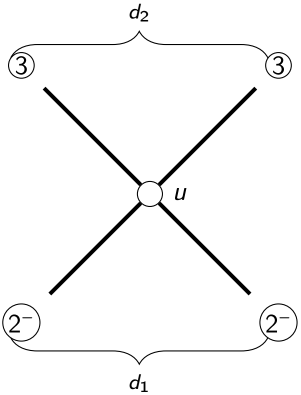


Assume $|E(G[\{n_0, n_1, n_2, n_3\}])| = 0$. We have a lemma.

Another Reduction

Configuration 8

Let G be a minimal CE, d_1, d_2 two integers such that $3d_1 + d_2 + 1 \geq 6$. Then G does not contain a $(3d_1 + d_2 + 1)^-$-vertex adjacent to d_2 3-vertices and d_1 2^--vertices.



We consider a labelling of $G - \{uv_1, \ldots, uv_{d_1}, uw_1, \ldots, uw_{d_2}\}$ and we extend it.

Another Reduction

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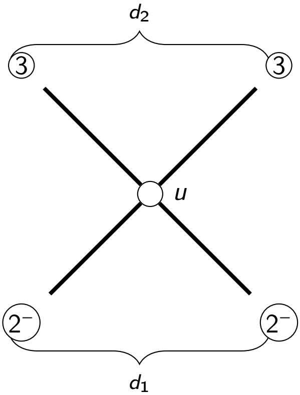
Let G be a minimal CE, d_1, d_2 two integers such that $3d_1 + d_2 + 1 \geq 6$. Then G does not contain a $(3d_1 + d_2 + 1)^-$-vertex adjacent to d_2 3-vertices and d_1 2^--vertices.



We consider a labelling of $G - \{uv_1, \ldots, uv_{d_1}, uw_1, \ldots, uw_{d_2}\}$ and we extend it. $G[\{w_1, \ldots, w_{d_2}\}]$ has maximum degree 2. Its connected component are either cycles, paths or isolated vertices.

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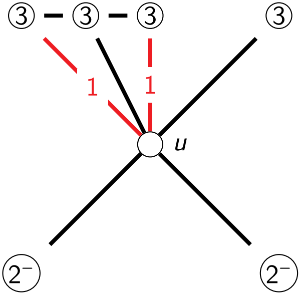
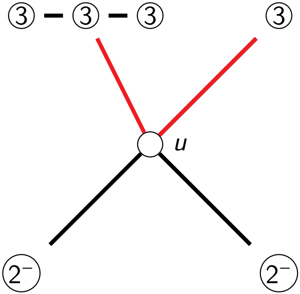


We consider a labelling of $G - \{uv_1, \ldots, uv_{d_1}, uw_1, \ldots, uw_{d_2}\}$ and we extend it. $G[\{w_1, \ldots, w_{d_2}\}]$ has maximum degree 2. Its connected component are either cycles, paths or isolated vertices. For each component with edges, we pick one arbitrary w_i and assign label 1 to every other uw_j of the component.

Another Reduction

Configuration 8

Let G be a minimal CE, d_1, d_2 two integers such that $3d_1 + d_2 + 1 \geq 6$. Then G does not contain a $(3d_1 + d_2 + 1)^-$-vertex adjacent to d_2 3-vertices and d_1 2^--vertices.

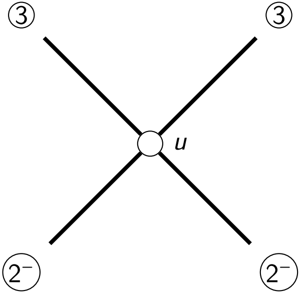


We consider a labelling of $G - \{uv_1, \dots, uv_{d_1}, uw_1, \dots, uw_{d_2}\}$ and we extend it. We assign a fitting label to every remaining uw_i.

Another Reduction

Configuration 8

Let G be a minimal CE, d_1, d_2 two integers such that $3d_1 + d_2 + 1 \geq 6$. Then G does not contain a $(3d_1 + d_2 + 1)^-$-vertex adjacent to d_2 3-vertices and d_1 2^--vertices.



We consider a labelling of $G - \{uv_1, \ldots, uv_{d_1}, uw_1, \ldots, uw_{d_2}\}$ and we extend it. There can be no cycle with one of the w_i. We only need to be sure that u does not have the same resulting sum as two of its neighbours.

Another Reduction

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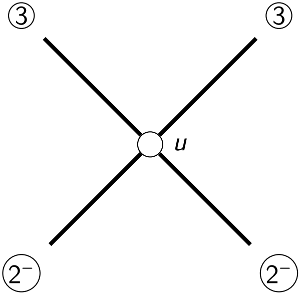
Let G be a minimal CE, d_1, d_2 two integers such that $3d_1 + d_2 + 1 \geq 6$. Then G does not contain a $(3d_1 + d_2 + 1)^-$-vertex adjacent to d_2 3-vertices and d_1 2^--vertices.



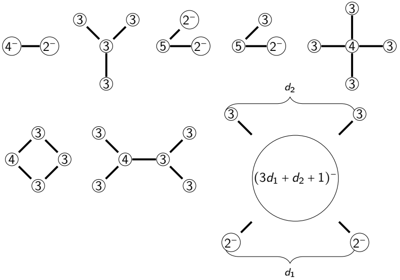
We consider a labelling of $G - \{uv_1, \ldots, uv_{d_1}, uw_1, \ldots, uw_{d_2}\}$ and we extend it.
We have a Lemma.

Our discharging process

Degree	1	2	3	4	5	6
Initial charge	$\frac{-7}{3}$	$\frac{-4}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{9}{3}$
Final charge	$\frac{-4}{3}$	$\frac{2}{3}$	0	0	0	0

Weak 3-vertices

A weak 3-vertex is a 3-vertex adjacent to exactly one 4^+-vertex, and it is a 4-vertex.



(R1) Every 5^+-vertex sends 1 to each of its 2^--neighbours.

(R2) Every 5^+-vertex sends $\frac{1}{3}$ to each of its 3-neighbours.

(R3) Every 4-vertex sends $\frac{1}{3}$ to each of its weak 3-neighbours.

(R4) Every 4-vertex sends $\frac{1}{6}$ to each of its non-weak 3-neighbours.

The final weights

Consider for instance v a 3-vertex. Note that $\omega(v) = -\frac{1}{3}$, and that we need $\omega^*(v) \geq 0$:

- If v is weak, then v has only one 4^+-neighbour, being a 4-vertex, which sent $\frac{1}{3}$ to v by Rule R3.
- If v is not weak, then either v neighbours at least one 5^+-vertex, or v neighbours at least two 4-vertices. In the former case, at least one 5^+-neighbour of v sent $\frac{1}{3}$ to v by Rule R2, while, in the latter case, at least two 4-neighbours of v both sent $\frac{1}{6}$ to v by Rule 4.

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Consider for instance v a 3-vertex. Note that $\omega(v) = -\frac{1}{3}$, and that we need $\omega^*(v) \geq 0$:

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Rule R3

Every 4-vertex sends $\frac{1}{3}$ to each of its weak 3-neighbours.

We prove the same result for every possible degree.

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Rule R2

Every 5^+-vertex sends $\frac{1}{3}$ to each of its 3-neighbours.

We prove the same result for every possible degree.

The final weights

Consider for instance v a 3-vertex. Note that $\omega(v) = -\frac{1}{3}$, and that we need $\omega^*(v) \geq 0$:

- If v is weak, then v has only one 4^+-neighbour, being a 4-vertex, which sent $\frac{1}{3}$ to v by Rule R3.
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We prove the same result for every possible degree.

The final weights

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Rule R4

Every 4-vertex sends $\frac{1}{6}$ to each of its non-weak 3-neighbours.

We prove the same result for every possible degree.

Results

In this presentation, we proved the degenerate 1-2 Conjecture:

- for 2-degenerate graphs;
- for bipartite graphs;
- for graphs of $mad < \frac{10}{3}$.

In fact, it also holds for graph of edge weight 7. We also have a corollary to the *mad* result:

Corollary

If G is a planar graph with $g(G) \geq 5$, then it admits a degenerate 2-labelling.

Perspectives

- Other classes of graph with vertex arboricity at most 2:
  - Graphs of maximum degree 4;
  - Graphs of degeneracy 3.
- Denser graphs (with bigger *mad*).

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- Other classes of graph with vertex arboricity at most 2:
  - Graphs of maximum degree 4;
  - Graphs of degeneracy 3.
- Denser graphs (with bigger *mad*).

Thank you for your attention !

Discharging

We will apply the following result to our minimal CE.

Theorem (Bonamy *et al.*, 2013)

Let G be a graph, m a value and (V_1, V_2) a partition of $V(G)$. Let ω be the charge function where $\omega(v) = d(v) - m$ for every $v \in V(G)$. If there is a discharging process resulting in a charge function ω^* such that:

- $\omega^*(v) \geq 0$ for every $v \in V_1$, and
- $\omega^*(V) \geq \omega(v) + d_{V_1}(v)$ for every $v \in V_2$,

Then $mad(G) \geq m$.

We consider V_2 the set of all the 2^--vertices of G. Note that by Configuration 1, $d_{V_1}(v) = d(v)$ for every $v \in V_2$.