

# Rankin-Selberg methods for String Amplitudes

Boris Pioline

CERN & LPTHE

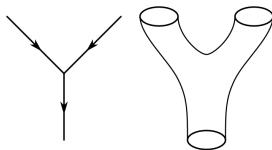


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*based on work with C. Angelantonj and I. Florakis,  
arXiv:1110.5318,1203.0566,1304.4271,1401.4265 and work in progress*

# String theory in a nutshell I

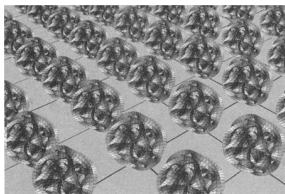
- **Quantum Field Theory** is the language in which we currently understand the fundamental laws of nature – except for **gravity**, which causes an infinite series of ultraviolet divergences upon naive perturbative quantization.



- **String Theory** attempts to solve this problem by viewing the graviton and all other fields of the Standard Model as low-energy excitations of relativistic 1+1-dimensional objects, with tension  $T \sim 1/l_s^2$ , whose interactions are governed by a single ‘topological’ coupling  $g_s$ .

# String theory in a nutshell II

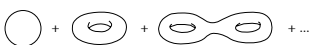
- Unlike pointlike particles, string interactions are **exponentially suppressed** at high energy (and fixed angle),  $\mathcal{A} \sim e^{-E^2 l_s^2}$ , which prevents any UV divergence.



- The price to pay is the existence of an **infinite tower of excitations** of transPlanckian mass  $M = \sqrt{N}/l_s$ , degeneracy  $e^{\mathcal{O}(1)\sqrt{N}}$  – and that **space-time must be 10-dimensional** (or 26 in bosonic string), with all but 4 of its dimensions curled up in a **compact space** of very small size.

# String theory in a nutshell III

- Technically, scattering amplitudes in string theory have a **genus expansion**, analogous to the loop expansion in QFT. For closed bosonic strings,

$$\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathcal{M}_h} d^\ell t \int_{(\Sigma_h)^{n-\kappa}} d\sigma_i \left\langle \prod_{j=1}^{\ell} |(b, q_j)|^2 \prod_{k=1}^{\kappa} |c(\sigma_k)|^2 \prod_{i=1}^n \mathcal{V}_i(\sigma_i) \right\rangle$$


- $\mathcal{M}_h$  is the **moduli space of Riemann surfaces**  $\Sigma_h$  of **genus**  $h$ , of complex dimension  $\ell = 3h - 3$  ( $h \geq 2$ ) and  $q_{j=1 \dots \mu}$  a basis of holomorphic quadratic Beltrami differentials
- $\mathcal{V}_i \sim e^{ik_i^\mu X^\mu}$  are **vertex operators** of dimension  $(1, 1)$  (for  $i \leq \kappa$ ) or  $(0, 0)$  (for  $i > \kappa$ ) creating the asymptotic states ( $\kappa = 3$  on sphere, 1 and torus, 0 otherwise),
- $\langle \dots \rangle$  denote correlators in the **conformal field theory** of  $D = 26$  free bosons  $X^\mu$  and the  $(b, c)$  ghost system

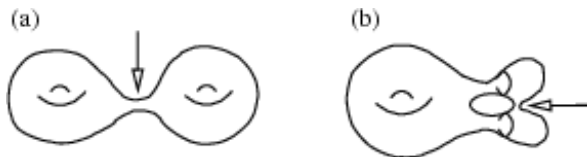
# String theory in a nutshell IV

- For **superstrings**, scattering amplitudes are given by a similar but more complicated correlator in the CFT of  $D = 10$  free bosons+fermions  $(X^\mu, \psi^\mu)$  and  $(b, c)$  and  $(\beta, \gamma)$  ghost systems, integrated over the **moduli space of super-Riemann surfaces**.
- After integrating over the positions  $\sigma_i$  and fermionic part of supermoduli, one is left with an integral over the bosonic moduli space  $\mathcal{M}_h$  (there is no canonical way of projecting the supermoduli space onto bosonic moduli space, different projections differ by total derivatives on  $\mathcal{M}_h$ ).
- $\mathcal{M}_h = \mathcal{T}_h/\Gamma_h$  is the quotient of the **Teichmüller space** by  $\Gamma_h$  the **mapping class group**. The integrand must be invariant under  $\Gamma_h$ ,

$$\mathcal{A}_h = \int_{\mathcal{T}_h/\Gamma_h} d\mu F_h$$

# String theory in a nutshell V

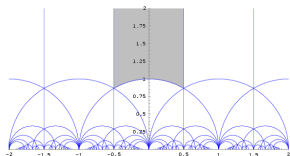
- $\mathcal{T}_h$  is the analog of the space of Schwinger/Feynman parameters in QFT, while  $\Gamma_h$  has no analog in QFT.
- Boundaries of  $\mathcal{M}_h$  only arise when  $\Sigma_h$  pinches off, corresponding to the exchange of on-shell massless states: the only possible divergences in string theory are **infrared divergences**, and are those of a QFT with the same massless spectrum !



which can be fixed by comparing with QFT.

# One-loop string amplitudes I

- At genus 1,  $\mathcal{T}_1$  is the Poincaré upper-half plane, parametrized by  $\tau = \tau_1 + i\tau_2$  and  $\Gamma_1$  is the modular group  $SL(2, \mathbb{Z})$ , acting via  $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ . The integrand  $F_1$  is a **modular form**, to be integrated over a fundamental domain of  $SL(2, \mathbb{Z})$ , e.g.



- $\tau_2$  can be interpreted as a **Schwinger parameter**,

$$\int d^D p_\mu \log(M^2 + p_\mu^2) = \int_0^\infty \frac{d\tau_2}{\tau_2^{1+D/2}} e^{-\pi M^2 \tau_2}$$

# One-loop string amplitudes II

- $\tau_1$  is (at least for  $\tau_2 > 1$ ) a **Lagrange multiplier** projecting the spectrum on level-matched states (with same number of left- and right-moving excitations)
- E.g. the one-loop vacuum amplitude in bosonic closed string theory in  $D = 26$  flat space time is

$$\mathcal{A}_1 = i V_D \int_{\mathcal{F}} \frac{d\tau_1 d\tau_2}{\tau_2^{1+D/2}} \frac{1}{|q|^2} \prod_{M,N=1}^{\infty} \frac{1}{(1 - q^M)^{2(D-2)} (1 - \bar{q}^N)^{2(D-2)}}$$

Recall that the Dedekind eta function  $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ , is a modular form of weight  $1/2$  ( $q = e^{2\pi i \tau}$ )

- Integrating over  $\mathcal{F}$  is in general untractable, and can only be done numerically (after regulating infrared divergences as  $\tau_2 \rightarrow \infty$ ).



# Higher loop string amplitudes I

- For genus 2 and 3, the Teichmüller space  $\mathcal{T}_h$  is still isomorphic to (an open set in) the **Siegel-Poincaré upper half plane**  $\mathcal{H}_h$ , parametrized by the **period matrix**  $\Omega$ , a complex  $h \times h$  symmetric matrix with positive definite imaginary part.  $F_h(\Omega)$  is a Siegel modular form for  $\Gamma_h = Sp(2h, \mathbb{Z})$ , acting as  $\Omega \mapsto (A\Omega + B) \cdot (C\Omega + D)^{-1}$ .
- For genus  $h \geq 4$ ,  $\mathcal{T}_h$  is a codimension  $\frac{1}{2}(h-2)(h-3)$  locus inside  $\mathcal{H}_h$  known as the **Schottky locus**. It is not clear how to extend  $F_h$  to a modular form on  $\mathcal{H}_h$ .
- Our goal is to develop methods to **compute integrals of Siegel modular forms over a fundamental domain of the Siegel upper-half plane analytically**.

# Rankin-Selberg method / unfolding trick I

- The key idea is to write the integrand as a **Poincaré series**,

$$F_h(\Omega) = \sum_{\gamma \in \Gamma_{h,\infty} \backslash \Gamma_h} f_h|_{\gamma}(\Omega)$$

where  $f_h|_{\gamma}(\Omega) = f_h(\gamma \cdot \Omega)$  and the ‘seed’  $f_h(\Omega)$  is invariant under a subgroup  $\Gamma_{h,\infty} \subset \Gamma_h$ . Typically,  $\Gamma_{h,\infty}$  is the **stabilizer of the cusp at infinity**, acting by integer shifts of  $\Omega_1$ .

- **Provided the sum is absolutely convergent**, one can exchange the sum and integral and obtain

$$\int_{\Gamma_h \backslash \mathcal{H}_h} d\mu_h F_h(\Omega) = \int_{\Gamma_{\infty,h} \backslash \mathcal{H}_h} d\mu_h f_h(\Omega) .$$

- We gain if  $\Gamma_{\infty,h} \backslash \mathcal{H}_h$  and  $f_h$  are simpler than  $\Gamma_h \backslash \mathcal{H}_h$  and  $F_h$  !

# Rankin-Selberg method / unfolding trick II

- This method is limited by our ability to represent the integrand as a Poincaré series. Not much is known in genus  $h > 1$ . In genus one, any **weakly, almost holomorphic modular form of negative weight** can be represented as a Poincaré series.
- We shall focus on a class of one-loop amplitudes of the form

$$\mathcal{A} = \int_{\mathcal{F}} d\mu \Gamma_{d+k,d} \Phi(\tau), \quad d\mu = \frac{d\tau_1 d\tau_2}{\tau_2^2}$$

- $\Gamma_{(d+k,d)} = \tau_2^{d/2} \sum_{p \in \Gamma} e^{-2\pi\tau_2 Q(p) + 2\pi i \tau_1 B(q)}$  : a **Siegel Theta series** for an **even self-dual lattice**  $(\Gamma, B)$  of signature  $(d+k, d)$ , equipped with a positive definite quadratic form  $Q$
- $\Phi(\tau)$  : a weakly, almost holomorphic modular form of weight  $w = -k/2$ , which I will call the **elliptic genus**

# Rankin-Selberg method / unfolding trick III

- Such modular integrals arise in certain **BPS-saturated amplitudes**, such as  $F^2, R^2, F^4, R^4$  in type II string theory ( $k = 0$ ) or heterotic string ( $k = 8, 16$ ) compactified on a torus  $T^d$ .
- $\mathcal{A}$  provides a function on the moduli space of lattices,

$$G_{d+k,d} = \frac{O(d+k, d)}{O(d+k) \times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a),$$

which is invariant under **T-duality**, i.e. under the automorphisms of the lattice. Mathematically,  $\Phi \mapsto \mathcal{A}$  is a **Theta correspondence** between  $SL(2, \mathbb{Z})$  and  $O(\Gamma_{d+k,d})$  automorphic forms.

- In the physics literature, such integrals are typically computed the **orbit method**, i.e. by applying the unfolding trick to  $\Gamma_{(d+k,d)}$ . Instead, we shall apply the unfolding trick to  $\Phi(\tau)$ , which has the advantage of keeping T-duality manifest throughout.

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- Consider the completed **non-holomorphic Eisenstein series**

$$E^*(\tau; s) = \zeta^*(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \tau_2^s |\gamma| = \frac{1}{2} \zeta^*(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau + d|^{2s}}$$

where  $\zeta^*(s) \equiv \pi^{-s/2} \Gamma(s/2) \zeta(s) = \zeta^*(1-s)$ .

- $E^*(\tau; s)$  is convergent for  $\operatorname{Re}(s) > 1$ , and has a meromorphic continuation to all  $s$ , invariant under  $s \mapsto 1-s$ , with simple poles at  $s = 0, 1$  with **constant residue**:

$$E^*(\tau; s) = \frac{1}{2(s-1)} + \frac{1}{2} \left( \gamma - \log(4\pi \tau_2 |\eta(\tau)|^4) \right) + \mathcal{O}(s-1),$$

- For any modular function  $F(\Omega)$  of rapid decay, consider the **Rankin-Selberg transform**

$$\mathcal{R}^*(F, s) = \int_{\mathcal{F}} d\mu E^*(\tau; s) F(\tau)$$

- By the unfolding trick,  $\mathcal{R}^*(F, s)$  is proportional to the **Mellin transform** of the constant term  $F_0(\tau_2) = \int_{-1/2}^{1/2} d\tau_1 F(\tau)$ ,

$$\begin{aligned} \mathcal{R}^*(F; s) &= \zeta^*(2s) \int_{\mathbb{R}^+ \times [-\frac{1}{2}, \frac{1}{2}]} d\mu \tau_2^s F(\tau) \\ &= \zeta^*(2s) \int_0^\infty d\tau_2 \tau_2^{s-2} F_0(\tau_2), \end{aligned}$$



- It inherits the meromorphicity and functional relations of  $E^*$ , e.g.  $\mathcal{R}^*(F; s) = \mathcal{R}^*(F; 1 - s)$ .
- Since the residue of  $E^*(\tau; s)$  at  $s = 0, 1$  is constant, the residue of  $\mathcal{R}^*(F; s)$  at  $s = 1$  is proportional to the modular integral of  $F$ ,

$$\text{Res}_{s=1} \mathcal{R}^*(F; s) = \frac{1}{2} \int_{\mathcal{F}} d\mu F$$

# Rankin-Selberg-Zagier method I

- This was extended by Zagier to the case where  $F^{(0)}$  is of power-like growth  $F^{(0)}(\tau) \sim \varphi(\tau_2)$  at the cusp: the **renormalized integral**

$$\text{R.N. } \int_{\mathcal{F}} d\mu F(\tau) = \lim_{T \rightarrow \infty} \left[ \int_{\mathcal{F}_T} d\mu F(\tau) - \hat{\varphi}(T) \right]$$

$$\varphi(\tau_2) = \sum_{\alpha} c_{\alpha} \tau_2^{\alpha}, \quad \hat{\varphi}(T) = \sum_{\alpha \neq 1} c_{\alpha} \frac{T_2^{\alpha-1}}{\alpha-1} + \sum_{\alpha=1} c_{\alpha} \log \tau_2$$

is related to the Mellin transform of the (regularized) constant term

$$\mathcal{R}^*(F; s) = \zeta^*(2s) \int_0^{\infty} d\tau_2 \tau_2^{s-2} \left( F^{(0)} - \varphi \right),$$

via  $\boxed{\text{R.N. } \int_{\mathcal{F}} d\mu F(\tau) = 2 \text{Res}_{s=1} \mathcal{R}^*(F; s) + \delta}$

$$\text{R.N. } \int_{\mathcal{F}} d\mu F(\tau) = 2 \operatorname{Res}_{s=1} \mathcal{R}^*(F; s) + \delta$$

- $\delta$  is a scheme-dependent correction which depends only on the leading behavior  $\varphi(\tau_2)$ ,

$$\delta = 2 \operatorname{Res}_{s=1} [\zeta^*(2s) h_{\mathcal{T}}(s) + \zeta^*(2s-1) h_{\mathcal{T}}(1-s)] - \hat{\varphi}(\mathcal{T}),$$

where  $h_{\mathcal{T}}(s) = \int_0^{\mathcal{T}} d\tau_2 \varphi(\tau_2) \tau_2^{s-2}$ .

- The RSZ method applies immediately to  $\mathcal{A} = \int_{\mathcal{F}} d\mu \Gamma_{d,d}(g, B)$ :

$$\begin{aligned}\mathcal{R}^*(\Gamma_{d,d}; \mathbf{s}) &= \zeta^*(2\mathbf{s}) \int_0^\infty d\tau_2 \tau_2^{s+d/2-2} \sum'_{p_L^2 - p_R^2 = 0} e^{-\pi\tau_2(p_L^2 + p_R^2)} \\ &= \zeta^*(2\mathbf{s}) \frac{\Gamma(\mathbf{s} + \frac{d}{2} - 1)}{\pi^{s + \frac{d}{2} - 1}} \mathcal{E}_V^d(g, B; \mathbf{s} + \frac{d}{2} - 1)\end{aligned}$$

where  $\mathcal{E}_V^d(g, B; \mathbf{s})$  is the **constrained Epstein series**

$$\mathcal{E}_V^d(g, B; \mathbf{s}) \equiv \sum_{\substack{(m_i, n^j) \in \mathbb{Z}^{2d} \setminus (0,0) \\ m_i n^j = 0}} \mathcal{M}^{-2s}, \quad \mathcal{M}^2 = p_L^2 + p_R^2$$

# Epstein series and BPS state sums I

- This is identified as a **sum over all BPS states** of momentum  $m_i$  and winding  $n^j$ , with mass

$$\mathcal{M}^2 = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

subject to the **BPS condition**  $m_i n^i = 0$ . Invariance under  $O(\Gamma_{d,d})$  is manifest.

- The constrained Epstein Zeta series  $\mathcal{E}_V^d(g, B; s)$  converges absolutely for  $\text{Re}(s) > d$ . The RSZ method shows that it admits a meromorphic continuation in the  $s$ -plane satisfying

$$\mathcal{E}_V^{d*}(s) = \pi^{-s} \Gamma(s) \zeta^*(2s - d + 2) \mathcal{E}_V^d(s) = \mathcal{E}_V^{d*}(d - 1 - s),$$

with a simple pole at  $s = 0, \frac{d}{2} - 1, \frac{d}{2}, d - 1$  (double poles if  $d = 2$ ).

# Epstein series and BPS state sums II

- The residue at  $s = \frac{d}{2}$  produces the modular integral of interest:

$$\text{R.N.} \int_{\mathcal{F}} d\mu \Gamma_{d,d}(g, B) = \frac{\Gamma(\frac{d}{2} - 1)}{\pi^{\frac{d}{2}-1}} \mathcal{E}_V^d(g, B; \frac{d}{2} - 1)$$

rigorously proving an old conjecture of Obers and myself (1999).

- For  $d = 2$ , the BPS constraint  $m_i n^i = 0$  can be solved, leading to

$$\mathcal{E}_V^{2*}(T, U; s) = 2 E^*(T; s) E^*(U; s)$$

hence to Dixon-Kaplunovsky-Louis famous result (1989)

$$\int_{\mathcal{F}} (\Gamma_{2,2}(T, U) - \tau_2) d\mu = -\log \left( T_2 U_2 |\eta(T) \eta(U)|^4 \right) + \text{cte}$$

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# Modular integrals with unphysical tachyons I

- For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon,  $\Phi(\tau) \sim 1/q^\kappa + \mathcal{O}(1)$  with  $\kappa = 1$ .
- In mathematical terms,  $\Phi(\tau) \in \mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta]$  is an **almost, weakly holomorphic modular** form with weight  $w = -k/2 \leq 0$ .
- The RSZ method fails, however the unfolding trick could still work provided  $\Phi(\tau)$  can be represented as a **uniformly convergent Poincaré series** with seed  $f(\tau)$  is invariant under  $\Gamma_\infty : \tau \rightarrow \tau + n$ ,

$$\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\tau)|_w \gamma$$

- Convergence requires  $f(\tau) \ll \tau_2^{1-\frac{w}{2}}$  as  $\tau_2 \rightarrow 0$ . The choice  $f(\tau) = 1/q^\kappa$  works for  $w > 2$  but fails for  $w \leq 2$ .



# Various Poincaré series representations I

- One option is to insert a **non-holomorphic convergence factor** à la Hecke-Kronecker, i.e. choose a seed  $f(\tau) = \tau_2^{s-\frac{w}{2}} q^{-\kappa}$ :

$$E(s, \kappa, w) \equiv \frac{1}{2} \sum_{(c,d)=1} \frac{(c\tau + d)^{-w} \tau_2^{s-\frac{w}{2}}}{|c\tau + d|^{2s-w}} e^{-2\pi i \kappa \frac{a\tau+b}{c\tau+d}}$$

*Selberg; Goldfeld Sarnak; Pribitkin*

- This converges absolutely for  $\text{Re}(s) > 1$ , but analytic continuation to desired value  $s = \frac{w}{2}$  is tricky, and in general **non-holomorphic**.
- Moreover,  $E(s, \kappa, w)$  is not an eigenmode of the Laplacian, rather

$$[\Delta_w + \frac{1}{2} s(1-s) + \frac{1}{8} w(w+2)] E(s, \kappa, w) = 2\pi\kappa (s - \frac{w}{2}) E(s+1, \kappa, w)$$

# Niebur-Poincaré series I

- We shall use another regularization which does not require analytic continuation: the **Niebur-Poincaré series**

$$\mathcal{F}(s, \kappa, w) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathcal{M}_{s,w}(-\kappa\tau_2) e^{-2\pi i \kappa \tau_1} |w \gamma$$

*Niebur; Hejhal; Bruinier Ono Bringmann...*

where  $\mathcal{M}_{s,w}(y)$  is proportional to a Whittaker function, so that

$$\left[ \Delta_w + \frac{1}{2} s(1-s) + \frac{1}{8} w(w+2) \right] \mathcal{F}(s, \kappa, w) = 0$$

- The seed  $f(\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2) e^{-2\pi i \kappa \tau_1}$  is uniquely determined by

$$f(\tau) \sim_{\tau_2 \rightarrow 0} \tau_2^{s - \frac{w}{2}} e^{-2\pi i \kappa \tau_1} \quad f(\tau) \sim_{\tau_2 \rightarrow \infty} \frac{\Gamma(2s)}{\Gamma(s + \frac{w}{2})} q^{-\kappa}$$

ensuring that  $\mathcal{F}(s, \kappa, w)$  converges absolutely for  $\text{Re}(s) > 1$ .

# Niebur-Poincaré series II

- Under raising and lowering operators,

$$D_w = \frac{i}{\pi} \left( \partial_\tau - \frac{iW}{2\tau_2} \right), \quad \bar{D}_w = -i\pi \tau_2^2 \partial_{\bar{\tau}},$$

the NP series transforms as

$$D_w \cdot \mathcal{F}(s, \kappa, w) = 2\kappa \left( s + \frac{w}{2} \right) \mathcal{F}(s, \kappa, w + 2),$$

$$\bar{D}_w \cdot \mathcal{F}(s, \kappa, w) = \frac{1}{8\kappa} \left( s - \frac{w}{2} \right) \mathcal{F}(s, \kappa, w - 2).$$

- Under Hecke operators,

$$H_{\kappa'} \cdot \mathcal{F}(s, \kappa, w) = \sum_{d|(\kappa, \kappa')} d^{1-w} \mathcal{F}(s, \kappa \kappa' / d^2, w).$$

- For congruence subgroups of  $SL(2, \mathbb{Z})$ , one can similarly define NP series  $\mathcal{F}_\alpha(s, \kappa, w)$  for each cusp.

# Niebur-Poincaré series III

- For  $s = 1 - \frac{w}{2}$ , the value relevant for weakly holomorphic modular forms, the seed simplifies to

$$f(\tau) = \Gamma(2 - w) \left( q^{-\kappa} - \bar{q}^{\kappa} \sum_{\ell=0}^{-w} \frac{(4\pi\kappa\tau_2)^{\ell}}{\ell!} \right)$$

- For  $w < 0$ , the value  $s = 1 - \frac{w}{2}$  lies in the convergence domain, but  $\mathcal{F}(1 - \frac{w}{2}, \kappa, w)$  is in general NOT holomorphic, but rather a **weakly harmonic Maass form**,

$$\Phi = \sum_{m=-\kappa}^{\infty} a_m q^m + \sum_{m=1}^{\infty} m^{w-1} \bar{b}_m \Gamma(1 - w, 4\pi m\tau_2) q^{-m}$$

- For any such form,  $\bar{D}\Phi = \tau_2^{2-w} \bar{\Psi}$  where  $\Psi = \sum_{m \geq 1} b_m q^m$  is a holomorphic cusp form of weight  $2 - w$ , the **shadow** of the Mock modular form  $\Phi^- = \sum_{m=-\kappa}^{\infty} a_m q^m$ .

# Niebur-Poincaré series IV

- If  $|w|$  is small enough, the negative frequency coefficients  $b_m$  vanish and  $\Phi$  is in fact a weakly holomorphic modular form:

$w$	$\mathcal{F}(1 - \frac{w}{2}, 1, w)$
0	$j + 24$
-2	$3! E_4 E_6 / \Delta$
-4	$5! E_4^2 / \Delta$
-6	$7! E_6 / \Delta$
-8	$9! E_4 / \Delta$
-10	$11! \Phi_{-10}$
-12	$13! / \Delta$
-14	$15! \Phi_{-14}$

Here  $\Phi_{-10}$  and  $\Phi_{-14}$  are genuine harmonic Maass forms with shadow  $2.8402... \times \Delta$  and  $1.3061... \times E_4 \Delta$ .

- Theorem (Bruinier) : any **weakly holomorphic** modular form of weight  $w \leq 0$  with polar part  $\Phi = \sum_{0 < m \leq \kappa} a_{-m} q^{-m} + \mathcal{O}(1)$  is a linear combination of Niebur-Poincaré series

$$\Phi = \frac{1}{\Gamma(2-w)} \sum_{0 < m \leq \kappa} a_{-m} \mathcal{F}\left(1 - \frac{w}{2}, m, w\right) + a'_0 \delta_{w,0}$$

(The same holds for congruence subgroups of  $SL(2, \mathbb{Z})$ , including contributions from all cusps)

- **Weakly almost holomorphic** modular forms of weight  $w \leq 0$  can similarly be represented as linear combinations of  $\mathcal{F}\left(1 - \frac{w}{2} + n, m, w\right)$  with  $0 < m \leq \kappa, 0 \leq n \leq p$  where  $p$  is the depth. This fails for positive weight, as such forms are not necessarily harmonic !

# Unfolding the modular integral

- By Bruinier's thm, any modular integral is a linear combination of

$$\mathcal{I}_{d+k,d}(s, \kappa) = \text{R.N.} \int_{\mathcal{F}} d\mu \Gamma_{d+k,d}(G, B, Y) \mathcal{F}(s, \kappa, -\frac{k}{2})$$

- Using the unfolding trick, one arrives at the **BPS state sum**

$$\begin{aligned} \mathcal{I}_{d+k,d}(s, \kappa) &= (4\pi\kappa)^{1-\frac{d}{2}} \Gamma(s + \frac{2d+k}{4} - 1) \\ &\times \sum_{\text{BPS}} {}_2F_1\left(s - \frac{k}{4}, s + \frac{2d+k}{4} - 1; 2s; \frac{4\kappa}{p_L^2}\right) \left(\frac{p_L^2}{4\kappa}\right)^{1-s-\frac{2d+k}{4}} \end{aligned}$$

*Bruinier; Angelantonj Florakis BP*

where  $\sum_{\text{BPS}} \equiv \sum_p \delta(p_L^2 - p_R^2 - 4\kappa)$ . This converges absolutely for  $\text{Re}(s) > \frac{2d+k}{4}$  and can be analytically continued to  $\text{Re}(s) > 1$  with a simple pole at  $s = \frac{2d+k}{4}$ .

# Unfolding the modular integral

- For  $s = 1 - \frac{w}{2} + n$ , the values relevant for almost holomorphic modular forms, the summand can be written using elementary functions, e.g.

$$\mathcal{I}_{2+k,2}(1 + \frac{k}{4}, \kappa) = -\Gamma(2 + \frac{k}{2}) \sum_{\text{BPS}} \left[ \log \left( \frac{p_R^2}{p_L^2} \right) + \sum_{\ell=1}^{k/2} \frac{1}{\ell} \left( \frac{p_L^2}{4\kappa} \right)^{-\ell} \right]$$

- The result is manifestly  $O(\Gamma_{d+k,d})$  invariant, and requires no choice of chamber in Narain modular space. Singularities on  $G_{d+k,d}$  arise when  $p_L^2 = 0$  for some lattice vector.



# Fourier-Jacobi expansion I

- For  $d = 2, k = 0$ , the Fourier expansion in  $T_1$  (or  $U_1$ ) can be obtained by solving the BPS constraint. E.g. for  $\kappa = 1$ , all solutions to  $m_1 n^1 + m_2 n^2 = 1$  are

$$\begin{cases} m_1 = b + dM, & n^1 = -c \\ m_2 = a + cM, & n^2 = d \end{cases}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash SL(2, \mathbb{Z}), M \in \mathbb{Z}$$

- After Poisson resummation over  $M$ , the sum over  $\gamma$  neatly produces Niebur-Poincaré series in  $U$ ,

$$\begin{aligned} \mathcal{I}(s, 1) &= 2^{2s} \sqrt{4\pi} \Gamma(s - \frac{1}{2}) T_2^{1-s} \mathcal{E}(U; s) \\ &+ 4 \sum_{N>0} \sqrt{\frac{T_2}{N}} K_{s-\frac{1}{2}}(2\pi N T_2) \left[ e^{2\pi i N T_1} \mathcal{F}(s, N, 0; U) + \text{cc} \right] \end{aligned}$$

# Fourier-Jacobi expansion II

- The same result is obtained by the usual orbit method. In fact, both methods require to compute the same integral,

$$\int_{\mathcal{H}} d\mu e^{-\frac{\pi T_2}{\tau_2 U_2} |U-\tau|^2} \mathcal{F}(\tau) = 2 T_2^{-1/2} e^{2\pi T_2} K_{s-\frac{1}{2}}(2\pi T_2) \mathcal{F}(U),$$

where  $\mathcal{F}(\tau)$  is the seed of the NP series in the unfolding method, or the full NP series  $\mathcal{F}(s, \kappa, 0; \tau)$  in the old orbit method.

- This formula works for any eigenmode of  $\Delta + \frac{1}{2}s(1-s)$ , irrespective of modular invariance. It generalizes the **average value property** of harmonic functions.

# Fourier-Jacobi expansion III

- Using  $\mathcal{F}(s, N, 0) = H_N \cdot \mathcal{F}(s, 1, 0)$ , one recovers the usual Borcherds products for  $s = 1$ ,

$$\begin{aligned}\mathcal{A} &= 8\pi \operatorname{Res}_{s=1} \left[ T_2^{1-s} \mathcal{E}(s; U) \right] + 2 \sum_{N>0} \left[ \frac{q_T^N}{N} H_N^{(U)} \cdot [j(U) + 24] + \text{cc} \right] \\ &= -24 \log \left[ T_2 U_2 |\eta(T)\eta(U)|^4 \right] - 2 \sum_{M,N} c(MN) [\log(1 - q_T^N q_U^M) + \text{c.c.}] \\ &= -24 \log \left[ T_2 U_2 |\eta(T)\eta(U)|^4 \right] - \log |j(T) - j(U)|^4\end{aligned}$$

where we have used  $\mathcal{F}(1, 1, 0; U) = j(U) + 24$ ,  $j(U) = \sum c(M)q^M$ .

*Borcherds; Harvey Moore*

# Fourier-Jacobi expansion IV

- For  $s = 1 + n$ , relevant for almost holomorphic modular forms of depth  $p \geq n$ , one can express  $\mathcal{I}_{2,2}(n+1, 1)$  as the iterated derivative of a generalized prepotential,

$$\mathcal{I}_{2,2}(n+1, 1) = 4 \operatorname{Re} \left[ \frac{(-D_T D_U)^n}{n!} f_n(T, U) \right]$$

where  $f_n$  is holomorphic in  $T$  but harmonic in  $U$ ,

$$\begin{aligned} f_n(T, U) = & 2(2\pi)^{2n+1} E(n+1, -2n; U) \\ & + \sum_{N>0} \frac{2q_T^N}{(2N)^{2n+1}} \mathcal{F}(n+1, N, -2n; U) \end{aligned}$$

- One can turn  $f_n$  into a holomorphic function  $\tilde{f}_n(T, U)$  by replacing  $E(n+1, -2n; U)$  by its analytic part  $\tilde{E}(n+1, -2n; U)$  without affecting the real part of its iterated derivative.

- The generalized holomorphic prepotential  $\tilde{f}_n(T, U)$  now transforms as an **Eichler integral** of weight  $(-2n, -2n)$  under  $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times (T \leftrightarrow U)$ ,

$$(cU + d)^{2n} \tilde{f}_n \left( T, \frac{aU + b}{cU + d} \right) = \tilde{f}_n(T, U) + P_\gamma(U),$$

where  $P_\gamma(U)$  is a computable polynomial of degree  $\leq 2n$ .

- The case  $n = 1$  describes the standard prepotential appearing in string vacua with  $\mathcal{N} = 2$  supersymmetry. Its modular anomaly was discussed by Antoniadis, Ferrara, Gava, Narain, Taylor in 1995, which is **the first occurrence of Eichler integrals in string theory !**
- The case  $n = 2$  has appeared in the context of 1/4-BPS amplitudes in  $Het/K_3$ . Higher  $n$  has not come up yet, but is suggestive of  $CY_{2n+1}$ -fold.

- 1 String theory in a nutshell
- 2 The Rankin-Selberg method
- 3 Niebur-Poincaré series and generalized prepotentials
- 4 Rankin-Selberg method at higher genus**

# Rankin-Selberg method at higher genus I

- String amplitudes at genus  $h \leq 3$  take the form

$$\mathcal{A}_h = \int_{\mathcal{F}_h} d\mu_h \Gamma_{d+k,d,h}(G, B, Y; \Omega) \Phi(\Omega), \quad d\mu_h = \frac{d\Omega_1 d\Omega_2}{[\det \Omega_2]^{h+1}}$$

- $\mathcal{F}_h$  is a fundamental domain of the action of  $\Gamma = Sp(2h, \mathbb{Z})$  on Siegel's upper half plane  $\{\Omega = \Omega^t \in \mathbb{C}^{h \times h}, \Omega_2 > 0\}$
- $\Gamma_{d+k,d,h}$  a Siegel-Narain theta series of signature  $(d+k, d)$

$$\Gamma_{d+k,d,h} = [\det \Omega_2]^{d/2} \sum_{(\Gamma_{d+k,d})^h} e^{i\pi \text{Tr}(\Omega P_L P_L^t) - i\pi \text{Tr}(\bar{\Omega} P_R P_R^t)}$$

- $\Phi(\Omega)$  a Siegel modular form of weight  $-k/2$ .
- We would like to generalize the previous methods to the case where  $\Phi(\Omega)$  is an almost holomorphic modular form with poles inside  $\mathcal{F}_h$ , such as  $1/\chi_{10}$ . As a first step, take  $k=0$ ,  $\Phi=1$ .

# Rankin-Selberg method at higher genus II

- The genus  $h$  analog of  $\mathcal{E}^*(s; \tau)$  is the non-holomorphic Siegel-Eisenstein series

$$\mathcal{E}_h^*(s; \Omega) = \zeta^*(2s) \prod_{j=1}^{[h/2]} \zeta^*(4s - 2j) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} |\Omega_2|^s |\gamma|$$

where  $\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix} \right\} \subset \Gamma$ ,  $|\Omega_2| = |\det \operatorname{Im} \Omega|$ .

- The sum converges absolutely for  $\operatorname{Re}(s) > \frac{h+1}{2}$  and can be meromorphically continued to the full  $s$  plane. The analytic continuation is invariant under  $s \mapsto \frac{h+1}{2} - s$ , and has a simple pole at  $s = \frac{h+1}{2}$  with constant residue  $r_h = \frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^*(2j+1)$



# Rankin-Selberg method at higher genus III

- For any modular function  $F(\Omega)$  of rapid decay, the Rankin-Selberg transform can be computed by unfolding the integration domain against the sum,

$$\begin{aligned}\mathcal{R}_h^*(F; s) &= \int_{\mathcal{F}_h} d\mu_h F(\Omega) \mathcal{E}_h^*(\Omega, s) \\ &= \zeta^*(2s) \prod_{j=1}^{[h/2]} \zeta^*(4s - 2j) \int_{GL(h, \mathbb{Z}) \backslash \mathcal{P}_h} d\Omega_2 |\Omega_2|^{s-h-1} F_0(\Omega_2)\end{aligned}$$

where  $\mathcal{P}_h$  is the space of positive definite real matrices, and  $F_0(\Omega_2) = \int_0^1 d\Omega_1 F(\Omega)$  is the constant term of  $F$ .

- The residue at  $s = \frac{h+1}{2}$  is proportional to the average of  $F$ ,

$$\text{Res}_{s=\frac{h+1}{2}} \mathcal{R}_h^*(F; s) = r_h \int_{\mathcal{F}_h} F.$$

# Rankin-Selberg method at higher genus IV

- The Siegel-Narain theta series is not a cusp form, instead its zero-th Fourier mode is

$$\Gamma_{d,d,h}^{(0)}(\mathbf{g}, \mathbf{B}; \Omega) = |\Omega_2|^{d/2} \sum_{(m_i^\alpha, n^{i\alpha}) \in \mathbb{Z}^{2d \times h}, m_i^{\alpha} n^{i\beta} = 0} e^{-\pi \text{Tr}(M^2 \Omega_2)}$$

where

$$M^{2;\alpha\beta} = (m_i^\alpha + B_{ik} n^{k\alpha}) g^{ij} (m_j^\beta + B_{jl} n^{l\beta}) + n^{i\alpha} g_{ij} n^{j\beta}$$

Terms with  $\text{Rk}(m_i^\alpha, n^{i\alpha}) < h$  do not decay rapidly at  $\Omega_2 \rightarrow \infty$ . For  $d < h$ , this is always the case.

- The Siegel-Eisenstein series  $\mathcal{E}_h^*(\Omega, \mathbf{s})$  similarly has non-decaying constant term of the form  $\sum_T e^{-\text{Tr}(T\Omega_2)}$  with  $\text{Rk}(T) < h$ .

# Rankin-Selberg method at higher genus V

- The regularized Rankin-Selberg transform is obtained by subtracting non-suppressed terms, and yields a field theory-type amplitude, with BPS states running in the loops,

$$\begin{aligned}\mathcal{R}_h(\Gamma_{d,d,h}; \mathbf{s}) &= \int_{GL(h, \mathbb{Z}) \backslash \mathcal{P}_h} \frac{d\Omega_2}{|\Omega_2|^{h+1-s-\frac{d}{2}}} \sum_{\text{BPS}} e^{-\pi \text{Tr}(M^2 \Omega_2)} \\ &= \Gamma_h\left(\mathbf{s} - \frac{h+1-d}{2}\right) \sum_{\text{BPS}} \left[\det M^2\right]^{\frac{h+1-d}{2}-s}\end{aligned}$$

$$\sum_{\text{BPS}} = \sum_{\substack{(m_i^\alpha, n^{i\alpha}) \in \mathbb{Z}^{2d \times h}, \\ m_i^{(\alpha} n^{i\beta)} = 0, \det M^2 \neq 0}}, \quad \Gamma_h(\mathbf{s}) = \pi^{\frac{1}{4}h(h-1)} \prod_{k=0}^{h-1} \Gamma\left(\mathbf{s} - \frac{k}{2}\right)$$

# Rankin-Selberg method at higher genus VI

- This is recognized as the Langlands-Eisenstein series of  $SO(d, d, \mathbb{Z})$  with infinitesimal character  $\rho - 2(s - \frac{h+1-d}{2})\lambda_h$ , associated to  $\Lambda^h V$  where  $V$  is the defining representation,

$$\mathcal{R}_h(\Gamma_{d,d}; s) \propto \mathcal{E}_{\Lambda^h V}^{SO(d,d)}(s - \frac{h+1-d}{2}) \quad (h > d)$$

- For  $h = d$ ,  $\Lambda^h V = S^2 \oplus C^2$  where  $S, C$  are spinor representations,

$$\mathcal{R}_h(\Gamma_{h,h,h}; s) \propto \mathcal{E}_S^{SO(h,h)}(2s - 1) + \mathcal{E}_C^{SO(h,h)}(2s - 1)$$

- The modular integral of  $\Gamma_{d,d,h}$  is proportional to the residue of  $\mathcal{R}_h(\Gamma_{d,d,h}; s)$  at  $s = \frac{h+1}{2}$ , up to a scheme dependent term  $\delta$ . For  $d < h$ , the entire result comes from  $\delta$ .

# Rankin-Selberg method at higher genus VII

- For  $d = 1$ , any  $h$ ,

$$\mathcal{A}_h = \mathcal{V}_h(R^h + R^{-h}), \quad \mathcal{V}_h = \int_{\mathcal{F}_h} d\mu_h = 2 \prod_{j=1}^h \zeta^*(2j)$$

- For  $h = d = 2$ , either by computing the BPS sum, or by unfolding the Siegel-Narain theta series, one finds

$$\begin{aligned} \mathcal{R}_2^*(\Gamma_{2,2}, \mathbf{s}) &= 2\zeta^*(2s)\zeta^*(2s-1)\zeta^*(2s-2) \\ &\quad \times [\mathcal{E}_1^*(T; 2s-1) + \mathcal{E}_1^*(U; 2s-1)] \end{aligned}$$

hence

$$\mathcal{A}_2 = 2\zeta^*(2) [\mathcal{E}_1^*(T; 2) + \mathcal{E}_1^*(U; 2)]$$

proving the conjecture by Obers and BP (1999).

- For  $h = d = 3$ ,

$$\mathcal{R}_3^*(\Gamma_{3,3}; s) = \zeta^*(2s) \zeta^*(2s-1) \zeta^*(2s-2) \zeta^*(2s-3) \\ \left[ \mathcal{E}_S^{*,SO(3,3)}(2s-1) + \mathcal{E}_C^{*,SO(3,3)}(2s-1) \right]$$

hence

$$\mathcal{A}_3 = 2\zeta^*(2)\zeta^*(4) \left[ \mathcal{E}_S^{*,SO(3,3)}(3) + \mathcal{E}_C^{*,SO(3,3)}(3) \right]$$

- Modular integrals can be efficiently computed using Rankin-Selberg type methods. The result is expressed as a field theory amplitude with BPS states running in the loop.
- T-duality and singularities from enhanced gauge symmetry are manifest. Fourier-Jacobi expansions can be obtained in some cases by solving the BPS constraint.
- The RSZ method also works at higher genus, at least for  $h = 2, 3$ . For computing modular integrals with  $\Phi \neq 1$  it will be important to develop Poincaré series representations for Siegel modular forms with poles at Humbert divisors, such as  $1/\Phi_{10}$ .
- Non-BPS amplitudes where  $\Phi$  is not almost weakly holomorphic are challenging ! So are amplitudes with  $h \geq 4$  !