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The Skew-Symmetric Ortho-Symmetric Solutions of the Matrix Equations $A^*XA = D$

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Abstract

In this paper, the following problems are discussed.
Problem 1. Given matrices $A \in C^{n\times m}$ and $D \in C^{m\times m}$,find $X \in SSC_p^n$ such that $A^*XA = D$, where $SSC_p^n = \{X \in SSC^{n\times n}/PX \in SC^{n\times n}$ for given $P \in OC^{n\times n}$ satisfying $P^* = P\}$.
Problem 2. Given a matrix $\tilde{X} \in C^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\left\|\tilde{X} - \widehat{X}\right\| = \inf_{X\in S_E} \left\|\tilde{X} - X\right\|,$$

where $\|.\|$ is the Frobenius norm, and $S_E$is the solution set of problem 1.
Expressions for the general solution of problem 1 are derived. Necessary and sufficient conditions for the solvability of Problem 1 are determined. For problem 2, an expression for the solution is given.

Mathematics Subject Classifications: 15A57, 65F10, 65F20

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1 Introduction

Let $C^{n\times m}$ denote the set of all $n\times m$complex matrices, and let $OC^{n\times n}$,$SC^{n\times n}$, $SSC^{n\times n}$ denote the set of all$n \times n$orthogonal matrices, the set of all$n \times n$complex symmetric matrices, the set of all $n \times n$complex skew-symmetric matrices, respectively. The symbol I_K will stand for the identity matrix of order K,$A^{\dagger}$ for the Moore-penrose generalized inverse of a matrix A, and rk(A) for the rank of matrix A. For matrices $A, B \in C^{n\times m}$, the expression $A * B$ will be the Hadamard product of A and B; also $\|.\|$ will denote the Frobenius norm. Defining the inner product (A, B)=tr(B^*A) for matrices $A, B \in C^{n\times m}$, $C^{n\times m}$ becomes a Hilbert space. The norm of a matrix generated by this inner product is the Frobenius norm. If $A = (a_{ij}) \in C^{n\times n}$, let $L_A = (l_{ij}) \in C^{n\times n}$ be defined as follows: $l_{ij} = a_{ij}$ whenever $i > j$ and $l_{ij} = 0$ otherwise $(i, j = 1.2...., n)$. Let e_i be the i-th column of the identity matrix $I_n(i = 1, 2, ..., n)$ and set $S_n = (e_n, e_{n-1}, ..., e_1)$. It is easy to see that

$$S_n^* = S_n, \quad S_n^* S_n = I_n.$$

An inverse problem [2]-[6] arising in the structural modification of the dynamic behaviour of a structure calls for the solution of the matrix equation

$$A^*XA = D, \tag{1.1}$$

where $A \in C^{n\times m}$, $D \in C^{m\times m}$, and the unknown X is required to be complex and symmetric, and positive semidefinite or possibly definite. No assumption is made about the relative sizes of m and n, and it is assumed throught that $A \neq 0$ and $D \neq 0$.
Equation (1.1) is a special case of the matrix equation

$$AXB = C. \tag{1.2}$$

. Consistency conditions for equation (1.2) were given by Penrose[7] (see also [1]). When the equation is consistent, a solution can be obtained using generalized inverses. Khatri and Mitra [8] gave necessary and sufficient conditions for the existence of symmetric and positive semidefinite solutions as well as explicit formulae using generalized inverses. In [9],[10] solvability conditions for symmetric and positive definite solutions and general solutions of Equation (1.2) were obtained through the use of generalized singular value decomposition [11]-[13].
For important results on the inverse problem $A^*XA = D$ associated with several kinds of different sets S, for instance, symmetric matrices, symmetric nonnegative definite matrices, bisymmetric (same as persymmetric) matrices, bisymmetric nonnegative definite matrices and so on, We refer the reader to [14]-[17].
For the case the unknown A is skew-symmetric ortho-symmetric,[18] has discussed the

inverse problem $AX = B$. However, for this case, the inverse problem $A^*XA = D$ has not been dealt with yet. This problem will be considered here.

Definition 1.1 *A matrix $P \in C^{n\times n}$ is said to be a symmetric orthogonal matrix if $P^* = P$, $P^*P = I_n$.*

In this paper, without special statement, we assume that P is a given symmetric orthogonal matrix.

Definition 1.2 *A Matrix $X \in C^{n\times n}$ is said to be a skew-symmetric ortho-symmetric matrix if $X^* = -X$, $(PX)^* = PX$. We denote the set of all $n \times n$ skew-symmetric ortho-symmetric matrices by SSC_p^n.*

The problem studied in this paper can now be described as follows.

Problem 1. Given matrices $A \in C^{n\times m}$ and $D \in C^{m\times m}$, find a skew-symmetric ortho-symmetric matrix X such that

$$A^*XA = D.$$

In this paper, we discuss the solvability of this problem and an expression for its solution is presented.

The Optimal approximation problem of a matrix with the above-given matrix restriction comes up in the processes of test or recovery of a linear system due to incomplete data or revising given data. A preliminary estimate $\widetilde{X}$ of the unknown matrix X can be obtained by the experimental observation values and the information of statistical distribution. The optimal estimate of X is a matrix $\hat{X}$ that satisfies the given matrix restriction for $X$and is the best approximation of $\widetilde{X}$, see [19]-[21].

In this paper, we will also considered the so-called optimal approximation problem associated with $A^*XA = D$. It reads as follows.

Problem 2. Given matrix $\widetilde{X} \in C^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\left\|\tilde{X} - \widehat{X}\right\| = \inf_{X\in S_E} \left\|\tilde{X} - X\right\|,$$

where S_E is the solution set of Problem 1.

We point out that if Problem 1 is solvable, then Problem 2 has a unique solution, and in this case an expression for the solution can be derived.

The paper is organized as follows. In section 2, we obtain the general form of S_E and the sufficient and necessary conditions under which problem 1 is solvable mainly by using the structure of SSC_p^n and orthogonal projection matrices. In section 3, the expression for the solution of the matrix nearness problem 2 will be determined.

2 The expression of the general solution of problem 1

In this section we first discuss some structure properties of symmetric orthogonal matrices. Then given such a matrix P, we consider structural properties of the subset SSC_p^n of $C^{n\times n}$. Finally we present necessary and sufficient conditions for the existence of and the expressions for the skew-symmetric ortho-symmetric (with respect to the given P) solutions of problem 1.

Lemma 2.1. *Assume P is a symmetric orthogonal matrix of size n, and let*

$$P_1 = \frac{1}{2}(I_n + P), \quad P_2 = \frac{1}{2}(I_n - P). \tag{2.1}$$

Then P_1 and P_2 are orthogonal projection matrices satisfying $P_1 + P_2 = I_n, P_1P_2 = 0$.

Proof. Since

$$P_1 = \frac{1}{2}(I_n + P), P_2 = \frac{1}{2}(I_n - P).$$

Then

$$P_1 + P_2 = \frac{1}{2}(I_n + P) + \frac{1}{2}(I_n - P) = \frac{1}{2}(I_n + P + I_n - P) = \frac{1}{2}(2I_n) = I_n.$$

$$P_1P_2 = \frac{1}{2}(I_n + P).\frac{1}{2}(I_n - P) = \frac{1}{4}(I_n - P + P - P^2) = \frac{1}{4}(I_n - P^2) = \frac{1}{4}(I_n - P.P^*) = \frac{1}{4}(I_n - I_n) = 0.$$

Lemma 2.2. *Assume P_1 and P_2 are defined as (2.1) and rank $(P_1) = r$. Then rank$(P_2) = n - r$, and there exists unit column orthogonal matrices $U_1 \in C^{n\times r}$ and $U_2 \in C^{n\times(n-r)}$ such that $P_1 = U_1U_1^*, P_2 = U_2U_2^*$, and $U_1^*U_2 = 0$ then $P = U_1U_1^* - U_2U_2^*$.*

Proof. Since P_1 and P_2 are orthogonal projection matrices satisfying $P_1 + P_2 = I_n$ and $P_1P_2 = 0$, the column space $R(P_2)$ of the matrix P_2 is the orthogonal complement of the column space $R(P_1)$ of the matrix P_1, in other words, $R^n = R(P_1) \oplus R(P_2)$. Hence, if rank $(P_1) = r$, then rank $(P_2) = n-r$. On the other hand, rank $(P_1) = r$, rank$(P_2) = n-r$, and P_1, P_2 are orthogonal projection matrices. Thus there exists unit column orthogonal matrices $U_1 \in C^{n\times r}$ and $U_2 \in C^{n\times(n-r)}$ such that $P_1 = U_1U_1^*, P_2 = U_2U_2^*$. Using $R^n = R(P_1) \oplus R(P_2)$, we have $U_1^*U_2 = 0$.
Substituting $P_1 = U_1U_1^*, P_2 = U_2U_2^*$, into (2.1), we have $P = U_1U_1^* - U_2U_2^*$.

Elaborating on Lemma 2.2 and its proof, we note that $U = (U_1, U_2)$ is an orthogonal matrix and that the symmetric orthogonal matrix P can be expressed as

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*. \tag{2.2}$$

Lemma 2.3. *The matrix $X \in SSC_P^n$ if and only if X can be expressed as*

$$X = U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*, \tag{2.3}$$

where $F \in C^{r\times(n-r)}$ and U is the same as (2.2).

proof Assume $X \in SSC_P^n$. By lemma 2.2 and the definition of SSC_P^n, We choose $p_1 = \frac{I+p}{2}, P_2 = \frac{I-P}{2}$

$$P_1XP_1 = \frac{I+P}{2}X\frac{I+P}{2} = \frac{1}{4}(X + PX + XP + PXP)$$

$$= \frac{1}{4}(U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* + PX + XP + U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*).$$

$$= \frac{1}{4}(U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* + PX + XP + U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*).$$

$$= \frac{1}{4}(U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* + PX + XP + U \begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*).$$

$$= \frac{1}{4}(U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* + PX + XP + U \begin{pmatrix} 0 & -F \\ F^* & 0 \end{pmatrix} U^*).$$

$$P_1XP_1 = \frac{1}{4}(XP + PX).$$

Similarly

$$P_2XP_2 = \frac{-1}{4}(XP + PX).$$

Hence,

$$X = (P_1 + P_2)X(P_1 + P_2) = P_1XP_1 + P_1XP_2 + P_2XP_1 + P_2XP_2$$

$$X = P_1XP_2 + P_2XP_1 \quad (since P_1XP_1 + P_2XP_2 = 0).$$

$$X = P_1XP_2 + P_2XP_1 = U_1U_1^*XU_2U_2^* + U_2U_2^*XU_1U_1^* \quad (since P_1 = U_1U_1^* and P_2 = U_2U_2^*)$$

$$= U_1FU_2^* + U_2GU_1^*$$

Let $F = U_1^*XU_2$ *and* $G = U_2^*XU_1$.
It is easy to verify that $F^* = -G$.

$$(since F = U_1^*XU_2, \quad F^* = (U_1^*XU_2^*)^* = U_2^*X^*U_1 = -U_2^XU_1 = -G)$$

Then we have

$$X = U_1FU_2^* + U_2GU_1^* = U \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix} U^*$$

$$X = U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*$$

Conversely, for any $F \in C^{r\times(n-r)}$, *Let*

$$X = U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*$$

It is easy to verify that $X^* = -X$

$$X = U_1FU_2^* + U_2GU_1^*, \quad X^* = (U_1FU_2^*)^* + (U_2GU_1^*)^* = U_2F^*U_1^* + U_1G^*U_2^*$$

$$= -U_2GU_1^* - U_1FU_2^* = -(U_1FU_2^* + U_2GU_1^*) = -X.$$

using (2.2), we have

$$PXP = PU \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*P$$

$$= U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*$$

$$= U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*$$

$$= U\begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}U^* = U\begin{pmatrix} 0 & -F \\ F^* & 0 \end{pmatrix}U^* = -X$$

Thus

$$X = U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* \in SSC_P^n.$$

Lemma 2.4. *Let $A \in C^{n\times n}, D \in SSC^{n\times n}$ and assume $A - A^* = D$. Then there is precisely one $G \in SC^{n\times n}$ such that $A = L_D + G$, and $G = \frac{1}{2}(A + A^*) - \frac{1}{2}(L_D + L_D^*)$.*

Proof.For given $A \in C^{n\times n}, D \in SSC^{n\times n}$ and $A - A^* = D$.
It is easy to verify that there exists unique

$$G = \frac{1}{2}(A + A^*) - \frac{1}{2}(L_D + L_D^*) \in SC^{n\times n},$$

and we have

$$A = \frac{1}{2}(A - A^*) + \frac{1}{2}(A + A^*) = \frac{1}{2}(L_D - L_D^*) + \frac{1}{2}(A + A^*)$$

$$A = \frac{1}{2}(A + A^*) + L_D - \frac{1}{2}(L_D + L_D^*)(since L_D = \frac{1}{2}(L_D + L_D^*) + \frac{1}{2}(L_D - L_D^*))$$

$$A = L_D + G.$$

Let $A \in C^{n\times m}$ and $D \in C^{m\times m}$, U defined in (2.2), Set

$$U^*A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, A_1 \in C^{r\times m}, A_2 \in C^{(n-r)\times m}. \tag{2.4}$$

The generalized singular value decomposition (see [11],[12],[13]) of the matrix pair $[A_1^*, A_2^*]$ is

$$A_1^* = M \sum A_1 W^*, A_2^* = M \sum A_2 V^*, \tag{2.5}$$

where $W \in C^{m\times m}$ is a nonsingular matrix, $W \in OC^{r\times r}, V \in OC^{(n-r)\times(n-r)}$ and

$$\sum A_1 = \begin{pmatrix} I_k & & \\ & S_1 & \\ & & O_1 \\ & \cdots\cdots\cdots & \\ & O & \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ \\ m-t \end{matrix} \tag{2.6}$$

$$\sum A_2 = \begin{pmatrix} O_2 & & \\ & S_2 & \\ & & I_{t-k-s} \\ & \cdots\cdots\cdots & \\ & O & \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ \\ m-t \end{matrix} \qquad (2.7)$$

where

$$t = rank(A_1^*, A_2^*), K = t - rank(A_2^*),$$

$$S = rank(A_1^*) + rank(A_2^*) - t$$

$$S_1 = diag(\alpha_1, ..., \alpha_s), S_2 = diag(\beta_1, ..., \beta_s),$$

with $1 > \alpha_1 \geq ... \geq \alpha_s > 0, \quad 0 < \beta_1 \leq ... \leq \beta_s < 1,$ *and* $\alpha_i^2 + \beta_i^2 = 1, i = 1, ..., s.$
$0, 0_1$ *and* 0_2 *are corresponding zero submatrices.*

Theorem 2.5. *Given* $A \in C^{n\times m}$ *and* $D \in C^{m\times m}, U$ *defined in (2.2), and* U^*A *has the partition form of (2.4), the generalized singular value decomposition of the matrix pair* $[A_1^*, A_2^*]$ *as (2.5). Partition the matrix* $M^{-1}DM^{-*}$ *as*

$$M^{-1}DM^{-*} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix}$$

$$k \quad s \quad t-k-s \quad m-t$$

then the problem 1 has a solution $X \in SSC_P^n$ if and only if
$D^* = -D, \quad D_{11} = 0, \quad D_{33} = 0, \quad D_{41} = 0, \quad D_{42} = 0, \quad D_{43} = 0, \quad D_{44} = 0.$
In that case it has the general solution

$$X = U \left(\begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) U^*, \tag{2.9}$$

where

$$F = W \begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G)S_2^{-1} & S_1^{-1}D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^*, \tag{2.10}$$

with $X_{11} \in C^{r\times(n-r+k-t)}$, $X_{21} \in C^{s\times(n-r+k-t)}$, $X_{31} \in C^{(r-k-s)\times(n-r+k-t)}$, $X_{32} \in C^{(r-k-s)\times s}$, $X_{33} \in C^{(r-k-s)\times(t-k-s)}$ and $G \in SC^{s\times s}$ are arbitrary matrices.
Proof. The Necessity :
Assume the equation (1.1) has a solution $X \in SSC_P^n$. By the definition of SSC_P^n, it is easy to verify that $D^* = -D$. Since $D = A - A^*$

$$D^* = (A - A^*)^* = -A + A^* = -(A - A^*) = -D,$$

and we have from lemma 2.3 that X can be expressed as

$$X = U \left(\begin{smallmatrix} 0 & F \\ -F^* & 0 \end{smallmatrix} \right) U^*, \tag{2.11}$$

where $F \in C^{r\times(n-r)}$.
Note that U is an orthogonal matrix, and the definition of $A_i (i = 1, 2)$, Equation(1.1) is equivalent to

$$A_1^* F A_2 - A_2^* F A_1 = D. \tag{2.12}$$

Substituting (2.5) in (2.12), then we have

$$\begin{gathered} M \textstyle\sum_{A_1} W^* F A_2 - M \sum_{A_2} V^* F A_1 = D \\ M \textstyle\sum_{A_1} W^* F V \sum_{A_2}^* M^* - M \sum_{A_2} V^* F A_1 = D \\ M \textstyle\sum_{A_1} W^* F V \sum_{A_2}^* M^* - M \sum_{A_2} V^* F W \sum_{A_1}^* M^* = D \\ M^{-1} M \textstyle\sum_{A_1} W^* F V \sum_{A_2}^* M^* M^{-*} - M^{-1} M \sum_{A_2} V^* F W \sum_{A_1}^* M^* M^{-*} = M^{-1} D M^{-*} \\ \textstyle\sum_{A_1} (W^* F V) \sum_{A_2}^* - \sum_{A_2} (V^* F W) \sum_{A_1}^* = M^{-1} D M^{-*} \\ \textstyle\sum_{A_1} (W^* F V) \sum_{A_2}^* - \sum_{A_2} (W^* F V)^* \sum_{A_1}^* = M^{-1} D M^{-*}, \end{gathered} \tag{2.13}$$

partition the matrix W^*FV *as*

$$W^*FV = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}, \tag{2.14}$$

where $X_{11} \in C^{r\times(n-r+k-t)}, X_{22} \in C^{s\times s}, X_{33} \in C^{(r-k-s)\times(t-k-s)}$.
Taking W^*FV *and* $M^{-1}DM^{-*}$, *in (2.13), We have*

$$\begin{pmatrix} 0 & X_{12}S_2 & X_{13} & 0 \\ -S_2X_{21}^* & S_1X_{22}S_2 - (S_1X_{22}S_2)^* & S_1X_{23} & 0 \\ -X_{13}^* & -X_{23}^*S_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix} \tag{2.15}$$

Therefore (2.15)holds if and only if (2.8) holds and

$$X_{12} = D_{12}S_2^{-1}, X_{13} = D_{13}, X_{23} = S_1^{-1}D_{23}$$

and

$$S_1X_{22}S_2 - (S_1X_{22}S_2)^* = D_{22}.$$

It follows from Lemma 2.4 that $X_{22} = S_1^{-1}(L_{D_{22}} + G)S_2^{-1}$, *where* $G \in SC^{S\times S}$ *is arbitrary matrix. Substituting the above into (2.14), (2.11), thus we have formulation (2.9) and (2.10).*
The sufficiency. Let

$$F_G = W \begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G)S_2^{-1} & S_1^{-1}D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^*.$$

obviously, $F_G \in C^{r\times(n-r)}$. *By Lemma 2.3 and*

$$X_O = U\begin{pmatrix} 0 & F_G \\ -F_G^* & 0 \end{pmatrix}U^*,$$

We have $X_0 \in SSC_P^n$. *Hence*

$$A^*X_0A = A^*UU^*X_0UU^*A = \begin{pmatrix} A_1^* & A_2^* \end{pmatrix}\begin{pmatrix} 0 & F_G \\ -F_G^* & 0 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\begin{pmatrix} -A_2^*F_G^* & A_1^*F_G \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = -A_2^*F_G^*A_1 + A_1^*F_GA_2 = D.$$

This implies that

$$X_0 = U\begin{pmatrix} 0 & F_G \\ -F_G^* & 0 \end{pmatrix}U^* \in SSC_P^n$$

is the skew-symmetric ortho-symmetric solution of equation (1.1). Hence the proof.

3 The expression of the solution of Problem 2.

To prepare for an explicit expression for the solution of the matrix nearness problem 2, we first verify the following lemma.

Lemma 3.1. *Suppose that* $E, F \in C^{s\times s}$, *and let* $S_a = diag(a_1, ..., a_s) > 0, S_b = diag(b_1, ..., b_s) > 0$. *Then there exists a unique* $S_s \in SC^{s\times s}$ *and a unique* $S_r \in SSC^{s\times s}$ *such that*

$$\|S_aSS_b - E\|^2 + \|S_aSS_b - F\|^2 = min. \qquad (3.1)$$

and

$$S_s = \Phi * [S_a(E+F)S_b + S_b(E+F)^*S_a], \qquad (3.2)$$

$$S_r = \Phi * [S_a(E+F)S_b - S_b(E+F)^*S_a], \qquad (3.3)$$

where

$$\Phi = (\psi_{ij}) \in SC^{s\times s}, \qquad \Psi_{ij} = \frac{1}{2(a_i^2b_j^2 + a_j^2b_i^2)}, \qquad 1 \le i, j \le s. \qquad (3.4)$$

Proof. *We prove only the existence of* S_r *and (3.3). For any* $S = (S_{ij}) \in SSC^{s\times s}$, $E = (e_{ij})$, $F = (f_{ij}) \in C^{s\times s}$, *since* $S_{ii} = 0$, $S_{ij} = -S_{ji}$,

$$\|S_aSS_b - E\|^2 + \|S_aSS_b - F\|^2 = \sum_{1\le i,j\le s}[(a_ib_jS_{ij} - e_{ij})^2 + (a_ib_jS_{ij} - f_{ij})^2]$$

$$= \sum_{1\le i<j\le s} [(a_ib_jS_{ij} - e_{ij})^2 + (-a_jb_iS_{ij} - e_{ji})^2 + (a_ib_jS_{ij} - f_{ij})^2 + (-a_jb_iS_{ij} - f_{ji})^2 + \sum_{1\le i<s} (e_{ij}^2 + f_{ij}^2).$$

Using $\frac{\partial g(s)}{\partial S_{ij}} = 0 \quad (1 \le i, j \le n)$, *We have*

$$= 2(a_ib_jS_{ij} - e_{ij})(a_ib_j) + 2(-a_jb_iS_{ij} - e_{ji})(-a_jb_i) + 2(a_ib_jS_{ij} - f_{ij})(a_ib_j) + 2(-a_jb_iS_{ij} - f_{ji})(-a_jb_i)$$
$$= 2a_i^2b_j^2s_{ij} - 2a_ib_je_{ij} + 2a_j^2b_i^2S_{ij} + 2a_jb_ie_{ji} + 2a_i^2b_j^2s_{ij} - 2a_ib_jf_{ij} + 2a_j^2b_i^2S_{ij} + 2a_jb_if_{ji}$$
$$= 4a_i^2b_j^2s_{ij} + 4a_j^2b_i^2S_{ij} - 2a_ib_j(e_{ij} + f_{ij}) + 2a_jb_i(e_{ji} + f_{ji})$$
$$- 4S_{ij}(a_i^2b_j^2 + a_j^2b_i^2) = -2a_ib_j(e_{ij} + f_{ij}) + 2a_jb_i(e_{ji} + f_{ji})$$

$$S_{ij} = \frac{a_ib_j(e_{ij} + f_{ij}) - a_jb_i(e_{ji} + f_{ji})}{2(a_i^2b_j^2 + a_j^2b_i^2)}, \qquad 1 \le i, j \le s.$$

Here

$$\Phi = (\Psi_{ij}) = \frac{1}{2(a_i^2b_j^2 + a_j^2b_i^2)}, \qquad 1 \le i, j \le s$$

Hence there exists a unique solution $S_r = \hat{S_{ij}} \in SSC^{s\times s}$ *for (3.1) such that*

$$\hat{S_{ij}} = \frac{a_ib_j(e_{ij} + fij) - a_jb_i(e_{ji} + fji)}{2(a_i^2b_j^2 + a_j^2b_i^2)}, \qquad 1 \le i, j \le s.$$

$$S_r = \phi * [S_a(E + F)S_b - S_b(E + F)^*S_a].$$

Theorem 3.2. *Let* $\tilde{X} \in C^{n\times n}$*, the generalized singular value decomposition of the matrix pair* $[A_1^*\ ,\ A_2^*]$ *as (2.5), Let*

$$U^*\tilde{X}U = \left(\begin{smallmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{smallmatrix} \right), \tag{3.5}$$

$$W^*Z_{12}^*V = \begin{pmatrix} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix},$$

$$W^*{Z_{21}^*}^*V = \begin{pmatrix} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & Y_{22}^* & Y_{23}^* \\ Y_{31}^* & Y_{32}^* & Y_{33}^* \end{pmatrix}. \tag{3.6}$$

If problem 1 is solvable, then problem 2 has a unique solution $\hat{X}$, which can be expressed as

$$\hat{X} = U \begin{pmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{pmatrix} U^*, \tag{3.7}$$

where

$$\tilde{F} = W \begin{pmatrix} \frac{1}{2}(X_{11}^* - Y_{11}^*) & D_{12}S_2^{-1} & D_{13} \\ \frac{1}{2}(X_{21}^* - Y_{21}^*) & S_1^{-1}(L_{D_{22}} + \tilde{G})S_2^{-1} & S_1^{-1}D_{23} \\ \frac{1}{2}(X_{31}^* - Y_{31}^*) & \frac{1}{2}(X_{32}^* - Y_{32}^*) & \frac{1}{2}(X_{33}^* - Y_{33}^*) \end{pmatrix} V^*,$$

$\tilde{G} = \phi * [S_1^{-1}(X_{22}{}^* - Y_{22}{}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})S_2^{-1} + S_2^{-1}(X_{22}{}^* - Y_{22}{}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})^* S_1^{-1}]$,
with

$$\phi = (\psi_{ij}) \in SC^{s\times s}, \psi_{ij} = \frac{a_i^2 a_j^2 b_i^2 b_j^2}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, 1 \leq i, j \leq s.$$

Proof. *Using the invariance of the Frobenius norm under unitary transformations, from (2.9), (3.5) and (3.6) we have,*
(2.9)implies that

$$X = U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^*,$$

(3.5)implies that

$$U^* \tilde{X} U = \begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix}$$

(3.6)implies that

$$W^* Z_{12}^* V = \begin{pmatrix} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix}$$

$$W^* Z_{21}^*{}^* V = \begin{pmatrix} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & Y_{22}^* & Y_{23}^* \\ Y_{31}^* & Y_{32}^* & Y_{33}^* \end{pmatrix}$$

$$\tilde{X} = U \begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix} U^{-1}$$

$$X - \tilde{X} = U \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} U^* - U \begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix} U^* = U \begin{pmatrix} -Z_{11}^* & F - Z_{12}^* \\ -F^* - Z_{21}^* & -Z_{22}^* \end{pmatrix} U^*$$

$$\left\|X-\tilde{X}\right\|^2=\|Z_{11}^*\|^2+\|F-Z_{12}^*\|^2+\|-F^*-Z_{21}^*\|^2+\|Z_{22}^*\|^2$$

$$=\|Z_{11}^*\|^2+\left\|\begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13}\\ X_{21} & S_1^{-1}(L_{D_{22}}+G)S_2^{-1} & S_1^{-1}D_{23}\\ X_{31} & X_{32} & X_{33}\end{pmatrix}-W^*Z_{12}^*V\right\|^2+\left\|\begin{pmatrix} X_{11} & D_{12}S_2^{-1} & D_{13}\\ X_{21} & S_1^{-1}(L_{D_{22}}+G)S_2^{-1} & S_1^{-1}D_{23}\\ X_{31} & X_{32} & X_{33}\end{pmatrix}+W^*Z_{21}^{*\,*}V\right\|^2+\|$$

Thus

$$\left\|\tilde{X}-\hat{X}\right\|=\inf_{X\in S_E}\left\|\tilde{X}-X\right\|$$

is equivalent to

$$\|X_{11}-X_{11}^*\|^2+\|X_{11}+Y_{11}^*\|^2=min,\quad \|X_{21}-X_{21}^*\|^2+\|X_{21}+Y_{21}^*\|^2=min,$$

$$\|X_{31}-X_{31}^*\|^2+\|X_{31}+Y_{31}^*\|^2=min,\quad \|X_{32}-X_{32}^*\|^2+\|X_{32}+Y_{32}^*\|^2=min,$$

$$\|X_{33}-X_{33}^*\|^2+\|X_{33}+Y_{33}^*\|^2=min,$$

$$\left\|S_1^{-1}GS_2^{-1}-(X_{22}^*-S_1^{-1}L_{D_{22}}S_2^{-1})\right\|^2+\left\|S_1^{-1}GS_2^{-1}+(Y_{22}^*++S_1^{-1})L_{D_{22}}S_2^{-1})\right\|^2=min.$$

From Lemma 3.1. We have,

$$X_{11}=\frac{1}{2}(X_{11}^*-Y_{11}^*),X_{21}=\frac{1}{2}(X_{21}^*-Y_{21}^*)$$

$$X_{31}=\frac{1}{2}(X_{31}^*-Y_{31}^*),X_{32}=\frac{1}{2}(X_{32}^*-Y_{32}^*),\quad X_{33}=\frac{1}{2}(X_{33}^*-Y_{33}^*)$$

and

$$G=\Phi*[S_1^{-1}(X_{22}^*-Y_{22}^*-2S_1^{-1}L_{D_{22}}S_2^{-1})S_2^{-1}+S_2^{-1}(X_{22}^*-Y_{22}^*-2S_1^{-1}L_{D_{22}}S_2^{-1})^*S_1^{-1}].$$

Taking $X_{11},X_{21},X_{31},X_{32},X_{33}$ and G into (2.9), (2.10), we obtain that the solution of (the matrix mearness) Problem 2 can be expressed as

$$\hat{X}=U\begin{pmatrix}0 & \tilde{F}\\ -\tilde{F}^* & 0\end{pmatrix}U^*.$$

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