Geometric Automated Theorem Proving

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Geometric Automated Theorem Proving (GATP)

GATPs—Two major lines of research [CGZ94, CG01, Wan96]:

- Synthetic methods; Seminar 1
- Algebraic methods. Seminar 2

Formalization & Automated Discovery: Seminar 3

- Formalisation;
- Automated Discovery.

Geometric Tools & Geometric Knowledge Management: Seminar 4

- Geometric Tools: DGS/GATP/CAS/RGK/eLearning;
- Geometric Knowledge Management.

Synthetic Methods

Synthetic methods attempt to automate traditional geometry proof methods, producing human-readable proofs.

Seminal paper of Gelernter et al. It was based on the human simulation approach and has been considered a landmark in the AI area [Gel59, GHL60].

- Geometric reasoning - small and easy to understand proofs.
- Use of predicates only allow reaching fix-points.
- numerical model;
- constructing auxiliary points;
- generating geometric lemmas.

In spite of the success and significant improvements with these methods, the results did not lead to the development of a powerful geometry theorem prover [BdC95, CP79, CP86, Gil70, KA90, Nev74, Qua89]

Gelernter's GATP

A long-range program directed at the problem of "intelligent" behaviour and learning in machines has attained its first objective in the simulation on a high-speed digital computer of a machine capable of discovering proofs in elementary Euclidean plane geometry without resorting to exhaustive enumeration or to a decision procedure. The particular problem of a theorem proving in plane geometry was chosen as representative of a large class of difficult tasks that seem to require ingenuity and intelligence for their successful completion.

The theorem proving program relies upon heuristic methods to restrain if from generating proof sequences that do not have a high a priori probability of leading to a proof for the theorem in hand.

H. Gelernter1959, Realization of a geometry-theorem proving machine, Computers & thought, MIT Press, 1995

Gelernter's GATP

Backward chaining approach.

$$\forall_{\text{geometric elements}}[(H_1 \wedge \cdots \wedge H_r) \Rightarrow G]$$

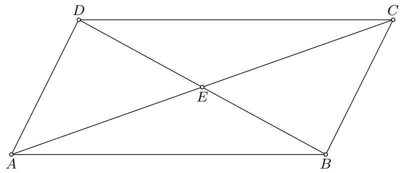
To prove G we search the *axiom rule set* to find a rule of the following form

$$[(G_1 \wedge \cdots \wedge G_r) \Rightarrow G]$$

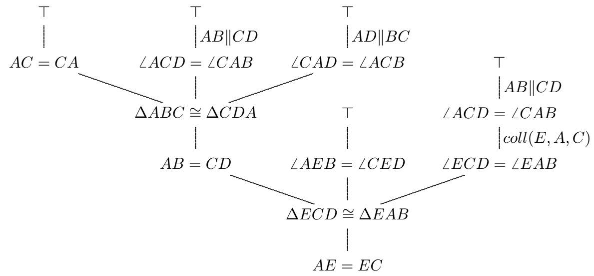
until the sub-goals are hypothesis.

The proof search will generate an and-or-proof-tree.

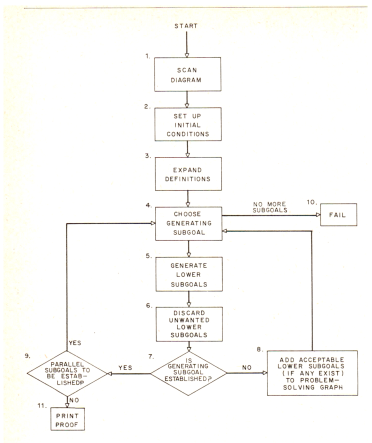
Example 1 - Gelernter

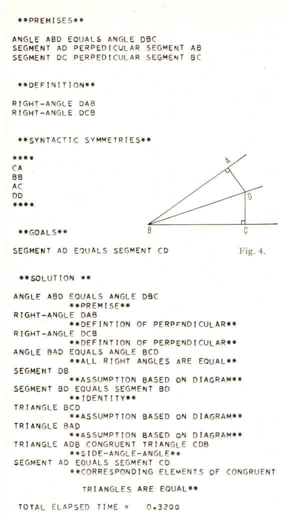


$$points(A,B,C) \wedge AB \| CD \wedge AD \| BC \wedge coll(E,A,C) \wedge coll(E,B,D) \Rightarrow AE = EC$$



Galernter (1959): Algorithm & Proof





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**PREMISES**

ANGLE ABD EQUALS ANGLE DBC
SEGMENT AD PERPEDICULAR SEGMENT AB
SEGMENT DC PERPEDICULAR SEGMENT BC

**DEFINITION**

RIGHT-ANGLE DAB
RIGHT-ANGLE DCB

**SYNTACTIC SYMMETRIES**

****
CA
BB
AC
DD
****

**GOALS**

SEGMENT AD EQUALS SEGMENT CD

**SOLUTION **

ANGLE ABD EQUALS ANGLE DBC
          **PREMISE**
RIGHT-ANGLE DAB
          **DEFINTION OF PERPENDICULAR**
RIGHT-ANGLE DCB
          **DEFINTION OF PERPENDICULAR**
ANGLE BAD EQUALS ANGLE BCD
          **ALL RIGHT ANGLES ARE EQUAL**
SEGMENT DB
          **ASSUMPTION BASED ON DIAGRAM**
SEGMENT BD EQUALS SEGMENT BD
          **IDENTITY**
TRIANGLE BCD
          **ASSUMPTION BASED ON DIAGRAM**
TRIANGLE BAD
          **ASSUMPTION BASED ON DIAGRAM**
TRIANGLE ADB CONGRUENT TRIANGLE CDB
          **SIDE-ANGLE-ANGLE**
SEGMENT AD EQUALS SEGMENT CD
          **CORRESPONDING ELEMENTS OF CONGRUENT

            TRIANGLES ARE EQUAL**

TOTAL ELAPSED TIME =   0.3200
```

A, D, B, C

Fig. 4.

GEOM — A "Coelho" out of the hat

Two uses of the geometric diagram as a model [CP86]:

- the diagram as a filter (a counter-example);
- the diagram as a guide (an example suggesting eventual conclusions).

Top-down or bottom-up directions? A general prover should be able to mix both directions of execution [CP86].

The introduction of new points can be envisaged as a means to make explicit more information in the model [CP86].

Although various strategies and heuristics were subsequently adopted and implement, the problem of search space explosion still remains and makes the methods of this type highly inefficient [CP79, CP86].

Example - GEOM

GEOM is a Prolog program that generates proofs for problems in high school plane geometry [CP86].

A user presents problems to GEOM by declaring the hypotheses, the optional diagram and the goal.

GEOM starts from the goal, top-down and with a depth-first strategy, outputing its deductions and reasons for each step of the proof.

The diagram works mostly as a source of counter-examples for pruning unprovable goals, and so proofs need not depend on it (...). However, the diagram may also be used in a positive guiding way.

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Example - GEOM

The geometric knowledge of GEOM, i.e. some of the axioms and theorems of elementary plane geometry, is embodied in nine procedures.

They are: equal angles (EAI), right angles (RAI), equal magnitude (EM, EM1), equal segments (ESI), midpoints (MP), parallel segments (PRI), parallelogram (PG), congruence (DIRCON) and diagram routines.

Because each procedure may call itself through others, **the search space can grow quite large**, in particular when the clause for differences of segments is used.

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GEOM: proof tree

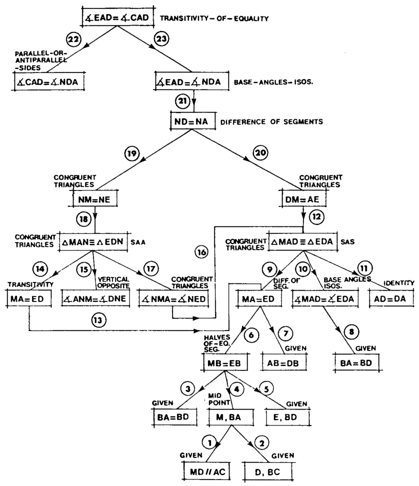

FIG. 22 — The proof tree of Problem 14 used by GEOM

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Coordinate-free Methods

Instead of coordinates, some basic geometric quantities, e.g. the ratio of parallel line segments, the signed area, and the Pythagorean difference (vector methods).

- Area method [CGZ93, JNQ12, QJ06b];
- Full-angle method [CGZ94, CGZ96b];
- Solid geometry [CGZ95].

Pros: Geometric proofs, small and human-readable.

Cons:

- not the "normal" high-school geometric reasoning;
- for many conjectures these methods still deal with extremely complex expressions.

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Area Method — Basic Geometric Quantities

Definition (Ratio of directed parallel segments)

For four collinear points P, Q, A, and B, such that $A \neq B$, the ratio of directed parallel segments, denoted $\frac{\overline{PQ}}{\overline{AB}}$ is a real number.

Definition (Signed Area)

The *signed area* of triangle ABC, denoted S_{ABC}, is the area of the triangle with a sign depending on its orientation in the plane.

Definition (Pythagoras difference)

For three points A, B, and C, the *Pythagoras difference*, is defined in the following way: $\mathcal{P}_{ABC} = \overline{AB}^2 + \overline{CB}^2 - \overline{AC}^2$.

Properties of the Ratio of Directed Parallel Segments

- $\frac{\overline{PQ}}{\overline{AB}} = -\frac{\overline{QP}}{\overline{AB}} = \frac{\overline{QP}}{\overline{BA}} = -\frac{\overline{PQ}}{\overline{BA}}$;
- $\frac{\overline{PQ}}{\overline{AB}} = 0$ iff $P = Q$;
- $(\dots)$

EL1 (The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$. Then it holds that $\frac{\overline{PM}}{\overline{QM}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{QAB}}$; $\frac{\overline{PM}}{\overline{PQ}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAQB}}$; $\frac{\overline{QM}}{\overline{PQ}} = \frac{\mathcal{S}_{QAB}}{\mathcal{S}_{PAQB}}$.

Properties of the Signed Area

- $\mathcal{S}_{ABC} = \mathcal{S}_{CAB} = \mathcal{S}_{BCA} = -\mathcal{S}_{ACB} = -\mathcal{S}_{BAC} = -\mathcal{S}_{CBA}$.
- $\mathcal{S}_{ABC} = 0$ iff A, B, and C are collinear.
- $PQ \parallel AB$ iff $\mathcal{S}_{PAB} = \mathcal{S}_{QAB}$, i.e., iff $\mathcal{S}_{PAQB} = 0$.
- Let $ABCD$ be a parallelogram, P and Q be two arbitrary points. Then it holds that $\mathcal{S}_{APQ} + \mathcal{S}_{CPQ} = \mathcal{S}_{BPQ} + \mathcal{S}_{DPQ}$ or $\mathcal{S}_{PAQB} = \mathcal{S}_{PDQC}$.
- Let R be a point on the line PQ. Then for any two points A and B it holds that $\mathcal{S}_{RAB} = \frac{\overline{PR}}{\overline{PQ}}\mathcal{S}_{QAB} + \frac{\overline{RQ}}{\overline{PQ}}\mathcal{S}_{PAB}$.
- $(\dots)$

Properties of the Pythagoras Difference

- $\mathcal{P}_{AAB} = 0$.
- $\mathcal{P}_{ABC} = \mathcal{P}_{CBA}$.
- If A, B, and C are collinear then, $\mathcal{P}_{ABC} = 2\overline{BA}\,\overline{BC}$.
- $AB \perp BC$ iff $\mathcal{P}_{ABC} = 0$.
- Let AB and PQ be two non-perpendicular lines, and Y be the intersection of line PQ and the line passing through A and perpendicular to AB. Then, it holds that

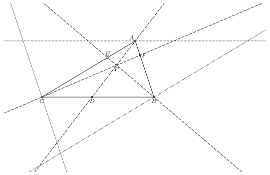
$$\frac{\overline{PY}}{\overline{QY}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{QAB}}, \quad \frac{\overline{PY}}{\overline{PQ}} = \frac{\mathcal{P}_{PAB}}{\mathcal{P}_{PAQB}}, \quad \frac{\overline{QY}}{\overline{PQ}} = \frac{\mathcal{P}_{QAB}}{\mathcal{P}_{PAQB}}.$$

- $(\dots)$

The Area Method — The Proof Algorithm

Express the hypothesis of a theorem using a set of constructive statements.

Each constructive statement introduces a new point.



The conclusion is expressed by a polynomial in some geometry quantities (defined above), without any relation to a given system of coordinates.

The proof is then developed by eliminating, in reverse order, the point introduced before, using for that purpose a set of lemmas.

Constructive Geometric Statements

ECS1 construction of an arbitrary point U; (...).

ECS2 construction of a point Y such that it is the intersection of two lines (LINE U V) and (LINE P Q);
ndg-condition: $UV \nparallel PQ$; $U \neq V$; $P \neq Q$.
degree of freedom for Y: 0

ECS3 construction of a point Y such that it is a foot from a given point P to (LINE U V); (...).

ECS4 construction of a point Y on the line passing through point W and parallel to (LINE U V), such that $\overline{WY} = r\overline{UV}$, (...).

ECS5 construction of a point Y on the line passing through point U and perpendicular to (LINE U V), such that $r = \frac{4\mathcal{S}_{UVY}}{\mathcal{P}_{UVU}}$, (...).

Forms of Expressing the Conclusion

property	in terms of geometric quantities
points A and B are identical	$\mathcal{P}_{ABA} = 0$
points A, B, C are collinear	$\mathcal{S}_{ABC} = 0$
AB is perpendicular to CD	$\mathcal{P}_{ABA} \neq 0 \wedge \mathcal{P}_{CDC} \neq 0 \wedge \mathcal{P}_{ACD} = \mathcal{P}_{BCD}$
AB is parallel to CD	$\mathcal{P}_{ABA} \neq 0 \wedge \mathcal{P}_{CDC} \neq 0 \wedge \mathcal{S}_{ACD} = \mathcal{S}_{BCD}$
O is the midpoint of AB	$\mathcal{S}_{ABO} = 0 \wedge \mathcal{P}_{ABA} \neq 0 \wedge \frac{\overline{AO}}{\overline{AB}} = \frac{1}{2}$
AB has the same length as CD	$\mathcal{P}_{ABA} = \mathcal{P}_{CDC}$
points A, B, C, D are harmonic	$\mathcal{S}_{ABC} = 0 \wedge \mathcal{S}_{ABD} = 0 \wedge \mathcal{P}_{BCB} \neq 0 \wedge \mathcal{P}_{BDB} \neq 0 \wedge \frac{\overline{AC}}{\overline{CB}} = \frac{\overline{DA}}{\overline{DB}}$
angle ABC has the same measure as DEF	$\mathcal{P}_{ABA} \neq 0 \wedge \mathcal{P}_{ACA} \neq 0 \wedge \mathcal{P}_{BCB} \neq 0 \wedge \mathcal{P}_{DED} \neq 0 \wedge \mathcal{P}_{DFD} \neq 0 \wedge \mathcal{P}_{EFE} \neq 0 \wedge \mathcal{S}_{ABC} \cdot \mathcal{P}_{DEF} = \mathcal{S}_{DEF} \cdot \mathcal{P}_{ABC}$
A and B belong to the same circle arc CD	$\mathcal{S}_{ACD} \neq 0 \wedge \mathcal{S}_{BCD} \neq 0 \wedge \mathcal{S}_{CAD} \cdot \mathcal{P}_{CBD} = \mathcal{S}_{CBD} \cdot \mathcal{P}_{CAD}$

Elimination Lemmas

EL2 Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (PRATIO Y W (LINE U V) r). Then it holds

$$G(Y) = G(W) + r(G(V) - G(U)).$$

EL3 Let $G(Y)$ be a linear geometric quantity and point Y is introduced by the construction (INTER Y (LINE U V) (LINE P Q). Then it holds

$$G(Y) = \frac{\mathcal{S}_{UPQ}G(V) - \mathcal{S}_{VPQ}G(U)}{\mathcal{S}_{UPVQ}}.$$

- (...)

Constructive Steps & Elimination Lemmas

		Geometric Quantities						
		$\mathcal{P}_{AYB}$	$\mathcal{P}_{ABY}$	$\mathcal{P}_{ABCY}$	$\mathcal{S}_{ABY}$	$\mathcal{S}_{ABCY}$	$\frac{\overline{AY}}{\overline{CD}}$	$\frac{\overline{AY}}{\overline{BY}}$
Constructive Steps	ECS2	EL5	EL3				EL11	EL1
	ECS3	EL6	EL4				EL12	
	ECS4	EL7	EL2				EL13	
	ECS5	EL10	EL9		EL8		EL14	
		Elimination Lemmas						

The Algorithm

→ $S = (C_1, C_2, \ldots, C_m, (E, F))$ is a statement in **C**.

← The algorithm tells whether S is true, or not, and if it is true, produces a proof for S.

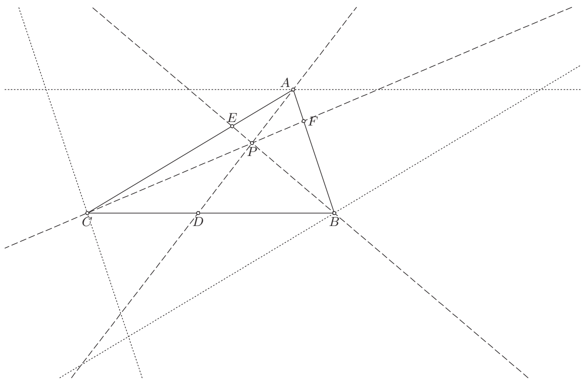
```
for (i=m;i==1;i--) {
   if (the ndg conditions of Ci is satisfied) exit;
   // Let G1,\ldots,Gn be the geometric quantities in E and F
   for (j=1;j<=n,j++) {
      Hj <- eliminating the point introduced
                     by construction Ci from Gj
      E <- E[Gj:=Hj]
      F <- F[Gj:=Hj]
   }
}
if (E==F) S <- true else S<-false
```

Adding to that it is needed to check the ndg condition of a construction (three possible forms).

An Example (Ceva's Theorem)

Let $\triangle ABC$ be a triangle and P be an arbitrary point in the plane. Let D be the intersection of AP and BC, E be the intersection of BP and AC, and F the intersection of CP and AB. Then:

$$\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} = 1$$



Example — Proof

The proof of a conjecture is based on eliminating all the constructed points, in reverse order, until an equality in only the free points is reached.

$$\begin{aligned}
\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} &= \frac{\mathcal{S}_{APC}}{\mathcal{S}_{BCP}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} && \text{the point } F \text{ is eliminated}\\
&= \frac{\mathcal{S}_{APC}}{\mathcal{S}_{BCP}}\frac{\mathcal{S}_{BPA}}{\mathcal{S}_{CAP}}\frac{\overline{CE}}{\overline{EA}} && \text{the point } D \text{ is eliminated}\\
&= \frac{\mathcal{S}_{APC}}{\mathcal{S}_{BCP}}\frac{\mathcal{S}_{BPA}}{\mathcal{S}_{CAP}}\frac{\mathcal{S}_{CPB}}{\mathcal{S}_{ABP}} && \text{the point } E \text{ is eliminated}\\
&= 1
\end{aligned}$$

Elimination Steps: 3; Geometric Steps: 6; Algebraic Steps: 23; Total Steps: 32; CPU Time 0.004s. The GATP also provide the ndg conditions.

Full-Angle Method

Intuitively, a full-angle $\angle[u, v]$ is the angle from line u to line v. Two full-angles $\angle[l, m]$ and $\angle[u, v]$ are equal if there exists a rotation K such that $K(l)\|u$ and $K(m)\|v$

Full-Angle is defined as an ordered pair of lines which satisfies the following rules [CGZ96b]:

R1 For all parallel lines $AB\|PQ$, $\angle[0] = \angle[AB, PQ]$ is a constant.

R2 For all perpendicular lines $AB \perp PQ$, $\angle[1] = \angle[AB, PQ]$ is a constant.

R7 If PX is parallel to UV, then $\angle[AB, PX] = \angle[AB, UV]$.

R8 If PX is perpendicular to UV, then $\angle[AB, PX] = \angle[1] + \angle[AB, UV]$.

Solid Geometry

Solid Geometry Method — For any points A, B, C and D in the space, the signed volume V_{ABCD} of the tetrahedron $ABCD$ is a real number which satisfies the following properties [CGZ95].

V.1 When two neighbor vertices of the tetrahedron are interchanged, the signed volume of the tetrahedron will change signs, e.g., $V_{ABCD} = -V_{ABDC}$.

V.2 Points A, B, C and D are coplanar iff $V_{ABCD} = 0$.

V.3 There exist at least four points A, B, C and D such that $V_{ABCD} \neq 0$.

V.4 For five points A, B, C, D and O, we have $V_{ABCD} = V_{ABCO} + V_{ABOD} + V_{AOCD} + V_{OBCD}$.

V.5 If A, B, C, D, E and F are six coplanar points and $S_{ABC} = \lambda SDEF$ then for any point T we have $V_{TABC} = \lambda V_{TDEF}$.

Algebraic Methods

Algebraic Methods: are based on reducing geometry properties to algebraic properties expressed in terms of Cartesian coordinates.

The biggest successes in automated theorem proving in geometry were achieved (i.e., the most complex theorems were proved) by algebraic provers based on:

- Wu's method [Cho85, Cho88];
- Gröbner bases method [Buc98, Kap86].

Decision procedures.

No readable, traditional geometry proofs, only a yes/no answer (with a corresponding algebraic argument).

Wu's Method

An elementary version of Wu's method is simple:
Geometric theorem T transcribed as polynomial equations and inequations of the form:

- H: $h_l = O, \ldots, h_s = O, d_1 \neq 0, \ldots, d_t \neq 0$;
- C: c=0.

Proving T is equivalent to deciding whether the formula

$$\forall_{x_l,\ldots x_n}[h_1 = 0 \wedge \cdots \wedge h_s \wedge d_1 \neq 0 \wedge \ldots \wedge d_t \neq 0 \Rightarrow c = 0] \quad (1)$$

is valid.

Wu's Method

Computes a characteristic set C of $\{h_1, \ldots, h_s\}$ and the pseudo-remainder r of c with respect to C.

If r is identically equal to 0, then T is proved to be true.

The subsidiary condition $J \neq 0$, where J is the product of initials of the polynomials in C are the ndg conditions [CG90, WT86, Wu00].

This is a decision procedure.

GCLC Implementation of Wu's Method

Let $\triangle ABC$ be a triangle and P be an arbitrary point in the plane. Let D be the intersection of AP and BC, E be the intersection of BP and AC, and F the intersection of CP and AB. Then: $\frac{\overline{AF}}{\overline{FB}} \frac{\overline{BD}}{\overline{DC}} \frac{\overline{CE}}{\overline{EA}} = 1$

$$
\begin{aligned}
p_1 &= -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1\\
p_2 &= u_5x_2 - u_4x_1\\
p_3 &= -u_3x_4 + u_2x_3\\
p_4 &= u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1\\
p_5 &= (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3)\\
p_6 &= 2x_6x_3^2x_1^3 - 3u_3x_6x_3^2x_1^2 + u_3^2x_6x_3^2x_1 - u_3x_6x_3x_1^3 + u_3^2x_6x_3x_1^2 -\\
&\quad -u_1x_3^2x_1^3 + 2u_3u_1x_3^2x_1^2 - u_3^2u_1x_3^2x_1
\end{aligned}
$$

GCLC Implementation of Wu's Method (cont)

Triangulation, step 1; step 2; step 3; step 4; step 5

Calculating final remainder of the conclusion:
$g = 2x_6x_3^2x_1^3 - 3u_3x_6x_3^2x_1^2 + u_3^2x_6x_3^2x_1 - u_3x_6x_3x_1^3 + u_3^2x_6x_3x_1^2 - u_1x_3^2x_1^3 + 2u_3u_1x_3^2x_1^2 - u_3^2u_1x_3^2x_1$
with respect to the triangular system.

Pseudo remainder with p_4 over variable x_6:
$g = (2u_5u_2 - u_5u_1 - 2u_4u_3 + u_3u_1)x_3^2x_1^3 + (-3u_5u_3u_2 + 2u_5u_3u_1 + 3u_4u_3^2 - 2u_3^2u_1)x_3^2x_1^2 + (u_5u_3^2u_2 - u_5u_3^2u_1 - u_4u_3^3 + u_3^3u_1)x_3^2x_1 + (-u_5u_3u_2 + u_4u_3^2)x_3x_1^3 + (u_5u_3^2u_2 - u_4u_3^3)x_3x_1^2$

$(\ldots)$

Pseudo remainder with p_0 over variable x_1: $g = 0$

Status: The conjecture has been proved.

...but all the calculations made, are not translatable to geometric reasoning.

Gröbner Basis

A Gröbner basis of an ideal is a special basis using which the membership problem of the ideal as well as the membership problem of the radical of the ideal can be easily decided.

(...) to decide whether a finite set of geometry hypotheses expressed as polynomial equations, in conjunction with a finite set of subsidiary conditions expressed as negations of polynomial equations which rule out degenerate cases, imply another geometry relation given as a conclusion.

Such a problem is shown to be equivalent to deciding whether a finite set of polynomials does not have a solution in an algebraically closed field. Using Hilbert's Nullstellensatz, this problem can be decided by checking whether 1 is in the ideal generated by these polynomials

This test can be done by computing a Gröbner basis of the ideal.

GCLC Implementation of Gröbner Basis Method

Let $\triangle ABC$ be a triangle and P be an arbitrary point in the plane. Let D be the intersection of AP and BC, E be the intersection of BP and AC, and F the intersection of CP and AB. Then: $\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} = 1$.

Conjecture $p_6 = 2x_6x_3^2x_1^3 - 3u_3x_6x_3^2x_1^2 + u_3^2x_6x_3^2x_1 - u_3x_6x_3x_1^3 + u_3^2x_6x_3x_1^2 - u_1x_3^2x_1^3 + 2u_3u_1x_3^2x_1^2 - u_3^2u_1x_3^2x_1$

The used proving method is Buchberger's method.
Input polynomial system is:

$p_0 = -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1$
$p_1 = u_5x_2 - u_4x_1$
$p_2 = -u_3x_4 + u_2x_3$
$p_3 = u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1$
$p_4 = (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3)$

GCLC Implementation of Gröbner Basis Method (cont)

iteration 1; iteration 2.

Gröbner basis has 7 polynomials:
$p_0 = -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1$
$p_1 = u_5x_2 - u_4x_1$
$p_2 = -u_3x_4 + u_2x_3$
$p_3 = u_5x_4 + (-u_4 + u_1)x_3 - u_5u_1$
$p_4 = (u_5 - u_3)x_6 + (-u_5u_2 + u_4u_3)$
$p_5 = (u_5u_2 - u_5u_1 - u_4u_3)x_1 + u_5u_3u_1$
$p_6 = (u_5u_2 - u_4u_3 + u_3u_1)x_3 - u_5u_3u_1$

$(\ldots)$

Status: The conjecture has been proved.
Space Complexity: The biggest polynomial obtained during proof process contained 259 terms.
Time Complexity: Time spent by the prover is 0.101 seconds.

"New" approaches

- An approach based on a deductive database and forward chaining works over a suitably selected set of higher-order lemmas and can prove complex geometry theorems, but still has a smaller scope than algebraic provers [CGZ94, CGZ00, YCG10b].
- Quaife used a resolution theorem prover to prove theorems in Tarski's geometry [Qua89].
- A GATP based on coherent-logic capable of producing both readable and formal proofs of geometric conjectures of certain sort [SPJ11].
- Probabilistic verification of elementary geometry statements [CFGG97, RGK99].
- Visual Reasoning/Proofs [Kim89, YCG10a, YCG10b].

Geometry Deductive Database

- In the general setting: structured deductive database and the data-based search strategy to improve the search efficiency.[1]
- Selection of a good set of rules; adding auxiliary points and constructing numerical diagrams as models automatically.

The result program can be used to find fix-points for a geometric configuration, i.e. the program can find all the properties of the configuration that can be deduced using a fixed set of geometric rules.

Generate ndg conditions.

Structured deductive database (graphs) reduce the size of the database in some cases by one thousand times.

[1] Semantic Graphs are an alternative!?

Deductive Databases

- Use canonical form for predicates;
- Use equivalent classes to represent some predicates;
- Use representative elements for equivalent classes;
- breadth-first forward chaining search:
  where D_0 is the hypotheses of the geometry statement and R is the rule set.

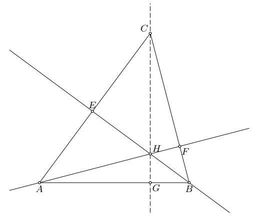
  For each rule r in R, apply it to D_0 to obtain new facts. Let D_1 be the union of D_0 and the set of new facts obtained.

  Repeat the above process for D_1 to obtain D_2, and so on.

  If at certain step $D_k = D_{k+1}$, we say that a fix-point for D_0 and R is reached.

Geometry Deductive Database – The Orthocenter Theorem

$points(A, B, C) \wedge coll(E, A, C) \wedge perp(B, E, A, C) \wedge coll(F, B, C) \wedge perp(A, F, B, C) \wedge coll(H, A, F) \wedge coll(H, B, E) \wedge coll(G, A, B) \wedge coll(G, C, H)$



The fix-point contains two of the most often encountered properties of this configuration:

- $perp(C, G, A, B)$;
- $\angle FGC = \angle CGE$

Quaife's GATP

Tarski axiomatic system: is, or rather its algebraic equivalent, complete and decidable.

Quaife developed a GATP for Euclidean plane geometry within the automated reasoning system OTTER (a resolution theorem prover) [Qua89].

(A1) Reflexivity axiom for equidistance.
$\rightarrow u \cdot v \equiv v \cdot u$

(A2) Transitivity axiom for equidistance.
$u \cdot v \equiv w \cdot x, u \cdot v \equiv y \cdot z \rightarrow w \cdot x \equiv y \cdot z$

(A4) Segment construction axiom, two clauses.
(A4.1) $\rightarrow B(u, v, Ext(u, v, w, x))$
(A4.2) $\rightarrow v \cdot Ext(u, v, w, x) \equiv w \cdot x$

(...)

Quaife's GATP

Heuristics

- maximum weight for retained clauses at 25,
- first attempt to obtain a proof in which no variables are allowed in any generated and retained clause.

The provers based upon Wu's algorithm, are able to prove quite more difficult theorems in geometry then those by Quaife's GATP.

However Wu's method only works with hypotheses and theorems that can be expressed as equations, and not with inequalities as correspond to the relation B in Quaife's resolution prover.

Coherent Logic GATP

Coherent Logic is a fragment of first-order logic with formulae of the following form:

$$A_1(x) \wedge \ldots \wedge A_n(x) \rightarrow \exists_{y_1} B(x, y_1) \vee \ldots \vee \exists_{y_m} B(x, y_m)$$

with a breath-first proof procedure sound and complete [BC05].

ArgoCLP (**C**oherent **L**ogic **P**rover of the Argo Group[2])

- new proof procedures;
- proof trace exportable to:
  - a proof object in Isabelle/Isar;
  - human readable (English/LaTeX).

not aimed at proving complex geometry theorems but rather at proving foundational theorems (close to the axiom level) [SPJ11].

[2] `http://argo.matf.bg.ac.rs/`

Probabilistic Verification

Probabilistic verification of elementary geometry statements [CFGG97, RGK99].

> *Cinderella (...) use (...) a technique called "Randomized Theorem Checking". First the conjecture (...) is generated. Then the configuration is moved into many different [random] positions and for each of these it is checked whether the conjecture still holds. (...) generating enough(!) random (!) examples where the theorem holds is at least as convincing as a computer-generated symbolic proof.*
>
> *User Manual for the Interactive Geometry Software Cinderella, Jürgen Richter-Gebert, Ulrich H. Kortenkamp*

Visual Reasoning/Representation

Visual Reasoning extend the use of diagrams with a method that allows the diagrams to be perceived and to be manipulated in a creative manner [Kim89].

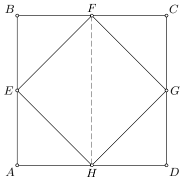
Visually Dynamic Presentation of Proofs linking the proof done by a synthetic method (full-angle) with a visual presentation of the proof [QSGB19, SQ10, YCG10a, YCG10b].

Visual Reasoning in Geometry Theorem Proving

> *We study the role of visual reasoning as a computationally feasible heuristic tool in geometry problem solving. We use an algebraic notation to represent geometric objects and to manipulate them.*
> *We show that this representation captures powerful heuristics for proving geometry theorems, and that it allows a systematic manipulation of geometric features in a manner similar to what may occur in human visual reasoning*
>
> *Michelle Y . Kim,*

An Example

Consider the problem in "Given a square $ABCD$, take the midpoints of the four sides, and prove that the two triangles ΔEEH and ΔGFH are congruent to each other."



To solve this problem, backward-chaining methods used by most of previous geometry-theorem proving systems [Gel59, CP86] would first set up a goal to prove that the two triangles arc congruent (...). A human mathematician, given the problem , may perceive an apparent symmetry in the diagram by observing a reflection across FH or across EG. As a symmetry is observed, it can be shown with little effort that the two triangles are congruent, and thus repeated proofs can be avoided.

Visually Dynamic Presentation of Proofs in Plane Geometry

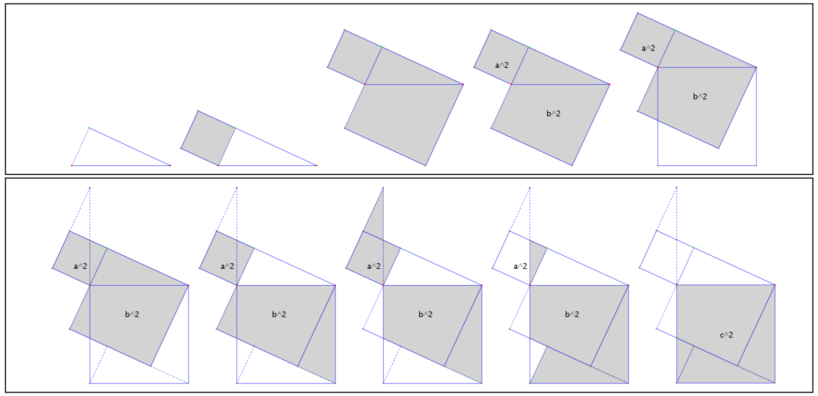


Figure: Pythagoras Theorem - Visual Proof

Formalisation

Full formal proofs mechanically verified by generic theorem proof assistants (e.g. Isabelle [Pau94, PN90], Coq [Tea09]).

- Hilbert's *Foundations of Geometry* [Hil77, MF03, DDS00];
- Jan von Plato's constructive geometry [Kah95, vP95];
- French high school geometry [Gui04];
- Tarski's geometry [Nar07b, BBN16];
- An axiom system for compass and ruler geometry [BNW18];
- Projective geometry [MNS11, FT11];
- Area Method [JNQ12, Nar06];
- Algebraic methods in geometry [MPPJ12].

Proof Assistants

Proof assistant (or interactive theorem prover) is a software tool to assist with the development of formal proofs by human-machine collaboration.

- Isabelle—`https://isabelle.in.tum.de/`—Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus.
- Coq—`https://coq.inria.fr/`—Coq is a formal proof management system. It provides a formal language to write mathematical definitions, executable algorithms and theorems together with an environment for semi-interactive development of machine-checked proofs.

Others: HOL Light; Lean; Mizar; ...

Area Method: Formalisation

Formalisation [JNQ12, Nar06, Nar09];

1. $\overline{AB} = 0$ if and only if the points A and B are identical
2. $\mathcal{S}_{ABC} = \mathcal{S}_{CAB}$
3. $\mathcal{S}_{ABC} = -\mathcal{S}_{BAC}$
4. If $\mathcal{S}_{ABC} = 0$ then $\overline{AB} + \overline{BC} = \overline{AC}$ (Chasles's axiom)
5. There are points A, B, C such that $\mathcal{S}_{ABC} \neq 0$ (dimension; not all points are collinear)
6. $\mathcal{S}_{ABC} = \mathcal{S}_{DBC} + \mathcal{S}_{ADC} + \mathcal{S}_{ABD}$ (dimension; all points are in the same plane)
7. For each element r of F, there exists a point P, such that $\mathcal{S}_{ABP} = 0$ and $\overline{AP} = r\overline{AB}$ (construction of a point on the line)
8. If $A \neq B$, $\mathcal{S}_{ABP} = 0$, $\overline{AP} = r\overline{AB}$, $\mathcal{S}_{ABP'} = 0$ and $\overline{AP'} = r\overline{AB}$, then $P = P'$ (unicity)
9. If $PQ \parallel CD$ and $\frac{\overline{PQ}}{\overline{CD}} = 1$ then $DQ \parallel PC$ (parallelogram)
10. If $\mathcal{S}_{PAC} \neq 0$ and $\mathcal{S}_{ABC} = 0$ then $\frac{\overline{AB}}{\overline{AC}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAC}}$ (proportions)
11. If $C \neq D$ and $AB \perp CD$ and $EF \perp CD$ then $AB \parallel EF$
12. If $A \neq B$ and $AB \perp CD$ and $AB \parallel EF$ then $EF \perp CD$
13. If $FA \perp BC$ and $\mathcal{S}_{FBC} = 0$ then $4\mathcal{S}^2_{ABC} = \overline{AF}^2\overline{BC}^2$ (area of a triangle)

Using this axiom system all the properties of the geometric quantities required by the area method were *formally verified* (within the *Coq* proof assistant), demonstrating the correctness of the system and eliminating all concerns about provability of the lemmas [Nar09].

Area Method: Formalization in Coq

```
Require Export field.
Require Import Classical.

Ltac Geometry := auto with Geom field_hints.

Parameter Point : Set. // The set of Points
Parameter S : Point -> Point -> Point -> F. // The signed area
Parameter DSeg : Point -> Point -> F. // The signed distance

Infix "**" := DSeg (left associativity, at level 20) : F_scope.

Definition Col (A B C : Point) : Prop := S A B C = 0.
Definition S4 (A B C D : Point) : F := S A B C + S A C D.
Definition parallel (A B C D : Point) : Prop := S4 A C B D = 0.

Axiom A1b : forall A B : Point, A ** B = 0 <-> A = B.

Axiom A2a : forall (A B : Point) (r : F),
    {P : Point | Col A B P /\ A ** P = r * A ** B}.
Axiom A2b : forall (A B P P1 : Point) (r : F),
    A <> B ->
    Col A B P ->
    A ** P = r * A ** B -> Col A B P1 -> A ** P1 = r * A ** B -> P = P1.

Axiom chasles : forall A B C : Point, Col A B C -> A ** B + B ** C = A ** C.
```


Automated Discovery

- Locus Generation: to determine the implicit equation of a locus set [BAE07, BA12].

  The set of points determined by the different positions of a point, the *tracer*, as a second point in which the tracer depends on, called the *mover*, runs along the one dimensional object to which it is restrained.

- Automated Finding of Theorems: the discovery of new facts about a given geometric configuration.

Finding locus equations

For most DGS a locus is basically a set of points in the screen with no algebraic information [BAE07, ABMR14].

- Numerical approach, based on interpolation (Cinderella, Cabri) [Bot02].
- Symbolic method, finding the equation of a locus [BL02, BA12, ABMR14].

Determine the equation of a locus set using remote computations on a server [EBA10].

Loci Finding: Algorithm

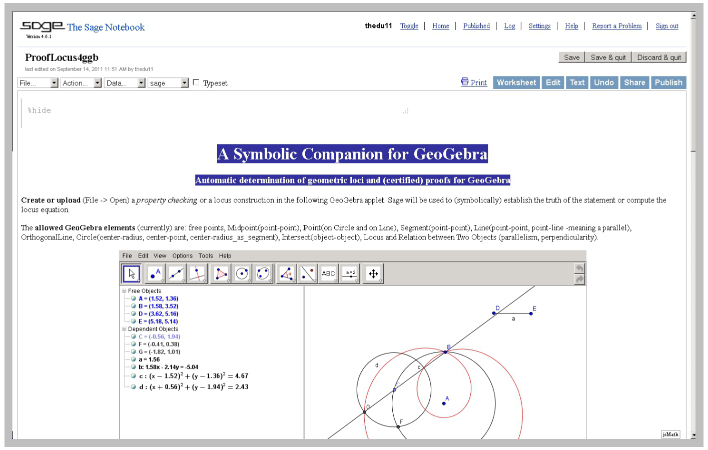
A statement is considered where the conclusion does not follow from the hypotheses.

Symbolic coordinates are assigned to the points of the construction (where every free point gets two new free variables u_i , u_{i+1} , and every bounded point gets up to two new dependent variables x_j, x_{j+1}) so the hypotheses and thesis are rewritten as polynomials $h_1, \ldots, h_n$ and $t$in $\mathbb{Q}[u, x]$.

Eliminating the dependent variables in the ideal $(hypotheses, thesis)$, the vanishing of every element in the elimination ideal $(hypotheses, thesis) \cap \mathbb{Q}[u]$ is a necessary condition for the statement to hold.

Locus Finding: Implementation

A *Sage* worksheet integrating *GeoGebra*



Implementation (cont.)

Two different tasks are performed over GeoGebra constructions:

- the computation of the equation of a geometric locus in the case of a locus construction;

  `LocusEquation( <Locus Point>, <Moving Point> )`

- the study of the truth of a general geometric statement included in the GeoGebra construction as a Boolean variable.

Both tasks are implemented using algebraic automatic deduction techniques based on Gröbner bases computations.

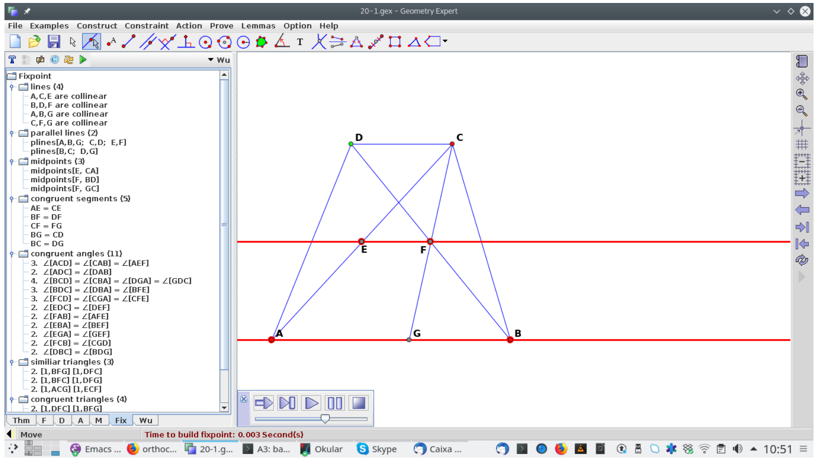
The algorithm, based on a recent work on the Gröbner cover of parametric systems, identifies degenerate components and extraneous adherence points in loci, both natural byproducts of general polynomial algebraic methods [BA12].

Automated Finding of Theorems

Deductive Database Approach. Forward chaining till reaching a fixed point.

> *An interesting application is to discover 'new' facts about a given geometric configuration.*
> *Our experiments show that our program can discover most of the well-known results and often some unexpected ones.*
> *Shang-Ching Chou, Xiao-Shan Gao, and Jing-Zhong Zhang. A deductive database approach to automated geometry theorem proving and discovering.*

JGEX: Automated Finding of Theorems



JGEX - example Manual, 20-1

Automated Geometer

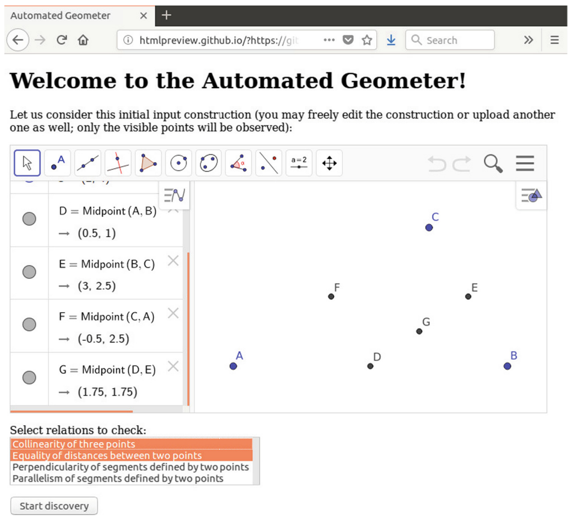
The Automated Geometer, AG, (also meaning Amateur Geometer) intends to be a GeoGebra module where pure automatic discovery is performed.

It includes a generator of further geometric elements from those of a given construction, and a set of rules for producing conjectures on the whole set of elements.

But the ultimate AG aim is not just performing a systematic exploration of the space of possible conjectures, but mimicking human thought when doing elementary geometry.

Francisco Botana, Zoltan Kovacs, and Tomas Recio. Towards an automated geometer. AISC 2018, LNCS 11110, Springer, 2018.

Automated Geometer / Amateur Geometer



http://prover-test.geogebra.org/~kovzol/ag/automated-geometer.html

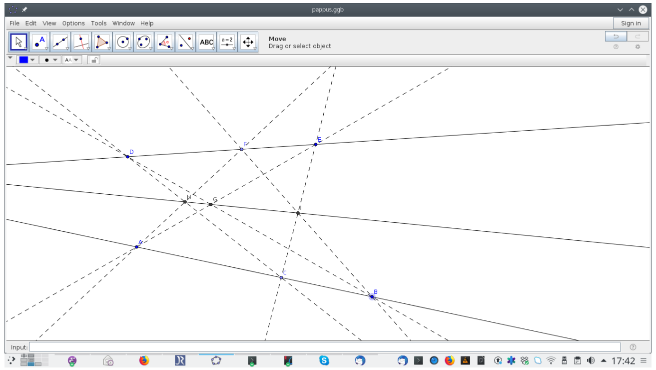
Geometric Tools

Geometric tools: Dynamic Geometry Software (DGS) & Geometry Automated Theorem Provers (GATP) & Computer Algebra Systems (CAS) & Repositories of Geometric Knowledge (RGK) & eLearning in Geometry.

- DGS: Cabri Geometry; C.a.R.; Cinderella; GCLC; GeoGebra; The Geometer's Sketchpad;
  JGEX [Gro11, CGY04, Hoh02, Jac01, Jan06, LS90, RGK99]; ...
- GATP; GCLC; OpenGeoProver; JGEX; GeoProof; ...
  - verification of the soundness of a geometric construction [JQ07].
  - reason about a given DGS construction [CGZ96a, JQ06, Nar07a, QJ06b].
  - human-readable proofs [JNQ12, QJ06a].
- RGK [QJ07, Qua11].
- eLearning [ABY86, HLY86, QJ06b, SQ08, QSM18, SQMC18]

Dynamic Geometry Software

DGS are computer environments which allow one to create and then manipulate geometric constructions, primarily in plane geometry.



Geometry Automated Theorem Provers: GCLC

Proving geometrical theorems by computer programs.

```
*** Ceva s theorem
point A 80 10
point B 50 80
point C 100 80
point P 75 65
line a B C
line b A C
line c A B
line pa P A
line pb P B
line pc P C
*** constructed point
intersec D a pa
intersec E b pb
intersec F c pc
*** conjecture
prove {equal{mult{mult{sratio A F F B}{sratio B D D C}}{sratio C E E A}}1}
```


Geometry Automated Theorem Provers: GCLC

```
A5exemplos: bash — Konsole
Ficheiro  Editar  Ver  Favoritos  Configuração  Ajuda
pedro@nomada:A5exemplos$ gclc ceva.gcl

GCLC 2015 (GC language (R) -> LaTeX  Converter)
Copyright (c) 1996-2015 by Predrag Janicic, University of Belgrade.
Licence Creative Commons CC BY-ND.

Input file: ceva.gcl
Output file: ceva.pic

Starting point number: 1

The theorem prover based on the area method used.
Number of elimination proof steps:      3
Number of geometric proof steps:        6
Number of algebraic proof steps:       23
Total number of proof steps:           32

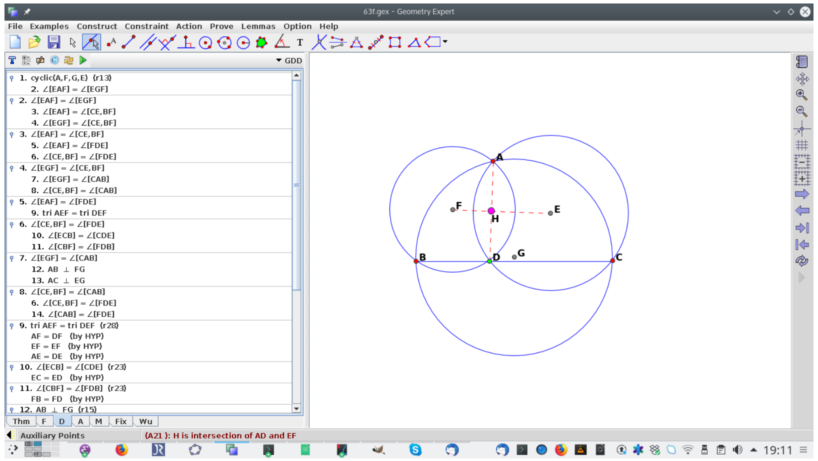
Time spent by the prover: 0.001 seconds
The conjecture successfully proved.
The prover output is written in the file ceva_proof.tex.

File 'ceva.gcl' successfully processed.
Ending point number: 151

Transcript written on gclc.log.

pedro@nomada:A5exemplos$
```


Geometry Automated Theorem Provers: JGEX



JGEX - GDD-FULL 63f

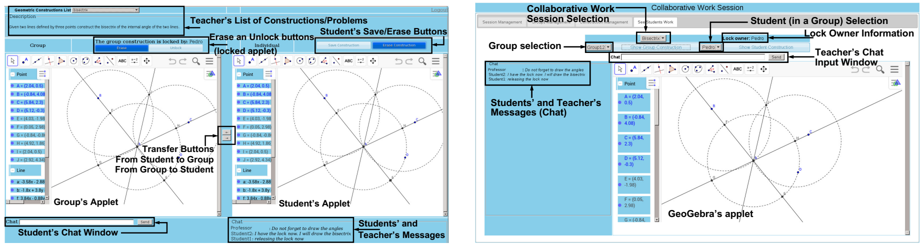
Integration: DGSs & GATPs

- GCLC/WinGCLC - A DGS tool that integrates three GATPs: Area Method, Wu's Method and Gröbner Bases Method [JQ06, Jan06].
- JGEX - is a software which combines a DGS and some GATPs (full angle, Wu's Method, Deductive Databases for the full angle) [YCG10a, YCG10b, CGY04].
- GeoProof - DGS tool that integrates three GATPs Area Method, Wu's Method and Gröbner Bases Method [Nar07a].
- GeoGebra - DGS + CAS + GATPs [ABK+16, BHJ+15, Kov15].
- Theorema Project - Theorema is a project that aims at supporting the entire process of mathematical theory exploration within one coherent logic and software system [BCJ+06]. Implementation of the Area Method[Rob02, Rob07].

Others: The Geometry Tutor, Mentoniezh, Defi, Chypre, Cabri-Euclide, Geometrix, Baghera, MMP-Geometer, Geometry Explorer, Cinderella.

Integration/eLearning (DGSs & GATPs & RGPs)

WebGeometryLab: A Web environment incorporating a DGS (GATPs) and a repository of geometric problems, that can be used in a synchronous and asynchronous fashion and with some adaptive and collaborative features. [QSM18, SQ08, SQMC18].



Others: Tabulae [MSB05]; GeoThink [MSM08]; Advanced Geometry Tutor [MV05]; AgentGeom [CFPR07]; geogebraTUTOR [RFHG07].

Integration Issues

Integrate a mosaic of tools into a coherent system.

- Intergeo Project [SHK+10];
- Deducation STREP Proposal [WSA+12];
- Road to an Intelligent Geometry Book, COST Proposal, OC-2019-1-XXXX.

Intergeo & I2GATP

The I2GATP format is an extension of the I2G (Intergeo) common format aimed to support conjectures and proofs produced by DGSs/GATPs.

XSD files contain the specification of the format:

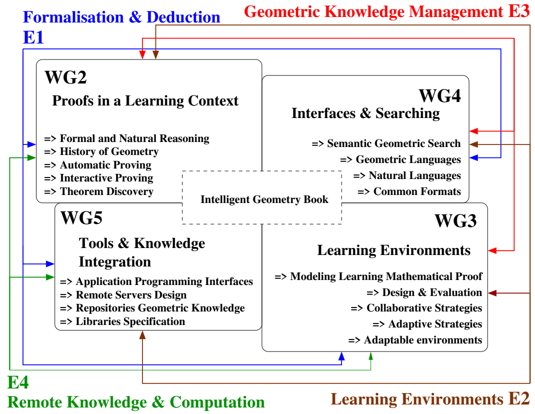
- `information.xsd` with the meta-information about a given geometric problem;
- `intergeo.xsd` no more than the XSD for the I2G format;
- `conjecture.xsd` with the specification of the conjectures;
- `proofInfo.xsd` with the meta-information about the proof(s).

All the XML files containing the information about a geometric problem and also other auxiliary files, are packaged in the I2GATP container, an extension of the I2G container.

A library of programs support the I2GATP format.

*i*GeometryBook

The "Road to an Intelligent Geometry Book" (COST) Action is dedicated to the study of how current developing methodologies and technologies of knowledge representation, management, and discovery in mathematics, can be incorporated effectively into the learning environments of the future.



Repositories of Geometric Problems

GeoThms: a Web-based framework for exploring geometrical knowledge that integrates Dynamic Geometry Software (DGS), Automatic Theorem Provers (ATP), and a repository of geometrical constructions, figures and proofs. [JQ06, QJ07].

TGTP: a Web-based library of problems in geometry to support the testing and evaluation of geometric automated theorem proving (GATP) systems [Qua11].

Sets of Examples and Comunities: Intergeo; GeoGebra; Geometriagon; examples in the DGSs/GATPs.)

TGTP

A comprehensive and easily accessible, library of GATP test problems [Qua11].

- Web-based, easily available to the research community. Easy to use.
- Tries to cover the different forms of automated proving in geometry, e.g. synthetic proofs and algebraic proofs.
- provides a mechanism for adding new problems.
- (...)

It is independent of any particular GATP system $\mapsto$ the I2GATP common format [QH12].

Proofs/Readble Proofs/Visual Proofs

Readable Proofs

- What is a readable proofs [QSGB19]?
- Can GATPs produce readable proofs [JNQ12]?

Visual Reasoning

- Proofs with a visual counterpart [QS19].
- Proofs done by "visual means" [YCG10a, YCG10b]

Readability of a Proof

- According to [CGZ94, p.442] a formal proof, done using the area method, is considered readable if one of the following conditions holds:
  - the maximal term in the proof is less than or equal to 5;
  - the number of deduction steps of the proof is less than or equal to 10;
  - the maximal term in the proof is less than or equal to 10 and the deduction step is less than or equal to 20.
- The de Bruijn factor [deB94, Wie00], the quotient of the *size* of corresponding informal proof and the *size* of the formal proof, could also be used as a measure of readability. Using this quotient a proof can be considered readable if the value is less than or equal to 2 (the formal proof is at most twice as larger then a given informal proof).

GATP, Readable Proofs: GCLC Area Method

(1) $\left(\left(\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1,$ by the statement

(2) $\left(\left(\left(-1 \cdot \frac{\overrightarrow{AF}}{\overrightarrow{BF}}\right) \cdot \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\right) \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right) = 1,$ by geometric simplifications

(3) $\left(-1 \cdot \left(\frac{\overrightarrow{AF}}{\overrightarrow{BF}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1,$ by algebraic simplifications

(4) $\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}}\right)\right)\right) = 1,$ by Lemma 8 (point F eliminated)

(5) $\left(-1 \cdot \left(\frac{S_{APC}}{S_{BPC}} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \left(-1 \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)\right)\right) = 1,$ by geometric simplifications

(6) $\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{AE}}\right)\right)}{S_{BPC}} = 1,$ by algebraic simplifications

(7) $\frac{\left(S_{APC} \cdot \left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{S_{BPC}} = 1,$ by Lemma 8 (point E eliminated)

(8) $\frac{\left(S_{APC} \cdot \left(\left(-1 \cdot \frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right) \cdot \frac{S_{CPB}}{S_{APB}}\right)\right)}{(-1 \cdot S_{CPB})} = 1,$ by geometric simplifications

(9) $\frac{\left(S_{APC} \cdot \frac{\overrightarrow{BD}}{\overrightarrow{CD}}\right)}{S_{APB}} = 1,$ by algebraic simplifications

(10) $\frac{\left(S_{APC} \cdot \frac{S_{BPA}}{S_{CPA}}\right)}{S_{APB}} = 1,$ by Lemma 8 (point D eliminated)

(11) $\frac{\left(S_{APC} \cdot \frac{S_{BPA}}{(-1 \cdot S_{APC})}\right)}{(-1 \cdot S_{BPA})} = 1,$ by geometric simplifications

(12) $1 = 1,$ by algebraic simplifications

GATP, Readable Proofs: Coherent Logic

Example: Proof Generated by ArgoCLP

Let us prove that $p = r$ by reductio ad absurdum.

1. Assume that $p \neq r$.
   2. It holds that the point A is incident to the line q or the point A is not incident to the line q (by axiom of excluded middle).
      3. Assume that the point A is incident to the line q.
         4. From the facts that $p \neq q$, and the point A is incident to the line p, and the point A is incident to the line q, it holds that the lines p and q intersect (by axiom ax_D5).
         5. From the facts that the lines p and q intersect, and the lines p and q do not intersect we get a contradiction.

         Contradiction.
      6. Assume that the point A is not incident to the line q.
         7. From the facts that the lines p and q do not intersect, it holds that the lines q and p do not intersect (by axiom ax_nint_l_l_21).
         8. From the facts that the point A is not incident to the line q, and the point A is incident to the plane α, and the line q is incident to the plane α, and the point A is incident to the line p, and the line p is incident to the plane α, and the lines q and p do not intersect, and the point A is incident to the line r, and the line r is incident to the plane α, and the lines q and r do not intersect, it holds that $p = r$ (by axiom ax_E2).
         9. From the facts that $p = r$, and $p \neq r$ we get a contradiction.

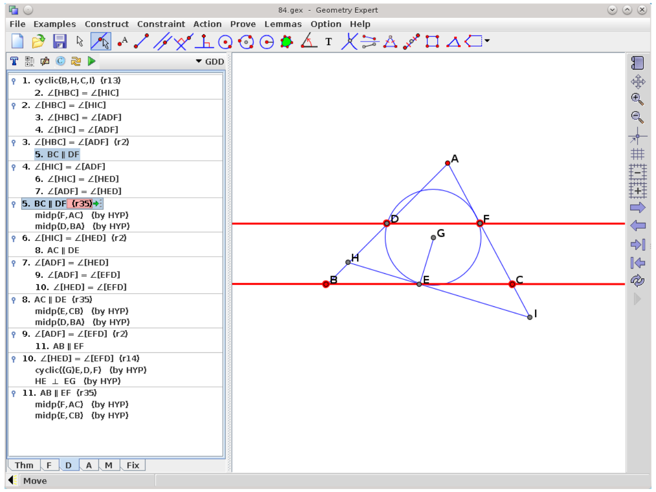
         Contradiction.

Therefore, it holds that $p = r$.

This proves the conjecture.

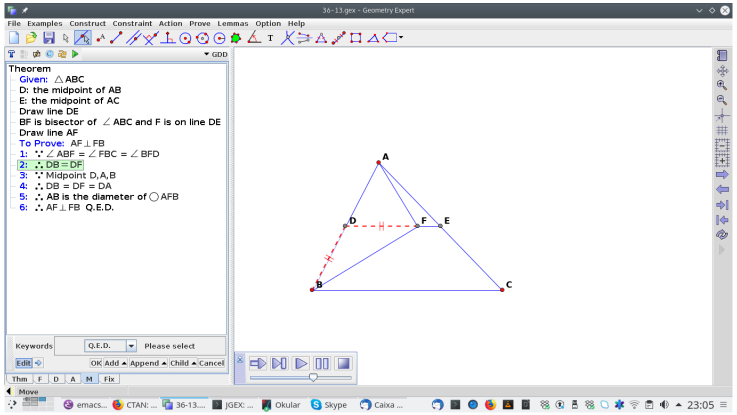
Sana Stojanović, Predrag Janičić Faculty of Mathematics Univ Automated Generation of Formal and Readable Proofs of Math

GATP, Proofs With Visual Support



JGEX – Example 84, Step 2

GATP, Visual Proofs



JGEX – Example 36-13 & PYTH-cnm14

Geometrography

Geometrography, "alias the art of geometric constructions" was proposed by Émile Lemoine between the late 1800s and the early 1900s [SBQ19, Mac93, Lem02, QSGB19].

Measure the complexity of ruler-and-compass geometric constructions.

Coefficient Simplicity: denoting the number of times any particular operation is performed.

Coefficient Exactitude: each time a drawing instrument is used, two types of error can be introduced in the image, systematic error and accidental errors due to personal operator's actions.

Geometrography

Considering the modifications proposed by Mackay [Mac93], the following ruler-and-compass constructions and the corresponding coefficients can be considered.

To place the edge of the ruler in coincidence with one point . . . . R_1
To place the edge of the ruler in coincidence with two points . . $2R_1$
To draw a straight line . . . . R_2
To put one point of the compasses on a determinate point . . . . . C_1
To put one point of the compasses on two determinate points . $2C_1$
To describe a circle . . . . C_2

For a given construction with $l_1R_1 + l_2R_2 + m_1C_1 + m_2C_2$ steps.

$cs = l_1 + l_2 + m_1 + m_2$, is called the coefficient of simplicity.
$ce = l_1 + m_1$ is called the coefficient of exactitude.

DGSs & Geometrography

Extrapolating (modernising) geometrography to DGS.

Coefficient of simplicity – must be adapted to new tools.
Coefficient of exactitude – loose its meaning (error free manipulations).
Coefficient of freedom – counts the degrees of freedom, gives a value for the dynamism of the construction.

Geometrography in GCLC (commands in the GCL language): a point in the plane (D), two degrees of freedom; a line defined by two points ($2C$); a point in a line D, one degree of freedom; etc.

Geometrography in GeoGebra: similar to GCLC, but using GeoGebra tools.

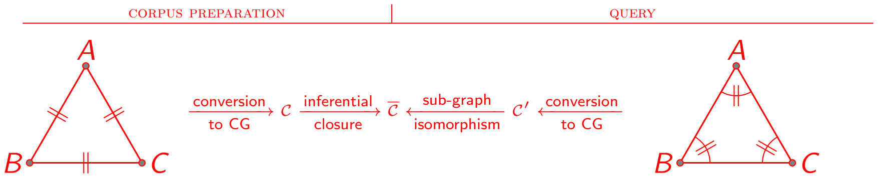
Geometrography as a way to measure the complexity and dynamism of a given construction, being able to compare between different solutions to a same goal

. . . and how about complexity of a proofs?

Geometric Search

When accessing RGK it should be possible to do geometric searches, i.e. we should be able to provide a geometric construction and look for similar constructions [QH12, HQ14, HQ18] .

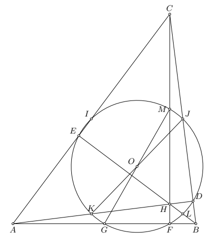
Given (in the RGK) a *triangle with three equal sides*, the query about a *triangle with three equal angles* (which is geometrically equivalent) should be successful.



Taxonomies for Geometry

The usefulness of repositories of geometric knowledge is directly related with the possibility of an easy retrieval of the information a given user is looking for [QSGB19, Qua18].

GEO0316—Nine Points Circle Prove that in any triangle midpoints of each side, feet of each altitude and midpoints of the segments of each altitude from its vertex to the orthocenter lie on a circle [Cho88].



MSC: 51M05, 70G55, 94B27.
GATP Provability: 1/3: *GCLC* area method, "The conjecture is out of scope of the prover"; *GCLC* Wu's method, "The conjecture successfully proved"; *GCLC* Gröbner basis method, "The conjecture not proved - timeout".
Readability [CGZ94]: non-synthetic proof: Wu's Method, 16 pages long proof.
Readability [deB94]: no readable proof: de Bruijn factor: 16/6.
Efficiency (CPU time): 0.17s
CCS: C.A.3; CO.A.1; CO.C.10; CO.D.12.
Construction Complexity: complex (cs=41). $cs = 3 \times D + 3 \times 2C + 3 \times 2C + 3 \times 2C + 2 \times 2C + 2C + 3 \times 2C + 2 \times 2C + 2C + 2C = 41$; $cf = 3 \times 2 = 6$.
Proofs in Education: Verification: good (0.17s); Explanation: no, only an algebraic, long (16 pages) GATP proof, exist.

MSC—Mathematics Subject Classification (`http://msc2010.org/`)
CCS—Common Core Standard (`http://www.corestandards.org/Math/`)

Conference & Journals

CADE (IJCAR/FLoC) International Conference on Automated Deduction `http://www.cadeinc.org/conferences`, every year.

ADG International Conference on Automated Deduction in Geometry, `http://adg2018.cc4cm.org/` (ADG2018), every two years.

ThEdu Theorem Proving Components for Educational Software `http://www.uc.pt/en/congressos/thedu/`, workshop at CADE, every year.

JAR Journal of Automated Reasoning, `https://link.springer.com/journal/10817`, Springer.

LNCS CADE and ADG proceedings, `https://www.springer.com/gp/computer-science/lncs`.

EPTCS ThEdu post-proceedings, `http://www.eptcs.org/`.

What to Do Next?

Integration of Methods integrate the study of logical, combinatorial, algebraic, numeric and graphical algorithms with heuristics, knowledge bases and reasoning mechanisms.

Applications design and implement integrated systems for computer geometry, integrating, in a modular fashion, DGSs, ITPs, GATPs, RGPs, etc. in research and/or educational environments.

Higher Geometry The existing algorithms should be extended and improved, new and advanced algorithms be developed to deal with reasoning in different geometric theories.

Axiom Systems Development of new axiom systems, motivated by machine formalisation. [ADM09]

Formalisation formalising geometric theories and methods.

Discovery Automated discovery of new results.

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