# ERROR ESTIMATES FOR THE DISCONTINUOUS GALERKIN METHODS FOR PARABOLIC EQUATIONS

K. CHRYSAFINOS[†] AND NOEL. J. WALKINGTON[†]

**Abstract.** We analyze the classical discontinuous Galerkin method for a general parabolic equation. Symmetric error estimates for schemes of arbitrary order are presented. The ideas we develop allow us to relax many assumptions freqently required in previous work. For example, we allow different discrete spaces to be used at each time step and do not require the spatial operator to be self adjoint or independent of time. Our error estimates are posed in terms of projections of the exact solution onto the discrete spaces and are valid under the minimal regularity guaranteed by the natural energy estimate. These projections are local and enjoy optimal approximation properties when the solution is sufficiently regular.

## 1. Introduction.

We consider the parabolic PDE of the form,

$$u_t + A(t)u = F(t), \qquad u(0) = u_0. \tag{1.1}$$

The operators act on Hilbert spaces related through the standard pivot construction, $U \hookrightarrow H \simeq H' \hookrightarrow U'$, where each embedding is continuous and dense. Then, $A(.) : U \to U'$ is a linear map and $F(.) \in U'$. Our goal is to analyze the classical discontinuous Galerkin (DG) scheme and derive fully-discrete error estimates under mimimal regularity assumptions. The class of DG schemes we consider are classical in the sense that the discrete solutions may be discontinuous in time but are conforming in space, i.e. are in (a subspace of) $U$ at each time.

Our techniques also apply to the more general implicit evolution equation [22, 23]

$$(M(t)u)_t + A(t)u = F(t), \qquad u(0) = u_0, \tag{1.2}$$

where $M(.) : H \to H$ is a self adjoint possitive definite operator. The extension of our analysis to this equation will be taken up seperatly. The analysis below addresses the following issues which have not yet been adequately considered in the literature.

- The operator $A(.)$ may depend upon time and is not required to be self adjoint. To date the sharpest estimates for DG approximations exploit classical spectral theory for self adjoint positive definite operators, so require $A$ to be such an operator and to be independent of time. When $A(.)$ is not self adjoint, multiplying (1.1) by $u_t$ does not give an estimate for the time derivative.
- The subspaces of $U$ used for the DG approximations may be different on each time interval $(t^{n-1}, t^n]$. This adds a significant complication to the analysis

[*]Mathematics subject classification: 65M60

[†]Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213. Supported in part by National Science Foundation Grants DMS–0208586, and ITR 0086093. This work was also supported by the NSF through the Center for Nonlinear Analysis.

which is present even when $A = 0$. Indeed, the first step in our analysis is to consider the DG scheme for ordinary differential equations in the Hilbert space $H$.

Different subspaces are an essential ingredient of adaptive strategies used in conjunction with a-posteriori error estimates to give guaranteed error bounds. Retriangulation is also necessary for many algorithms based upon a Lagrangian coordinate system; below we present such an example.

- DG approximations of equations of the form (1.2) have not been considered in the past. Below we show that equations of this form arise when Lagrangian schemes are constructed for the convection diffusion equation [6, 7, 8].
- The operator $A(.)$ is not required to be strictly coercive; we only require an assumption of the form $\langle A(.)u, u\rangle \geq c|u|_U^2 - C\|u\|_H^2$. Here $|.|_U$ is a semi-norm such that $\|.\|_U^2 = |.|_U^2 + |.|_H^2$. This causes significant problems in the analysis of DG schems since the classical Gronwall argument, used for the continuous problem, fails in the discrete setting. This failure is due to the elemenary observation that functions of the form $\chi_{[0,\hat{t})}u$ are not polynomial in time unless $\hat{t}$ is a partition point, so are not available as test functions in the discrete setting.* In the past this problem has been circumvented by bounding temporal derivatives of the solution [5, 24] so that the solution between the partition points can be controlled by the values at these points. This line of argument fails for solutions having minimal regularity. Below we circumvent these issues by constructing polynomial approximations to the characteristic functions $\chi_{[0,\hat{t})}$.

As stated above, our analysis does not require any regularity above and beyound the natural bounds that follow from the usual energy estimate. This is essential for control problems where solutions of the dual problem typically will not enjoy any additional regularity. Our estimates show that the error can be bounded by the "local truncation error" of the ordinary differental equation obtained by setting $A = 0$. These local truncation errors can also be viewed as the approximation error of local projections of the solution onto the discrete subspaces. In our analysis we are careful to keep track of how the various constants depend upon the coercivity constant on $A(.)$. This is important for the analysis of problems like the convection diffusion equation where the coercivity constant is small.

We present and example of equations which can be analyzed within the general framework developed here but fall outside of the theory developed, for example, in Thomée's text [24].

**Convection Diffusion Equation:** The classical convection diffusion equation is

$$\bar{u}_t + \mathbf{V}.\nabla\bar{u} - \epsilon\Delta\bar{u} = 0,$$

and the problems that arise when $\epsilon$ is small are notorious. To address these problem this equation is sometimes considered in a Lagrangian variable. Specifically, let $\tilde{\mathbf{V}}$ be a

*Here $\chi_{[0,\hat{t})}$ is the characteristic function equal to 1 on $[0, \hat{t})$ and zero otherwise.

(numerical) approximation of $\mathbf{V}$ and let $x = \chi(t, X)$ be the change of variables defined by the flow map associated with $\tilde{\mathbf{V}}$, i.e.

$$\dot{x}(t, X) = \tilde{\mathbf{V}}(t, x(t, X)), \qquad x(0, X) = X.$$

If $u(t, X) = \bar{u}(t, x(t, X))$ then

$$u_t + (\mathbf{V} - \tilde{\mathbf{V}}).(F^{-T}\nabla_X u) - \epsilon(1/J)div_X\big(JF^{-1}F^{-T}\nabla_X u\big) = 0,$$

where $F_{ij} = \partial x_i/\partial X_j$ is the Jacobian of the mapping and $J = det(F)$. The natural weak problem for this equation is

$$\int_\Omega \Big(\big(u_t + (\mathbf{V} - \tilde{\mathbf{V}}).(F^{-T}\nabla_X u)\big)v + \epsilon(F^{-T}\nabla_X u).(F^{-T}\nabla_X v)\Big)J = 0.$$

Using the properties of determinants we find

$$u_t J = (Ju)_t - J'u = (Ju)_t - J\, div(\tilde{\mathbf{V}})u.$$

If $div(\tilde{\mathbf{V}}) = 0$ then $J$ is constant the transformed problem takes the form of (1.1); otherwise, it takes the form of (1.2) with $M(.)u = Ju$, and

$$A(.)u = -div(\tilde{\mathbf{V}})uJ + (\mathbf{V} - \tilde{\mathbf{V}}).(F^{-T}\nabla_X u)J - \epsilon\, div_X\big(JF^{-1}F^{-T}\nabla_X u\big).$$

This statement of the problem generalizes the idea behind the "characteristic Galerkin" scheme introduced by Douglas and Russel in [6] and DuPont in [7].

This change of variables reduces the effective Peclet number, $|\mathbf{V} - \tilde{\mathbf{V}}|/\epsilon$, to order $\mathcal{O}(1)$ if $\tilde{\mathbf{V}}$ is a sufficiently accurate approximation of $\mathbf{V}$. This will eliminate many of the numerical difficulties encountered by algorithms based upon the classical statement; however, other problems arise. While the Jacobian of the transformation satisfies $F(0, X) = I$, its condition number grows exponentially if $\tilde{\mathbf{V}}$ is anything other than a rigid motion. In the context of a numerical scheme this problem is circumvented by reinitalizing the transformation at each (or every few) time step(s). This reinitialization corresponds to changing the subspace for the numerical solution every (few) time step(s). In essence, a trianglular mesh in the $X$ coordinate system will be a distorted mesh in the $x$-coordinate system, and reinitializing the transform corresponds to projecting the solution onto a (straight sided) triangular mesh in the $x$ coordinates. This gives rise to different subspaces at each time step.

**1.1. Related Results.** The discontinuous Galerkin method was first introduced to model and simulate neutron transport by Lasaint and Raviart in [15]. There is an abundant literature concerning applications of the DG scheme in hyperbolic problems, see e.g. [4, 14, 25] and references within. The DG method for ordinary differential equations was considered by Delfour, Hager and Trochu in [5]. They showed that the DG scheme was super convergent at the partition points (order $2k+2$ for polynomials of degree $k$). The super convergence results (and better rates in the $H$ norm) use a duality argument, and space considerations do not permit us to develop the corresponding estimates in the current setting.

In the context of parabolic equations DG schemes were first analyzed for linear parabolic problems by Jamet in [13] where $\mathcal{O}(k^q)$ results were proved and by Eriksson, Johnson and Thomée in [11] where $\mathcal{O}(k^{2q-1})$ results were proved for "smooth" initial data among others. An excellent exposition of their results and, more generally, the DG method for parabolic equations, can be found in Thomée's book [24]. In [24] nodal and interior estimates are presented in various norms. One may also consult [18] for the analysis of a related formulation based on the backward Euler scheme. The relation between the DG scheme and adaptive techniques was studied in [9] and [10]. Finally, some results concerning the analysis of parabolic integro-differential equations by discontinuous Galerkin method are presented in [16] (see also references therein).

In [8] DuPont and Liu introduce the concept of "symmetric error estimates" for parabolic problems. They define such an error estimate to be one of the form

$$|||u - u_h||| \leq C \inf_{w_h \in \mathcal{U}_h} |||u - w_h|||,$$

where $u$ and $u_h$ are the exact and approximate solutions respectively, $|||.|||$ is an appropriate norm, and $\mathcal{U}_h$ is the discrete subspace in which approximation solutions are sought. While estimates of this form are standard for elliptic problems, this is not the case for evolution problems. For example, error estimates for evolution problems approximated by the implicit Euler scheme frequently involve terms of the form $\|u_{tt}\|_{L^2(\Omega)}$. Symmetric error estimates are useful for problems where the solution $u$ may not be very regular, such as control problems, and are used to develop a-posteriori error estimates for adaptive schemes. Symmetric error estimates for moving mesh finite element methods were studied in [8, 17] (see also references within). Mesh modification techniques for finite elements have also been introduced in [19] and [20]. For some earlier work on convection-dominated problems based on the methods of characteristics and mesh modification one may consult [6] and [7] respectively.

An alternative to the symmetric error estimates are estimates of the form

$$|||u - u_h||| \leq C\, |||\, u - \mathbb{P}_h u|||, \tag{1.3}$$

where $\mathbb{P}_h : \mathcal{U} \to \mathcal{U}_h$ is a projection which exhibits optimal interpolation properties if $u$ is sufficiently smooth. Estimates of this form enjoy the same advantages of those proposed by DuPont and Liu. Below we construct an estimate of the form (1.3) for parabolic equations of the form (1.1); where the projection $\mathbb{P}_h u$ is the numerical approximation of an ODE, so is not local. However,

$$|||u - \mathbb{P}_h u||| \leq |||u - \mathbb{P}_h^{loc} u\, ||| + |||\, \mathbb{P}_h u - \mathbb{P}_h^{loc} u|||,$$

where $\mathbb{P}_h^{loc}$ is a local projection, so the first term can be estimated using classical interpolation theory. The second term $|||\mathbb{P}_h u - \mathbb{P}_h^{loc} u|||$ vanishes if the same subspace of $U$ is used in each partition $(t^{n-1}, t^n)$; otherwise, it depends solely upon the jump in the interpolant of the exact solution at the partition points $\{t^n\}_{n=0}^N$. The size of the constant $C$ in (1.3), and its dependence on various constants, play an important role; below we are careful to state the dependence of the constant upon the various coercivity constants and bounds assumed for the operator $A$.

Error estimates for Lagrange-Galerkin approximations of convection dominated problems for divergence-free velocity fields vanishing on the boundary are presented in [3]. Issues related to the stability of Lagrange-Galerkin approximations are also discussed in [21]. Recently there has been a lot of work on the development and analysis of discontinuous Galerkin methods for elliptic problems. A comprehensive survey and comparison of this work can be found in [2] which contains many references related to this approach.

**1.2. Outline.** In Section 2, we formulate and analyze the DG scheme for the ordinary differential equation corresponding to $A(.) \equiv 0$. In this section we focus on the difficulties that arise when different subspaces of $U$ may be used at every time step. Error estimates are first derived at times corresponding to the partition point. Additional arguments using "discrete characteristic functions" are developed to estimate the error and at times inbetween the partition points. The arguments we use appear to be new.

A priori estimates for the DG approximations of (1.1) are developed in Section 3. Estimates are derived in the natural norms associated with the parabolic problem; by "natural" we mean norms that arise in the natural energy estimates obtained by multiplying (1.1) by $u$. The results of Section 2 are used in an essential fashion. Indeed, the difficulties associated with different subspaces of $U$ at each time step are circumvented by comparing the discrete solution of the parabolic equation with and appropriate solution of an ODE. By using the "discrete characteristic functions" developed in Section 2 we can avoid the self adjoint assumptions typically imposed upon $A(.)$.

When the same discrete subspace of $U$ is used for each time step our techniques generalize, and to some extend simplify, the classical analysis. The reader interested in this case only needs to read Sections 2.3 and 2.5 on the construction of discrete characteristic functions, and Definition 2.2 for $\mathbb{P}_h^{loc}$ from Section 2 before proceeding to Section 3. Remark 4 in Section 3 amplifies upon this.

**1.3. Notaton.** Throughtout we assume that the evolution of the solution to (1.1) takes place in a Hilbert space $H$ and the operators $A(.)$ are defined on another Hilbert space $U$ with $U \hookrightarrow H \simeq H' \hookrightarrow U'$, where each of the embeddings are dense and continuous. The inner product on $H$ is denoted by $(.,.)$ and the induced duality pairing between $U$ and $U'$ will be dentoted by $\langle .,. \rangle$. The norm on $H$ is often denoted by $|.| \equiv \|.\|_H$, and we assume that the norm on $U$ can be written as $\|.\|_U^2 = |.|_U^2 + \|.\|_H^2$ where $|.|_U$ is a semi-norm on $U$ (the "principle" part) and is often denoted by $\|.\|$; $\|.\|_U^2 = \|.\|^2 + |.|^2$. Standard notation of the form $L^2[0,T;U]$, $H^1[0,T;U']$ etc. is used to indicate the temporal regularity of functions with values in $U$, $U'$ etc.

Approximations of (1.1) will be constructed on a partition $0 = t^0 < t^1 < \ldots < t^N = T$ of $[0,T]$. On each interval of the form $(t^{n-1}, t^n]$ a subspace $U_h^n$ of $U$ is specified, and the approximate solutions will lie in the space

$$\mathcal{U}_h = \{u_h \in L^2[0,T;U] \mid u_h|_{(t^{n-1},t^n]} \in \mathcal{P}_k(t^{n-1}, t^n; U_h^n)\}.$$

Here $\mathcal{P}_k(t^{n-1}, t^n; U_h^n)$ is the space of polynomials of degree $k$ or less having values in $U_h^n$. Notice that, by convention, we have chosen functions in $\mathcal{U}_h$ to be left continuous with right limits. We will write $u^n$ for $u_h(t^n) = u_h(t^n_-)$, and let $u^n_+$ dentote $u(t^n_+)$. This notation will is also used with functions like the error $e = u - u_h$. We always assume the exact solution, $u$, is in $C[0,T;H]$ so that the jump in the error at $t^n$, dentoted by $[e^n]$ is equal to $[u^n] = u^n_+ - u^n$.

## 2. DG scheme for an ODE.

**2.1. Background.** In this section we address the issues that arise when different discrete subspaces are used for each step of a time of the DG scheme. It suffices to consider such DG schemes in the context of an ODE in a Hilbert space since the additional terms appearing in the error estimates are the same for both the ODE and parabolic PDE case. Also, the solution of the ODE will be used to obtain error estimates for the parabolic PDE under minimal regularity assumptions.

We consider the problem of recovering a function $u \in C[0,T;H] \cap H^1[0,T;U']$ given the initial value $u(0)$ and its derivative $f = u_t$. Specifically, we consider the DG finite element approximations for the initial value problem

$$u_t = f, \qquad u(0) = u_0. \tag{2.1}$$

Recall that $H$ and $U$ are related through a pivot space construction, $U \hookrightarrow H \simeq H' \hookrightarrow U'$. In this situation there exists a unique $u \in C[0,T;H] \cap H^1[0,T;U']$ which is the solution of the weak problem,

$$(u(T), v(T)) - \int_0^T \langle u, v_t \rangle = (u_0, v(0)) + \int_0^T \langle f, v \rangle \qquad \forall\, v \in C[0,T;H] \cap H^1[0,T;U']. \tag{2.2}$$

Recall that we write $|\cdot|$ for the norm $|\cdot|_H$ and write the inner product in $H$ as $(.,.)$. We also write $\|u\|^2_U = |u|^2 + \|u\|^2$ where $\|.\|$ is a semi-norm on $U$ (the "principle" part). The $(U', U)$ duality pairing is denoted by $\langle \cdot, \cdot \rangle$

To approximate the solution of (2.2) we introduce a partition $0 = t^0 < t^1 < ... < t^N = T$ of $[0,T]$ and a collection $\{U^n_h\}_{n=0}^N$ of subspaces of $U$. The DG method constructs an approximate solution

$$u_h \in \mathcal{U}_h \equiv \{u \in L^2[0,T;H] \mid u|_{(t^{n-1},t^n]} \in \mathcal{P}_k(t^{n-1}, t^n; U^n_h)\}$$

such that

$$(u^n, v^n) - \int_{t^{n-1}}^{t^n} (u_h, v_{ht}) - (u^{n-1}, v^{n-1}_+) = \int_{t^{n-1}}^{t^n} \langle f, v_h \rangle, \tag{2.3}$$

for all $v_h \in \mathcal{U}_h$ and each $n = 1, 2, \ldots N$. Recall that $u^n \equiv u_h(t^n) = u_h(t^n_-)$ and we employ the standard notation $u^n_+, u^n_-$ for the traces from above and below respectively. Integration by parts gives the following alternative form of (2.3)

$$\int_{t^{n-1}}^{t^n} (u_{ht}, v_h) + (u^{n-1}_+ - u^{n-1}, v^{n-1}_+) = \int_{t^{n-1}}^{t^n} \langle f, v_h \rangle. \tag{2.4}$$

**2.2. Error Estimate at Partition Points.** In this elementary context it is possible to explicitly write an expression for the error at each nodal (partition) point $t^n$, see equation (2.5) below. A simple consequence of this formula is the following theorem which provides a decomposition of the error into the errors due to the changing of the spaces and the initial projection error.

THEOREM 2.1. *Let $u_h \in \mathcal{U}_h$ be the approximate solution of (2.1) computed using the discontinuous Galerkin shceme (2.3)and let $P_n : H \to U_h^n$ denote the projection operator in $H$, and write $\hat{e}^n = P_n u(t^n) - u^n$. Then*

$$\hat{e}^n = \sum_{i=1}^{n} \Big( \prod_{j=i}^{n} P_j \Big)(I - P_{i-1})u(t^{i-1}) + \big(P_n \circ P_{n-1} \circ ... \circ P_1\big)\hat{e}^0. \tag{2.5}$$

*In particular,*

$$|\hat{e}^n| \leq \sum_{i=1}^{n} |P_i(I - P_{i-1})u(t^{i-1})| + |\hat{e}^0|. \tag{2.6}$$

REMARK 1. *(1) Note that $P_i(I - P_{i-1}) = 0$ when $U_h^i \subset U_h^{i-1}$, so $|\hat{e}^n| \leq |\hat{e}^0|$ when the same discrete subspace is used at each time.*

*(2) If $e^n = u(t^n) - u^n$ is the total error at time $t^n$ then*

$$|e^n| \leq |(I - P_n)u(t^n)| + |\hat{e}^n|.$$

*When the same space is used at each step the first term becomes $(I - P_0)u(t^n)$ which is useful only if $u(t^n)$ can be well approximated in $U_h^0$ at all times. When this is not the case, an ideal strategy would chose the spaces $\{U_h^n\}$ to "track" the solution so that both $(I - P_n)u(t^n)$ and the jump terms in equation (2.6) are small.*

*Proof.* Let $e = u - u_h$ be the total error and note that the Galerkin orthogonality gives,

$$(e^n, v^n) - \int_{t^{n-1}}^{t^n} (e, v_{ht}) - (e^{n-1}, v_+^{n-1}) = 0. \tag{2.7}$$

If $v_h(t) \equiv v^n$ is independent of time, then the middle term vanishes to give

$$(e^n, v^n) = (e^{n-1}, v^n), \qquad v^n \in U_h^n.$$

It follows that

$$\hat{e}^n = P_n e^{n-1} \tag{2.8}$$

so that

$$\begin{aligned} \hat{e}^n &= P_n e^{n-1} = P_n(u(t^{n-1}) - u^{n-1}) \\ &= P_n\Big(u(t^{n-1}) \pm P_{n-1}u(t^{n-1}) - u^{n-1}\Big) \\ &= P_n(I - P_{n-1})u(t^{n-1}) + P_n\hat{e}^{n-1}. \end{aligned}$$

Using the above relation inductively yields (2.5). □

**2.3. Discrete Characteristic Functions.** To compute the error at arbitrary times $t \in [t^{n-1}, t^n)$ we would like to substitute $v_h = \chi_{[t^{n-1},t)} u_h$ into equation (2.4) where $\chi_{[t^{n-1},t)}$ is the characteristic function on $[t^{n-1}, t)$. Clearly this function is not in $\mathcal{U}_h$ so in this section we construct discrete approximations of such characteristic functions to circumvent this problem.

The construction of the discrete characteristic functions is invariant under translation so it is convinient to work on the interval $[0, \tau)$ with $\tau = t^n - t^{n-1}$. We begin by considering polynomials $p \in \mathcal{P}_k(0, \tau)$. A discrete approximation of $\chi_{[0,t)} p$ is the polynomial $\tilde{p} \in \{\tilde{p} \in \mathcal{P}_q(0, \tau) | \tilde{p}(0) = p(0)\}$ satisfying

$$\int_0^\tau \tilde{p} q = \int_0^t p q \qquad \forall\, q \in \mathcal{P}_{k-1}(0, \tau).$$

The above construction is motivated by the fact that we may put $q = p'$ to obtain $\int_0^\tau p' \tilde{p} = \int_0^t p p' = p^2(t) - p^2(0)$.

We next extend this elementary construction to approximate functions of the form $\chi_{[0,t)} v_h$ for $v_h \in \mathcal{P}_k(0, \tau; U_h)$ where $U_h$ is any subspace of $H$. If $v_h \in \mathcal{P}_k(0, \tau; U_h)$ we can write $v_h = \sum_{i=0}^k p_i(t) v_i$ where $\{p_i\} \subset \mathcal{P}_k(0, \tau)$ and $\{v_i\} \subset U_h$. If we define $\tilde{v}_h = \sum_{i=0}^k \tilde{p}_i(t) v_i$ it is clear that $\tilde{v}_h \in \mathcal{P}_k(0, \tau; U_h)$ satisfies

$$\tilde{v}_h(0) = v_h(0), \text{ and } \int_0^\tau (\tilde{v}_h, w_h) = \int_0^t (v_h, w_h) \qquad \forall\, w_h \in \mathcal{P}_{k-1}(0, \tau; U_h). \tag{2.9}$$

In the ODE setting we could have directly defined $\tilde{v}_h$ directly from equation (2.9) instead of the two stage construction given here. However, for parabolic equations it is useful to observe that $v_h$ is independent of the choice of the space $U_h$.

**2.4. Error Estimates at Arbitrary Times.** To estimate the error at an arbitrary time $t \in [t^{n-1}, t^n)$ we use the projection operator $\mathbb{P}_n^{loc}$ introduced in [11].

DEFINITION 2.2. *(1) The projection* $\mathbb{P}_n^{loc} : C[t^{n-1}, t^n; H] \to \mathcal{P}_k(t^{n-1}, t^n; U_h^n)$ *satisfies* $(\mathbb{P}_n^{loc} u)^n = P_n u(t^n)$, *and*

$$\int_{t^{n-1}}^{t^n} (u - \mathbb{P}_n^{loc} u, v_h) = 0, \qquad \forall\, v_h \in \mathcal{P}_{k-1}(t^{n-1}, t^n; U_h^n).$$

*Here we have used the convention* $(\mathbb{P}_n^{loc} u)^n \equiv (\mathbb{P}_n^{loc} u)(t^n)$ *and* $P_n : H \to U_h^n$ *is the projection operator onto* $U_h^n \subset H$.

*(2) The projection* $\mathbb{P}_h^{loc} : C[0, T; H] \to \mathcal{U}_h$ *satisfies*

$$\mathbb{P}_h^{loc} u \in \mathcal{U}_h \textit{ and } (\mathbb{P}_h^{loc} u)|_{(t^{n-1}, t^n]} = \mathbb{P}_n^{loc}(u|_{[t^{n-1}, t^n]}).$$

This projection satisfies the the standard approximation properties, [24, Theorem 12.1], and can be thought of as the one step DG approximation of $u_t = f$ on the interval $(t^{n-1}, t^n]$ with exact inital data $u(t^{n-1})$. To see this, write $u_h = \mathbb{P}_n^{loc} u$ and write test the function as the derivative $v_{ht}$ of a function $v_h \in \mathcal{P}_k(t^{n-1}, t^n; U_h^n)$. Integration by parts then gives

$$(u^n, v^n) - \int_{t^{n-1}}^{t^n} (u_h, v_{ht}) - (u(t^{n-1}), v_+^{n-1}) = \int_{t^{n-1}}^{t^n} \langle u_t, v_h \rangle$$

which is identical in form to (2.3). For the parabolic problem we will use the analogus *global* projection, ($\mathbb{P}_h$ below) which is the DG approximation of $u_t = f$ on all of $[0, T]$. We are now ready to prove the main result of this section which shows that the error estimate of Theorem 2.1, which held at the discrete times, holds for every time.

THEOREM 2.3. *Let $u_h \in \mathcal{U}_h$ be the approximate solution of (2.1) computed using the discontinuous Galerkin shceme (2.3). Let $\hat{e} = \mathbb{P}_h^{loc} u - u_h$ where $\mathbb{P}_h^{loc}$ is the projection defined in Definition 2.2. Then*

$$|\hat{e}(t)| \leq \sum_{i=1}^{n} |P_i(I - P_{i-1})u(t^{i-1})| + |\hat{e}^0|, \qquad t \in (t^{n-1}, t^n],$$

*and*

$$|[\hat{e}^{n-1}]| \leq \sum_{i=1}^{n} |P_i(I - P_{i-1})u(t^{i-1})| + |\hat{e}^0|.$$

*where $[\hat{e}^{n-1}] = \hat{e}_+^{n-1} - \hat{e}^{n-1}$ is the jump in $\hat{e}$ at $t^{n-1}$.*

*More generally, if $(.,.)_V$ is a (semi) inner product on $U_h^n$ then*

$$\|\hat{e}(t)\|_V \leq \|\hat{e}^n\|_V = \|\sum_{i=1}^{n} \Big(\prod_{j=i}^{n} P_j\Big)(I - P_{i-1})u(t^{i-1}) + \big(P_n \circ P_{n-1} \circ ... \circ P_1\big)\hat{e}^0\|_V,$$

*for $t \in (t^{n-1}, t^n]$.*

REMARK 2. *(1) In Theorem 2.1 we used the notation $\hat{e}^n$ to denote $P_n e^n = P_n e(t_-^n)$. This is consistant with the notation for $\hat{e}(t)$ above, since by construction $(\mathbb{P}_n^{loc} e)^n = P_n e^n$.*

*(2) Notice that discreteness plays an essential role in the last inequality. We have not assumed, for example, that $u \in C[0, T; U]$, yet the last inequality gives an expression for estimate the error of $P_n u$ in this norm.*

*Proof.* Recalling the Galerkin orthogonality relation (2.7) and applying the definition of the projection $\mathbb{P}_n^{loc}$ shows that

$$(\hat{e}^n, v^n) - \int_{t^{n-1}}^{t^n} (\hat{e}, v_{ht}) - (e^{n-1}, v_+^{n-1}) = 0,$$

or equivalently,

$$\int_{t^{n-1}}^{t^n} (\hat{e}_t, v_h) + (\hat{e}_+^{n-1} - e^{n-1}, v_+^{n-1}) = 0.$$

Define $z_h \in \mathcal{P}(t^{n-1}, t^n; U_h^n)$ to be the "discrete Laplacian" of $\hat{e}$; that is, for each $t \in [t^{n-1}, t^n]$

$$(z_h(t), w_h) = (\hat{e}(t), w_h)_V, \qquad \text{for all } w_h \in U_h^n.$$

Then set $v_h = \tilde{z} \simeq \chi_{[t^{n-1},t)} z_h$ to be the approximate characteristic function of $z_h$ constructed in Section 2.3. This choice of test function gives

$$(1/2)\Big(\|\hat{e}(t)\|_V^2 - \|\hat{e}_+^{n-1}\|_V^2\Big) + (\hat{e}_+^{n-1} - Pe^{n-1}, \hat{e}_+^{n-1})_V = 0,$$

so that

$$(1/2)\|\hat{e}(t)\|_V^2 + (1/2)\|\hat{e}_+^{n-1}\|_V^2 - (P_n e^{n-1}, \hat{e}_+^{n-1})_V = 0. \qquad (2.10)$$

From equation (2.8) we find $\hat{e}^n = P_n e^{n-1}$, so an application of the Cauchy Schwarz inequality and $ab \leq (1/2)(a^2 + b^2)$ shows $\|\hat{e}(t)\|_V \leq \|\hat{e}^n\|_V$ as claimed.

To establish the bound on the jump term we set $V = H$ in (2.10) and rearrange the terms to get

$$(1/2)|\hat{e}(t)|^2 + (1/2)|\hat{e}_+^{n-1}|^2 - (\hat{e}^{n-1}, \hat{e}_+^{n-1}) = ((I - P_{n-1})u(t^{n-1}), \hat{e}_+^{n-1}).$$

Next let $t \searrow t_+^{n-1}$ and complete the square on the left to obtain

$$(1/2)|\hat{e}_+^{n-1}|^2 + (1/2)|\hat{e}_+^{n-1} - \hat{e}^{n-1}|^2 = (1/2)|\hat{e}^{n-1}|^2 + ((I - P_{n-1})u(t^{n-1}), \hat{e}_+^{n-1}).$$

It follows that

$$\begin{aligned} |\hat{e}_+^{n-1} - \hat{e}^{n-1}|^2 &\leq |\hat{e}^{n-1}|^2 + |P_n(I - P_{n-1})u(t^{n-1})|^2 \\ &\leq \big(|\hat{e}^{n-1}| + |P_n(I - P_{n-1})u(t^{n-1})|\big)^2, \end{aligned}$$

and again we use the estimate established in Theorem 2.1 to complete the proof. □

The following definition and corollary provide a concise synopsis of the results of this section in a form useful for the analysis of the parabolic problem in the next section.

Definition 2.4. *The projection $\mathbb{P}_h : C[0,T;H] \cap H^1[0,T;U'] \to \mathcal{U}_h$ is the discontinuous Galerkin approximation of the function reconstructed from it's derivative and initial data. That is, if $u_h = \mathbb{P}_h u$, then $u_h$ is the solution of (2.3) where $f = u'$.*

The previous theorem can then be interprated as an estimate of the difference between the global projection $\mathbb{P}_h u$ and the local projection $\mathbb{P}_h^{loc} u$.

Corollary 2.5. *Let $u \in C[0,T;H] \cap H^1[0,T;U']$, then*

$$|(\mathbb{P}_h u - \mathbb{P}_h^{loc} u)(t)| \leq \sum_{i=1}^n |P_i(I - P_{i-1})u(t^{i-1})| + |\hat{e}^0|, \qquad t \in (t^{n-1}, t^n],$$

*and*

$$|[(\mathbb{P}_h u - \mathbb{P}_h^{loc} u)^{n-1}]| \leq \sum_{i=1}^n |P_i(I - P_{i-1})u(t^{i-1})| + |\hat{e}^0|,$$

*where $\hat{e}^0 = P_0 u(0) - u^0$. More generally, if $(.,.)_V$ is a (semi) inner product on $U_h^n$ then*

$$\|(\mathbb{P}_h u - \mathbb{P}_h^{loc} u)(t)\|_V \leq \|\sum_{i=1}^n \big(\prod_{j=i}^n P_j\big)(I - P_{i-1})u(t^{i-1}) + \big(P_n \circ P_{n-1} \circ \ldots \circ P_1\big)\hat{e}^0\|_V,$$

*for* $t \in (t^{n-1}, t^n]$.

REMARK 3. *An alternative to using the last expression to estimate the error in other norms is to postulate an inverse inequality. For example, if* $\|v_h\| \le C_{inv}(h)|v_h|$ *for all* $v_h \in \cup_n U_h^n$, *then for* $t \in (t^{n-1}, t^n]$

$$\|(\mathbb{P}_h u - \mathbb{P}_h^{loc} u)(t)\| \le C_{inv}(h)\Big(\sum_{i=1}^{n} |P_i(I - P_{i-1})u(t^{i-1})| + |\hat{e}^0|\Big).$$

*If the solution has sufficent regularity projection errors in the different norms typically satisfy* $C_{inv}(h)|e| \sim \|e\|$.

**2.5. Estimates for Discrete Characteristic Functions.** Our construction of the discrete characteristic functions in Section 2.3 was purely algebraic and, for the ODE, no bounds were required for the mapping $e \mapsto \tilde{e}$. The analysis of the parabolic problem requires the bounds developed next.

In the next two lemmas, the function $p \mapsto \tilde{p}$ will refer to the map constructed in Secton 2.3.

LEMMA 2.6. *The mapping* $p \mapsto \tilde{p}$ *in* $\mathcal{P}_k(0, \tau)$ *is linear, continuous and there exists a constant* $\hat{C}_k$ *depending only upon* $k$ *such that* $\|\tilde{p} - p\|_{L^2(0,\tau)} \le \hat{C}_k \|p\|^2_{L^2(t,\tau)}$. *Moreover,*

$$\|\tilde{p} - \chi_{[0,t)} p\|_{L^2(0,\tau)} \le \|\tilde{p} - p\|_{L^2(0,\tau)} + \|p - \chi_{[0,t)} p\|_{L^2(0,\tau)} \le (1 + \hat{C}_k)\|p\|_{L^2(t,\tau)}$$

*and* $\|\tilde{p}\|_{L^2(0,\tau)} \le (1 + \hat{C}_k)\|p\|_{L^2(0,\tau)}$.

*Proof.* Since $\tilde{p}(0) = p(0)$ we can write $\tilde{p} - p = t\bar{p}$ with $\bar{p} \in \mathcal{P}_{k-1}(0, \tau)$. Then using the definition of $\tilde{p}$,

$$\int_0^\tau t\bar{p}q = \int_0^\tau (\tilde{p} - p)q = -\int_t^\tau pq \qquad \forall\, q \in \mathcal{P}_{k-1}(0, \tau).$$

Setting $q = \bar{p}$ we obtain,

$$c_k\tau \int_0^\tau \bar{p}^2 \le \int_0^\tau t\bar{p}^2 = -\int_t^\tau p\bar{p},$$

where we use the equivalence of norms on $\mathcal{P}_k$, and scale to make $c_k$ independent of $\tau$. It follows that the matrix corresponding to the bilinear form is non-singular, so solutions exist. The Cauchy-Schwarz inequality gives

$$(c_k\tau)^2 \int_0^\tau \bar{p}^2 \le \int_t^\tau p^2,$$

which implies,

$$c_k^2 \int_0^\tau (\tilde{p} - p)^2 = c_k^2 \int_0^\tau t^2\bar{p}^2 \le c^2\tau^2 \int_0^\tau \bar{p}^2 \le \int_t^\tau p^2.$$

□

We next consider the induced mapping $v \mapsto \tilde{v}$ on $\mathcal{P}_k(0,\tau;V)$ where $V$ is any (semi) inner product space. Recall that if $v = \sum_{i=0}^k p_i(t)v_i$ then $\tilde{v} = \sum_{i=0}^k \tilde{p}_i(t)v_i$. Since this construction is purely algebraic it follows that $\tilde{v} \in \mathcal{P}_k(0,\tau;V)$ satisfies

$$\tilde{v}(0) = v(0) \text{ and } \int_0^\tau (\tilde{v}, w)_V = \int_0^t (v, w)_V \quad \forall\, w \in \mathcal{P}_{k-1}(0,\tau;V)$$

LEMMA 2.7. *Let $V$ be a semi-inner product space, then the mapping $\sum_{i=0}^k p_i(t)v_i \mapsto \sum_{i=0}^k \tilde{p}_i(t)v_i$ on $\mathcal{P}_k(0,\tau;V)$ is continuous in $\|\cdot\|_{L^2[0,\tau;V]}$. In particular,*

$$\|\tilde{v}\|_{L^2[0,\tau;V]} \le C_k \|v_h\|_{L^2[0,\tau;V]},$$

*and*

$$\|\tilde{v} - \chi_{[0,t)} v_h\|_{L^2[0,\tau;V]} \le C_k \|v_h\|_{L^2[0,\tau;V]},$$

*where $C_k = (k+1)^{1/2}(1+\hat{C}_k)$.*

*Proof.* Without loss of generality write $v = \sum_{i=0}^k p_i(t)v_i$ where $\{p_i\}$ form an orthonormal basis of $\mathcal{P}_k(0,\tau)$ in $L^2(0,\tau)$, so that $\|v\|^2_{L^2[0,\tau;V]} = \sum_{i=0}^k \|v_i\|_V^2$. The lemma then follows by direct computation:

$$\begin{aligned}
\int_0^\tau \|\tilde{v}\|_V^2 &= \int_{t^{n-1}}^{t^n} \sum_{i,j=0}^k \tilde{p}_i(t)\tilde{p}_j(t)(v_i, v_j)_V \\
&\le \sum_{i,j=0}^k \|\tilde{p}_i\|_{L^2(0,\tau)} \|\tilde{p}_j\|_{L^2(0,\tau)} \|v_i\|_V \|v_j\|_V \\
&\le (1+\hat{C}_k)^2 \sum_{i,j=0}^k \|v_i\|_V \|v_j\|_V \\
&\le (1+\hat{C}_k)^2 (k+1) \left( \sum_{i=0}^k \|v_i\|_V^2 \right) \\
&\le (1+\hat{C}_k)^2 (k+1) \int_0^\tau \|v_h\|_V^2.
\end{aligned}$$

The second estimate follows similarly. □

## 3. DG Scheme for Parabolic PDE's.

**3.1. Formulation of the DG scheme..** We turn our attention on the approximation of (1.1) using the discontinuous Galerkin scheme. In order to derive optimal error estimates we extend the ideas introduced in Section 2. We denote by $a(\cdot,\cdot)$ the natural bilinear form associated with $A(\cdot)$. We assume $a(.,.)$ satisfies the following continuity and coercivity conditions.

ASSUMPTION 1. *There exist non-negative constants $C_a$, $C_\alpha$, $c_a$, $c_\alpha$ such that*

1. *Continuity of the bilinear form and data:*

$$|a(t;u,v)| \le \big(c_a\|u\|^2 + C_a|u|^2\big)^{\frac{1}{2}}\big(c_a\|v\|^2 + C_a|v|^2\big)^{\frac{1}{2}}$$

*and*

$$|\langle f,v\rangle| \le \|f\|_*\big(c_a\|u\|^2 + C_a|u|^2\big)^{\frac{1}{2}}$$

2. *Corecivity of the bilinear form:*

$$a(t;u,u) \ge c_\alpha\|u\|^2 - C_\alpha|u|^2$$

In this context the natural weak formulation of (1.1) is to find $u \in \mathcal{U} \equiv L^2[0,T;U] \cap H^1[0,T;U']$ such that

$$(u(T),v(T)) + \int_0^T \Big(\langle -u, v_t\rangle + a(t,u,v)\Big) = (u_0, v(0)) + \int_0^T \langle F, v\rangle, \tag{3.1}$$

for all $v \in \mathcal{U}$. We emphasize that $\mathcal{U}$ is the natural space to seek convergence, so ideally we would like to derive estimates in a related norm. To approximate the solution of the above weak formulation we introduce a partition $0 = t^0 < t^1 < ... < t^N = T$ of $[0,T]$ and on each partition we construct a closed subspace $U_h^n \subset U$. The discontinuous Galerkin method constructs an approximate solution $u_h|_{(t^{n-1},t^n]} \in \mathcal{P}_k(t^{n-1},t^n;U_h^n)$ satisfying

$$\begin{aligned}(u^n,v^n) &+ \int_{t^{n-1}}^{t^n} \Big(-\langle u_h, v_{ht}\rangle + a(\cdot;u_h,v_h)\Big) \\ &= (u^{n-1},v_+^{n-1}) + \int_{t^{n-1}}^{t^n} \langle F, v_h\rangle \qquad \forall\, v_h \in \mathcal{P}_k(t^{n-1},t^n;U_h^n).\end{aligned} \tag{3.2}$$

Integration of the temporal term by parts yields the representation:

$$\begin{aligned}\int_{t^{n-1}}^{t^n} &\Big(\langle u_{ht}, v_h\rangle + a(\cdot;u_h,v_h)\Big) + (u_+^{n-1} - u^{n-1}, v_+^{n-1}) \\ &= \int_{t^{n-1}}^{t^n} \langle F, v_h\rangle \qquad \forall\, v_h \in \mathcal{P}_k(t^{n-1},t^n;U_h^n).\end{aligned} \tag{3.3}$$

**3.2. Preliminary Estimates.** Classical bounds for the parabolic equation (3.2) are obtained upon selecting $u = v$ in equation (3.1); the discrete analogue would be to set $v_h = u_h$ in (3.3). Upon observing that

$$\int_{t^{n-1}}^{t^n} \langle u_{ht}, u_h\rangle + (u_+^{n-1} - u^{n-1}, u_+^{n-1}) = \frac{1}{2}|u^n|^2 - \frac{1}{2}|u^{n-1}|^2 + \frac{1}{2}|u_+^{n-1} - u^{n-1}|^2,$$

standard enery arguments use the continuity and coercivity assumptions 1 lead to the inequality

$$\begin{aligned}|u^n|^2 &+ c_\alpha \int_{t^{n-1}}^{t^n} \|u_h\|^2 + |u^{n-1} - u_+^{n-1}|^2 \\ &\le |u^{n-1}|^2 + \int_{t^{n-1}}^{t^n} \big((1 + c_a/c_\alpha)\|F\|_*^2 + (2C_\alpha + C_a)|u_h|^2\big).\end{aligned} \tag{3.4}$$

When $u_h \in \mathcal{P}_k(t^{n-1}, t^n; U_h^n)$ with $k \geq 2$ this inequality does not control $|u_h(s)|$ for $s \in (t^{n-1}, t^n)$. When $u_h$ is piecewise constant or linear in time ($k = 0$ or 1) the terms on the left hand side will dominate the last term on the right for sufficiently small time steps [24, Theorem 12.4]. The case $k > 2$ has not been completly addressed previouly; typically strict coercivity is assumed so that $C_\alpha \leq 0$. In this situation it is possible to write

$$\int_{t^m-1}^{t^n} \langle F, u_h \rangle \leq \int_{t^m-1}^{t^n} (1/2\epsilon)\|F\|_*^2 + (\epsilon/2)(|u_h|^2 + \|u_h\|^2)$$

and using the Poincare inequality to control the last term.

Similar difficulties are encountered with error estimates when $k \geq 2$. Letting $e = u - u_h$ the orthogonality condition becomes

$$(e^n, v^n) + \int_{t^{n-1}}^{t^n} \Big( - \langle e, v_{ht} \rangle + a(\cdot; e, v_h) \Big) = (e^{n-1}, v_+^{n-1})$$

for all $v_h \in \mathcal{P}_k(t^{n-1}, t^n; U_h^n)$. Writing

$$e = u - u_h = (u - \mathbb{P}_h u) + (\mathbb{P}_h u - u_h) \equiv e_h + e_p,$$

where $\mathbb{P}_h$ is the projection defined in Definition 2.4 we compute

$$\begin{aligned}(e_h^n, v^n) &+ \int_{t^{n-1}}^{t^n} (-\langle e_h, v_{ht} \rangle + a(\cdot; e_h, v_h)) - (e_h^{n-1}, v_+^{n-1}) \\ &= -(e_p^n, v^n) + \int_{t^{n-1}}^{t^n} \langle e_p, v_{ht} \rangle + (e_p^{n-1}, v_+^{n-1}) - \int_{t^{n-1}}^{t^n} a(\cdot; e_p, v_h))\end{aligned}$$

Since $\mathbb{P}_h u$ is the discontinuous Galerkin solution of an ordinary differential equation it follows that $e_p = u - \mathbb{P}_h u$ satisfies the orghogonality condition (2.7). We then conclude that all but the last term on the right hand side of the above expression vanishes so that

$$\begin{aligned}(e_h^n, v_h^n) + \int_{t^{n-1}}^{t^n} \Big( \langle -e_h, v_{ht} \rangle + a(\cdot; e_h, v_h) \Big) \\ = (e_h^{n-1}, v_{h+}^{n-1}) - \int_{t^{n-1}}^{t^n} a(\cdot; e_p, v_h).\end{aligned} \tag{3.5}$$

This expresion is identical in form to the original scheme (3.2) for $u_h$ with $F(.) = -a(e_p, .)$. Putting $v_h = e_h$ we obtain the analogue of equation (3.4),

$$\begin{aligned}|e_h^n|^2 &+ c_\alpha \int_{t^{n-1}}^{t^n} \|e_h\|^2 + |e_h^{n-1} - e_{h+}^{n-1}|^2 \\ &\leq |e_h^{n-1}|^2 + \int_{t^{n-1}}^{t^n} \Big( (1 + c_a/c_\alpha) \left( c_a \|e_p\|^2 + C_a |e_p|^2 \right) + (2C_\alpha + C_a)|e_h|^2 \Big).\end{aligned} \tag{3.6}$$

Again the natural energy arguments for the DG scheme fail to control $e_h(t)$ for $t \in (t^{n-1}, t^n)$.

Remark 4. *The projection constructed using the discrete solution of the corresponding ODE is necessary when different subspaces are used in each time step, i.e., $U_h^n \neq U_h^{n-1}$. Using the "standard" projection, $\mathbb{P}_h^{loc} u$ in place of $\mathbb{P}_h$ (as in [24]) gives*

$$
\begin{aligned}
(e_h^n, v_h^n) + \int_{t^{n-1}}^{t^n} \Big( \langle -e_h, v_{ht} \rangle + a(\cdot; e_h, v_h) \Big) \\
= (e_h^{n-1}, v_{h+}^{n-1}) - \int_{t^{n-1}}^{t^n} a(\cdot; e_p^{n-1}, v_h) + (e_p, v_{h+}^{n-1}).
\end{aligned}
$$

*with $e_p = u - \mathbb{P}_h^{loc} u$. Note that the last term is equal to $(e_p^{n-1}, v_{h+}^{n-1} - w_-)$ for every $w_- \in U^{n-1}$, so when $U_h^n = U_h^{n-1}$ we may pick $w_- = v_{h+}^{n-1}$ to obtain (3.5).*

**3.3. Stability and Error Estimates.** In this section we show how the estimates in (3.4) and (3.6) can be augmented to provide bounds on the solution and the error for all times; in particular, for the intermediate times $t \in (t^{n-1}, t^n)$. To do this we use the discrete characteristic functions developed in Section 2.3. Since equations (3.2) for $u_h$ and (3.5) for $e_h$ are identical in form, the same line of argument can be applied to either of them. For this reason we will only prove the theorem for the error estimate and will simply state the analogous bound for $u_h$.

Theorem 3.1. *Let $U \hookrightarrow H \hookrightarrow U'$ be a dense embedding of Hilbert spaces and $\mathcal{U}_h$ be the subspace of $L^2[0,T;U]$ defined in Section 2.1 and let the bilinear form $a : U \times U \to \mathbb{R}$ and linear form $F : U \to \mathbb{R}$ satisfy Assumptions 1. Let $u \in L^2[0,T;U] \cap H^1[0,T;U']$ be the solution of (3.1) and $u_h \in \mathcal{U}_h$ be the approximate solution computed using the discontinuous Galerkin scheme (3.2) on the partition $0 = t^0 < t^1 < \ldots < t^N = T$ and set $\tau \equiv \max_n t^n - t^{n-1}$.*

*Then there exists a constant $C > 0$ depending only on $k$ (through the constant $C_k$ of Lemma 2.7), the constants $C_a$, $C_\alpha$, and the ratio $c_a/c_\alpha$, such that*

$$
\begin{aligned}
(1-\lambda)|u^n|^2 + \lambda \sup_{t^{n-1} \le s \le t^n} |u(s)|^2 + \sum_{i=0}^{n-1} e^{C(t^{n-1}-t^i)} |u^i - u_+^i|^2 \\
+ (1-\lambda) \frac{c_\alpha}{2} \int_0^{t^n} \mathrm{e}^{C(t^n - s)} \|u_h(s)\|^2 ds \\
\le \big(1 + T\mathcal{O}(\tau)\big) \left( \mathrm{e}^{Ct^n} |u_h^0|^2 + C\lambda \int_0^{t^n} \mathrm{e}^{C(t^n-s)} \|F(s)\|_*^2 \, ds \right),
\end{aligned}
$$

*and*

$$
\begin{aligned}
(1-\lambda)|e_h^n|^2 + \lambda \sup_{t^{n-1} \le s \le t^n} |e_h(s)|^2 + \sum_{i=0}^{n-1} e^{C(t^{n-1}-t^i)} |e_h^i - e_{h+}^i|^2 \\
+ (1-\lambda) \frac{c_\alpha}{2} \int_0^{t^n} \mathrm{e}^{C(t^n - s)} \|e_h(s)\|^2 ds \\
\le \big(1 + T\mathcal{O}(\tau)\big) \left( \mathrm{e}^{Ct^n} |e_h^0|^2 + C\lambda \int_0^{t^n} \mathrm{e}^{C(t^n-s)} \big(c_a \|e_p(s)\|^2 + C_a |e_p(s)|^2\big) \, ds \right),
\end{aligned}
$$

*provided* $C\tau < 1$. *Here* $\lambda = 1/(2C_k + 4C_k c_a/c_\alpha + 1) \in (0,1)$, *and* $e_p = u - \mathbb{P}_h u$ *and* $e_h = \mathbb{P}_h u - u_h$ *where* $\mathbb{P}_h : C[0,T;H] \cap H^1[0,T;U'] \to \mathcal{U}_h$ *is the projection defined in Definition 2.4.*

*Proof.* Since the line of argument to prove each inequality is identical we only prove the second.

Let $\tilde{e}_h \in \mathcal{P}_k(t^{n-1}, t^n; U_h^n)$ be the discrete approximation of $\chi_{[t^n,t)} e_h(t)$ constructed in Lemma 2.7. Setting $v_h = \tilde{e}_h$ in equation (3.5) and moving the term $a(.; e_h, \tilde{e}_h)$ to the right hand side gives

$$\frac{1}{2}|e_h(t)|^2 + \frac{1}{2}|e^{n-1} - e_+^{n-1}|^2 = \frac{1}{2}|e^{n-1}|^2 - \int_{t^{n-1}}^{t^n} a(.; e_p, \tilde{e}_h) + a(\cdot; e_h, \tilde{e}_h).$$

We estimate the each of the last two terms seperatly.

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} a(.; e_h, \tilde{e}_h) &\leq \int_{t^{n-1}}^{t^n} (c_a\|e_h\|^2 + C_a|e_h|^2)^{1/2}(c_a\|\tilde{e}_h\|^2 + C_a|\tilde{e}_h|^2)^{1/2} \\
&\leq \left(\int_{t^{n-1}}^{t^n} c_a\|e_h\|^2 + C_a|e_h|^2\right)^{1/2} \left(\int_{t^{n-1}}^{t^n} c_a\|\tilde{e}_h\|^2 + C_a|\tilde{e}_h|^2\right)^{1/2} \\
&\leq C_k \int_{t^{n-1}}^{t^n} \Big(c_a\|e_h\|^2 + C_a|e_h|^2\Big).
\end{aligned}$$

Lemma 2.7 was used to bound $\tilde{e}_h$ in terms of $e_h$ in the last line. A similar computation shows

$$\int_{t^{n-1}}^{t^n} a(.; e_p, \tilde{e}_h) \leq \frac{C_k}{2} \int_{t^{n-1}}^{t^n} \Big((c_a/c_\alpha + 1)(c_a\|e_p\|^2 + C_a|e_p|^2) + c_\alpha\|e_h\|^2 + C_a|e_h|^2\Big).$$

Combining the above gives

$$\begin{aligned}
&|e_h(t)|^2 + |e^{n-1} - e_+^{n-1}|^2 \leq |e^{n-1}|^2 \\
&+C_k \int_{t^{n-1}}^{t^n} \Big((c_a/c_\alpha + 1)(c_a\|e_p\|^2 + C_a|e_p|^2) + c_\alpha(1 + 2c_a/c_\alpha)\|e_h\|^2 + 3C_a|e_h|^2\Big)
\end{aligned} \tag{3.7}$$

We now consider the convex combination of $(1-\lambda)$ equation (3.4) and $\lambda$ equation (3.7). We choose the coefficient, $\lambda$, so that the term involving $\|e_h\|^2$ on the right of (3.7) is dominated by the corresponding term on the left of (3.4). Speciffically, let

$$\lambda C_k(1 + 2c_a/c_\alpha) = (1/2)(1 - \lambda), \qquad \text{or} \qquad \lambda = \frac{1}{(2C_k + 4C_k c_a/c_\alpha + 1)}.$$

This gives an estimate of the form

$$\begin{aligned}
(1-\lambda)|e_h^n|^2 + \lambda|e_h(t)|^2 + (1-\lambda)\frac{c_\alpha}{2}\int_{t^{n-1}}^{t^n} \|e_h\|^2 + |e_h^{n-1} - e_{h+}^{n-1}|^2 \\
\leq |e_h^{n-1}|^2 + C\lambda \int_{t^{n-1}}^{t^n} \Big(c_a\|e_p\|^2 + C_a|e_p|^2 + |e_h|^2\Big)
\end{aligned} \tag{3.8}$$

where $C$ depends only upon the coercivity constant $c_\alpha$ and $c_a$ through the ratio $c_a/c_\alpha$. Bound the first and last terms on the right by

$$|e_h^{n-1}|^2 \le (1-\lambda)|e_h^{n-1}|^2 + \lambda \sup_{t^{n-2}<s\le t^{n-1}} |e_h(s)|^2$$

and

$$\int_{t^{n-1}}^{t^n} |e_h|^2 \le \tau^n \sup_{t^{n-1}<s\le t^n} |e_h(s)|^2, \qquad \tau^n \equiv t^n - t^{n-1},$$

respectivly, and select the time $t$ on the left so that $|e_h(t)| = \sup_{t^{n-1}<s\le t^n} |e_h(s)|$ to get

$$(1-\lambda)|e_h^n|^2 + \lambda(1-C\tau^n) \sup_{t^{n-1}<s\le t^n} |e_h(t)|^2 + (1-\lambda)\frac{c_\alpha}{2}\int_{t^{n-1}}^{t^n} \|e_h\|^2 + |e_h^{n-1} - e_{h+}^{n-1}|^2$$
$$\le (1-\lambda)|e_h^{n-1}|^2 + \lambda \sup_{t^{n-2}<s\le t^{n-1}} |e_h(s)|^2 + C\lambda \int_{t^{n-1}}^{t^n} (c_a\|e_p\|^2 + C_a|e_p|^2).$$

Upon introducing a factor $(1-C\tau^n)$ in front of the first term, this inequality takes the form

$$(1-C\tau^n)\alpha^n + \beta^n \le \alpha^{n-1} + f^n,$$

and the theorem follows from the discrete Gronwall inequality. □

Remark 5. *Note that the above estimate consists of norms that measure the behaviour of the solution at the nodal, interior and jump points. All norms included are the "natural" ones and no additional regularity is required on the right hand side.*

Theorem 3.1 essentially shows that the error for the parabolic equation can be bounded by the error of the ODE. If the norms $|||.|||_\infty$, $|||.|||_2$ and jump term $J_N(e)$ are defined by

$$|||v|||_\infty^2 = \sup_{0\le s\le T} |v(s)|^2 + c_\alpha \int_0^T \mathrm{e}^{C(T-s)} \|v(s)\|^2\, ds$$

$$|||v|||_2^2 = \int_0^T \mathrm{e}^{C(T-s)} |v(s)|^2\, ds + c_a \int_0^T e^{C(T-s)} \|v(s)\|^2\, ds,$$

and

$$J_N^2(v) = \sum_{i=0}^{N-1} \mathrm{e}^{C(T-t^i)} |v^i - v_+^i|^2,$$

then Theorem 3.1 states

$$|||\mathbb{P}_h u - u_h\,|||_\infty^2 + J_N^2(\mathbb{P}_h u - u_h) \le C(T)\Big(|P_0 u(0) - u^0|^2 + |||u - \mathbb{P}_h u\,|||_2^2\,\Big).$$

Since $\int_0^T \mathrm{e}^{C(T-s)}|e_p(s)|^2 \le \mathrm{e}^{CT} \sup_{0\le s\le T} |e_p(s)|^2$ various symetric error estimates follow. For example, setting $u^0 = P_0 u(0)$ and using the triangle inequlality gives

$$|||u - u_h|||_2 \le C(T)\, |||\, u - \mathbb{P}_h u|||_2, \quad \text{and} \quad |||\, u - u_h|||_\infty \le C(T)\, |||\, u - \mathbb{P}_h u\,|||_\infty\,.$$

Since $\mathbb{P}_h u$ is not a local projection, classical interpolation theory does not immediatly yield rates of convergence for a specific problem. However, we can use the results of

Section 2.4 to estimate the right hand sides of the above in terms of the local projection $\mathbb{P}_h^{loc} u$.

THEOREM 3.2. *Under the assumptions in theorem 3.1 there exists a positive constant $C(T)$ depending only on $k$ (through the constant $C_k$ of Lemma 2.7), the constants $C, \lambda, C_a$, $C_\alpha$, and the ratio $c_a/c_\alpha$, and the final time $T$ such that the following estimate holds*

$$\begin{aligned}|||u - u_h|||_\infty &\leq C(T)\big(|e_h^0| + |||u - \mathbb{P}_h^{loc} u\,|||_2 + |||\,\mathbb{P}_h u - \mathbb{P}_h^{loc} u\,|||_\infty\,\big)\\ &\leq C(T)\Big(|e_h^0| + |||u - \mathbb{P}_h^{loc} u\,|||_2 + \sum_{i=1}^{N} |P_i(I - P_{i-1})u(t^{i-1})|\\ &\quad + \sqrt{c_a} \max_{1\leq n\leq N} \|\sum_{i=1}^{n} \big(\prod_{j=i}^{n} P_j\big)(I - P_{i-1})u(t^{i-1}) + (P_n \circ ... \circ P_1)e_h^0\|\Big),\end{aligned}$$

*where $e_h^0 = P_0 u(0) - u^0$, and $\mathbb{P}_h^{loc} u$ is the local projection defined in Definition 2.2, and $P_n : H \to U_h^n$ is the orthogonal projection.*

REMARK 6. *1) Upon assuming an inverse inequality of the form $\|v_h\| \leq C_{inv}(h)|v_h|$ for all $v_h \in U_h^n$, $n = 0, 1, \ldots,$ and $h > 0$, the above estimate simplifies to*

$$|||u-u_h|||_\infty \leq C(T) \left( |||u - \mathbb{P}_h^{loc} u\,|||_2 + \big(1 + \sqrt{c_a} C_{inv}(h)\big)\Big( \sum_{i=1}^{N} |P_i(I - P_{i-1})u(t^{i-1})| + |e_h^0|\Big) \right).$$

*2) Estimates for the jump terms $J_N(u - u_h)$ can also be obtained; however, they will converge at a reduced rate; specifically,*

$$\begin{aligned}J_N(u - u_h) &\leq C(T)\Big(\,|||\,u - \mathbb{P}_h^{loc} u\,|||_2 + J_N(u - \mathbb{P}_h^{loc} u)\\ &\quad + \sqrt{N}\Big(\sum_{i=1}^{N} |P_i(I - P_{i-1})u(t^{i-1})| + |e_h^0|\Big)\\ &\quad + \sqrt{c_a} \max_{1\leq n\leq N} \|\sum_{i=1}^{n} \big(\prod_{j=i}^{n} P_j\big)(I - P_{i-1})u(t^{i-1}) + (P_n \circ ... \circ P_1)e_h^0\|\Big),\end{aligned}$$

**Appendix A. Discrete Gronwall Inequality.** If $(1 - C\tau^n)a^n + b^n \leq a^{n-1} + f^n$ the discrete Gronwall inequality states that if $\max_n C\tau^n < 1$ then

$$a^N + \sum_{n=1}^{N} \frac{b^n}{\prod_{i=n}^{N}(1 - C\tau^i)} \leq \frac{a^0}{\prod_{i=1}^{N}(1 - C\tau^i)} + \sum_{n=1}^{N} \frac{f^n}{\prod_{i=n}^{N}(1 - C\tau^i)}.$$

Since

$$\exp\left(\sum_{i=n}^{N} C\tau^i\right) \leq \frac{1}{\prod_{i=n}^{N}(1 - C\tau^i)} \leq \left(1 - \sum_{i=n}^{N} (C\tau^i)^2\right)^{-1} \exp\left(\sum_{i=n}^{N} C\tau^i\right)$$

and $(1 - \sum_{i=n}^{N}(C\tau^i)^2) \geq 1 - C^2 T\tau$, where $\tau = \max_n \tau^n$ we may write

$$a^N + \sum_{n=1}^{N} e^{C(t^N - t^n)} b^n \leq \left(1 + T\mathcal{O}(\tau)\right) \left( \mathrm{e}^{Ct^N} a^0 + \sum_{n=1}^{N} \mathrm{e}^{C(t^N - t^n)} f^n \right),$$

where $t^n = \sum_{i=1}^{n} \tau^n$.

REFERENCES


[1] R. ADAMS, Sobolev Spaces, Academic Press, New York, 1975.

[2] D.N. ARNOLD, F. BREZZI, B. COCKBURN AND D.L. MARINI, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Num. Anal.* **39**, 2001-02, pp 1749-1779.

[3] M. BAUSE AND P. KNABNER, Uniform error analysis for Lagrange-Galerkin approximations of convection-dominated problems, *SIAM J. Num. Anal.* **39**, pp 1954-1984.

[4] B. COCKBURN, G. E. KARNADIAKIS, AND C. W. SHU, The development of discontinuous Galerkin methods, in Discontinuous Galerkin methods (Newport, RI, 1999), Springer, Berlin, 2000, pp 3-50.

[5] M. DELFOUR, W. HAGER AND F. TROCHU, Discontinuous Galerkin Methods for Ordinary Differential Equations, *Math. Comp*, **36**, 1981, pp 455-473.

[6] J. DOUGLAS JR. AND T. F. RUSSEL, Numerical methods for convection dominated difussion problems based on combining the method of characteristics with finite element or finite difference procedures, *SIAM J. Numer. Anal.*, **19**, 1982, pp 871-885.

[7] T. DUPONT, Mesh modification for evolutionary equations, *Math. Comp.*, **39**, 1982, pp 85-107.

[8] T. F. DUPONT AND YINGJIE LIU, Symmetric error estimates for moving mesh Galerkin methods for advection-diffusion equations, *SIAM J. Numer. Anal.* **40**, 2002, pp 914-927.

[9] K. ERIKSSON AND C. JOHNSON, Adaptive finite element methods for parabolic problems. I. A linear model problem, *SIAM J. Numer. Anal.*, **28**, 1991, pp. 43-77.

[10] K. ERIKSSON AND C. JOHNSON, Adaptive finite element methods for parabolic problems. II. Optimal error estimates in $L_\infty(L^2)$ and $L_\infty(L_\infty)$, *SIAM J. Numer. Anal.*, **32**, 1995, pp. 706-740.

[11] K. ERIKSSON, C. JOHNSON AND V. THOMÉE, Time discretization of parabolic problems by the discontinuous Galerkin method, *RAIRO Modél. Math. Anal. Numér.*, **29**, 1985, pp 611-643.

[12] D. ESTEP AND S. LARSSON, The discontinuous Galerkin method for semilinear parabolic equations, *RAIRO Modél. Math. Anal. Numér.*, **27**, 1993, pp 35-54.

[13] P. JAMET, Galerkin-type approximations which are discontinuous in time for parabolic equations in a varible domain, *SIAM J. Numer. Anal.* **15**, 1978, pp 912-928.

[14] C. JOHNSON, Numerical Solution of Partial Differential Equations by the Finite Element Method, Cambridge, 1987.

[15] P. LASAINT AND P.-A. RAVIART, On a finite element method for solving the neutron transport equation, in Mathematical aspects of finite elements in partial differential equations, C. de Boor, ed., Academic Press, New York, 1974, pp. 89-123.

[16] S. LARSSON, V. THOMÉE AND L.B. WALHBIN, Numerical solution of parabolic intregro-differential equations by the discontinuous Galerkin method, *Math. Comp.* **67**, 1998, pp 45-71.

[17] YINGJIE LIU, R.E. BANK, T.F DUPONT, S. GARCIA AND R.P. SANTOS, Symmetric error estimates for moving mesh mixed methods for advection-diffusion equations, *SIAM J. Numer. Anal.* **40**, 2003, pp 2270-2291.

[18] M. LUSKIN AND R. RANNACHER, On the smoothing property of the Galerkin method for parabolic equations, *SIAM J. Numer. Anal.* **19**, 1982, pp 93-113.

[19] K. MILLER, Moving finite elements II, *SIAM J. Numer. Anal.* **18**, 1981, pp 1033-1057.

[20] K. MILLER AND R.N. MILLER, Moving finite elements I, *SIAM J. Numer. Anal.* **18**, 1981, pp 1019-1032.

[21] K.W. MORTON, A. PRIESTLEY AND E. SÜLI, Stability of the Lagrange-Galerkin method with nonexact integration, *RAIRO Modél. Math. Anal. Numér.*, **22**, 1988, pp 625-653.

[22] SHOWALTER, R. E., *Hilbert Space Methods for Partial Differential Equations*, Pitman, 1979.

[23] SHOWALTER, R. E., *Monotone operators in Banach space and nonlinear partial differential equations*, American Mathematical Society, Providence, RI, 1997.

[24] V. THOMÉE, Galerkin finite element methods for parabolic problems, Spinger-Verlag, Berlin, 1997

[25] N. J. WALKINGTON, Convergence of the discontinuous Galerkin method for discontinuous solutions, preprint.