Math 128

Problem set #3

Feb. 19, 2004, due Feb 26

Contents

1 The Heisenberg group and algebra.

1. Let H be the group of all real three by three matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider the algebrea of all operators on functions of one real variable. Let $\mathbf{h}$ denote three dimensional space of operators spanned by the identity operator, $\mathbf{1}$, the operator $\mathbf{x}$ consisting of multiplication by x, and the operator d/dx:

$$\begin{aligned} \mathbf{1}: &\qquad f \mapsto f \\ \mathbf{x}: &\qquad f(x) \mapsto xf(x) \\ \frac{d}{dx}: &\qquad f \mapsto \frac{df}{dx}. \end{aligned}$$

Show that $\mathbf{h}$ is a Lie subalgebra of the algebra of operators on smooth functions defined on $\mathbf{R}$. (It is called the Heisenberg algebra.) Show that $\mathbf{h}$ is isomorphic to the Lie algebra of H.

2 Poisson bracket.

We consider an even dimensional space with coordinates $q_1, q_2, \ldots, p_1, p_2, \ldots$. The polynomials have a Poisson bracket

$$\{f,g\} := \sum \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right). \tag{1}$$

This is clearly anti-symmetric, and direct computation will show that the Jacobi identity is satisfied. Here is a more interesting proof of Jacobi's identity: Notice that if f is a constant, then $\{f,g\} = 0$ for all g. So in doing bracket computations we can ignore constants. On the other hand, if we take g to be successively $q_1, \ldots, q_n, p_1, \ldots, p_n$ in (1) we see that the partial derivatives of f are completely determined by how it brackets with all g, in fact with all linear g. If we fix f, the map

$$h \mapsto \{f,h\}$$

is a **derivation**, i.e. it is linear and satisfies

$$\{f, h_1h_2\} = \{f,h_1\}h_2 + h_1\{f,h_2\}.$$

This follows immediately from from the definition (1). Now Jacobi's identity amounts to the assertion that

$$\{\{f,g\},h\} = \{f,\{g,h\}\} - \{g,\{f,h\}\},$$

i.e. that the derivation

$$h \mapsto \{\{f,g\},h\}$$

is the commutator of the of the derivations

$$h \mapsto \{f,h\} \quad \text{and} \qquad h \mapsto \{g,h\}.$$

It is enough to check this on linear polynomials h, and hence on the polynomials q_j and p_k. If we take $h = q_j$ then

$$\{f,q_j\} = \frac{\partial f}{\partial p_j}, \quad \{g,q_j\} = \frac{\partial g}{\partial p_j}$$

so

$$\begin{aligned}
\{f,\{g,q_j\}\} &= \sum \left(\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q_i \partial p_j} - \frac{\partial f}{\partial q_i}\frac{\partial^2 g}{\partial p_i \partial p_j} \right) \\
\{f,\{f,q_j\}\} &= \sum \left(\frac{\partial g}{\partial p_i}\frac{\partial^2 f}{\partial q_i \partial p_j} - \frac{\partial g}{\partial q_i}\frac{\partial^2 f}{\partial p_i \partial p_j} \right) \text{ so} \\
\{f,\{g,q_j\}\} - \{g,\{f,q_j\}\} &= \frac{\partial}{\partial p_j}\{f,g\} \\
&= \{\{f,g\},q_j\}
\end{aligned}$$

as desired, with a similar computation for p_k.

3 The Heisenberg algebra again.

2. Consider the subspace spanned by the functions $1, q_1, \cdots, q_n, p_1, \cdots, p_n$. In other words, the space of inhomogeneous linear functions. Show that this is a Lie subalgebra of the algebra of all functions under Poisson bracket. Show that for $n = 1$ this subalgebra is isomorphic to the Lie algebra of Problem **1**.

So now we generalize the notation of Problem **1** and call this $2n+1$ dimensional algebra the Heisenberg algebra $\mathbf{h}$.

4 The symplectic algebra.

3. Show that the space of all inhomogeneous quadratic polynomials, i.e. the space spannec by $1, q_1, \ldots, q_n, p_1, \ldots, p_n, q_i, q_j.q_ip_j, p_1p_j, (i.j = 1, \ldots n)$ is a Lie subalgebra under Poisson bracket and that $\mathbf{h}$ is an ideal in this algebra. Furthermore, the subspace consisting of homogeneous quadratic polynomials is a subalgebra complementary to $\mathbf{h}$.

The **symplectic algebra** $sp(2n)$ is defined to be the subalgebra consisting of all homogeneous quadratic polynomials.

4. Show that $sp(2)$ is isomorphic to $sl(2)$.

For the rest of this problem set we will let $\mathbf{g}$ denote the symplectic algebra $sp(2n)$.

Let

$$E := \frac{1}{2}(p_1q_1 + \cdots + p_nq_n).$$

5. Show that

$$\mathbf{g} = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1$$

where

$$[E, X] = X, \quad X \in \mathbf{g}_1, \qquad [E, X] = 0 \quad X \in {}_0, \qquad [E, X] = -X. \quad X \in \mathbf{g}_{-1}$$

and describe each of these subspaces. Show that this makes $\mathbf{g}$ into a graded Lie algebra with $\operatorname{ad} E$ as the degree derivation. Show that $\mathbf{g}_0$ is isomorphic to the Lie algebra $gl(n)$ of all $n \times n$ matrices and describe the action of $\mathbf{g}_0$ on $\mathbf{g}_{-1}$ and on $\mathbf{g}_1$. Conclude that $sp(2n)$ is simple.

5 The symplectic group.

We have shown that the symplectic algebra is simple, but we haven't really explained what it is. Consider the space of V of homogenous linear polynomials, i.e all polynomials of the form

$$\ell = a_1q_1 + \cdots + a_nq_n + b_1p_q + \cdots + b_np_n.$$

Define an anti-symmetric bilinear form ω on V by setting

$$\omega(\ell, \ell') := \{\ell, \ell').$$

From the formula (1) it follows that the Poisson bracket of two linear functions is a constant, so ω does indeed define an antisymmetric bilinear form on V, and we know that this bilinear form is non-degenerate. Furthermore, if f is a homogenous quadratic polynomial, and ℓ is linear, then $\{f, \ell\}$ is again linear, and if we denote the map

$$\ell \mapsto \{f, \ell\}$$

by $A = A_f$, then Jacobi's identity translates into

$$\omega(A\ell, \ell') + \omega(\ell A\ell') = 0 \tag{2}$$

since $\{\ell, \ell'\}$ is a constant. Condition (2) can be interpreted as saying that A belongs to the Lie algebra of the group of all linear transformations R on V which preserve ω, i.e. which satisfy

$$\omega(R\ell, R\ell') = \omega(\ell, \ell').$$

This group is known as the **symplectic group**. The form ω induces an isomorphism of V with V^* and hence of $\mathrm{Hom}(V, V) = V \otimes V^*$ with $V \otimes V$, and this time the image of the set of A satisfying (2) consists of all symmetric tensors of degree two, i.e. of $S^2(V)$. (Just as in the orthogonal case we got the anti-symmetric tensors). But the space $S^2(V)$ is the same as the space of homogenous polynomials of degree two. In other words, the symplectic algebra as defined above is the same as the Lie algebra of the symplectic group.

It is an easy theorem in linear algebra, that if V is a vector space which carries a non-degenerate anti-symmetric bilinear form, then V must be even dimensional, and if dim $V = 2n$ then it is isomorphic to the space constructed above. We will not pause to prove this theorem.

6 The root structure of the symplectic algebra.

Let $\mathfrak{h}$ consist of all linear combinations of

$$p_1q_1, \ldots, p_nq_n.$$

6. Show that $\mathfrak{h}$ is a maximal commutative subalgebra of $\mathbf{g} = sp(2n)$. In other words, show that any two elements of $\mathfrak{h}$ commute, and that if $X \in sp(2n)$ commutes with all elements of $v = \mathfrak{h}$ then $X \in \mathfrak{h}$.

As usual, we will let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$, In other words, $\mathfrak{h}^*$ consists of the space of all linear functions on $\mathfrak{h}$. Let L_i be defined by

$$L_i\left(a_1 p_1 q_1 + \cdots + a_n p_n q_n\right) = a_i$$

so $L_1, \ldots, L_n$ is the basis of $\mathbf{h}^*$ dual to the basis $p_1 q_1, \ldots, p_n q_n$ of $\mathbf{h}$.

Since $\mathfrak{h}$ is commutative, we know that

$$(\operatorname{ad} X)(\operatorname{ad} Y) = (\operatorname{ad} Y)(\operatorname{ad} X) \qquad \forall\ X, Y \in \mathfrak{h}.$$

So it makes sense to look for simultaneous eigenvectors for the operators $\operatorname{ad} X,\ \ X \in \mathfrak{h}$. That is, we look for elements $Z \in \mathbf{g}$ such that there is an $\alpha \in \mathfrak{h}^*$ so that

$$[X, Z] = \alpha(X) Z \quad \forall X \in \mathfrak{h}.$$

For example, if $Z \in \mathfrak{h}$ then the above equation holds with $\alpha \equiv 0$. We want to find the non-zero α (and Z) satisfying the above equation. Such a non-zero α satisfying the above equation with an appropriate Z is called a **root**.

7. Show that If $X = a_1 p_1 q_1 + \cdots + a_n p_n q_n$ then

$$\begin{aligned} [X, q^i q^j] &= (a_i + a_j) q^i q^j \\ [X, q^i p^j] &= (a_i - a_j) q^i p^j \\ [X, p^i p^j] &= -(a_i + a_j) p^i p^j \end{aligned}$$

so

$$\pm(L_i + L_j) \text{ all } i, j \ \text{ and } \ L_i - L_j \ \ i \neq j$$

are roots, and that these are all the roots.

We let Φ denote the collection of all roots. We can divide the set Φ into two subsets, the "positive roots" and the "negative roots" by setting

$$\Phi^+ = \{L_i + L_j\}_{\text{all } ij} \cup \{L_i - L_j\}_{i<j}$$

and

$$\Phi^- = -\Phi^+.$$

Define

$$\alpha_1 := L_1 - L_2, \ldots, \alpha_{n-1} := L_{n-1} - L_n, \alpha_n := 2L_n.$$

8. Show that every positive root can be written in a unique way as a linear combination of the α_i with non-negative integer coefficients.

If we set

$$h_1 := p_1 q_1 - p_2 q_2, \ldots, h_{n-1} := p_{n-1} q_{n-1} - p_n q_n,\ \ h_n := p_n q_n$$

then

$$\alpha_i(h_i) = 2$$

while for $i \neq j$

$$\begin{aligned} \alpha_i(h_{i\pm 1}) &= -1, \quad i = 1, \ldots, n-1 \\ \alpha_i(h_j) &= 0, \ j \neq i \pm 1, i = 1, \ldots, n \\ \alpha_n(h_{n-1}) &= -2. \end{aligned} \tag{3}$$