# MATH 217C NOTES

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These notes were taken in Stanford's Math 217c class in Winter 2015, taught by Eleny Ionel. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Ikshu Neithalath for finding a few errors.

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*"A differential geometer whose work often uses the simplifications obtained by considering the complex domain explained to me that the additional structure of complex manifolds makes them more interesting, just as two sexes are more interesting than one, but various aspects of this argument are open to debate." – Michael Spivak*

## 1. Introduction, Complex Manifolds and Holomorphic Maps: 1/6/15

The audience of the class has quite a varied background; some days, some people will be quite bored and others will find it quite difficult. We'll try to stay at an elementary level; this is an introductory course in complex differential geometry, supposed to be a second-year graduate-level course. In particular, this is not a topics course.

We're going to be most interested in complex manifolds and related structures, and we'll approach them with differential geometry and complex analysis. There's another approach which uses algebra, which we're not going to talk about as much. We're not going to follow any textbook particularly closely; here are the course topics (possibly out of order).

- Complex structures, almost complex structures, and integrability.
- Hermitian and Kähler metrics.
- Connections.
- Complex vector bundles, holomorphic vector bundles, connections and curvature.
- Chern classes and Chern-Weil theory.
- Many examples; in particular, the physicists will be interested in Calabi-Yau manifolds and Kähler-Einstein metrics and the relationship with physics.
- Cohomology theories: Hodge theory and Dolbeault cohomology.
- Vanishing theorems.
- Deformation theory and Kodaira-Spencer theory.

Since this is a second-year class, there will be no final and no midterm, but there will be homeworks; as usual, the best way to understand the material is to work through the examples.

We mentioned that there are the two analytic or algebraic approaches to complex differential geometry; the best results follow from combining the two. Here are some references for the class, recommended but not followed exactly.

- Moroianu, *Lectures on Kähler Geometry.* These lecture notes are short and to the point, and thus serve as a useful introduction. The first six lectures cover elementary differential geometry, which is a useful reference, since it will be assumed in this class.
- Huybrechts, *Complex Geometry.* This is a more detailed reference, and uses more complex analysis, and a little algebra. This is a cleaned-up version of Griffiths' and Harris' *Principles of Algebraic Geometry* (which is nonetheless still mostly analytic).
- Voisin, *Hodge Theory and Complex Algebraic Geometry: Volume I.* This is less relevant to the course, but its algebraic approach is useful and interesting.
- Demailly, *Complex Analytic and Differential Geometry.* This is also less relevant to the course; it mixes the analytic and algebraic approaches.

Now, let us enter the world of mathematics.

In this class, we want to study complex manifolds; we'll define them precisely in a moment, but one can think of these as spaces covered by charts, where each chart locally looks like $\mathbb{C}^n$, and the change-of-charts maps are *holomorphic*, i.e. complex differentiable; the linearization is complex linear. Complex manifolds are special cases of real manifolds, but there are real manifolds which cannot be complexified.

*Kähler manifolds* are a special class of complex manifolds; not every complex manifold is Kähler. $\mathbb{C}^n$ and $\mathbb{CP}^n$ are Kähler, as are complex submanifolds of $\mathbb{CP}^n$ and Calabi-Yau manifolds. These will have relatively nice properties.

A real manifold may have several complex structures (which will play a role in deformation theory), and similarly a complex manifold may have several Kähler structures.

This class will assume some knowledge of real differential geometry, and the study of real manifolds. One studies a real manifold by adding structure, e.g. a metric (turning it into a Riemannian manifold). Adding a metric isn't canonical, but forgetting it is. Similarly, we'll view a complex manifold as a complex structure on a real manifold, and we can add additional struture, such as a Hermitian metric (where the inner product is Hermitian), producing what is called a *Hermitian manifold.* Like a Riemannian metric, there is always such a metric and generally many choices, and forgetting the complex structure, one obtains a Riemannian metric. *Kähler metrics* (and their associated Kähler manifolds) are special cases of Hermitian manifolds.

We'll develop tools to prove our main results, mostly from differential geometry (connections, bundles, curvature), but also a little from algebraic topology (typically cohomology). The idea is that a complex structure places restrictions on the underlying topology of the manifold.

We will have three types of main results.

(1) The Hodge and Lefschetz theorems impose strong restrictions on the cohomology of Kähler manifolds, and in particular implies that there are complex manifolds that do not admit a Kähler structure.

(2) Vanishing results, e.g. the Kodaira vanishing theorem and its applications, such as Kodaira embedding. These results can also be stated in terms of cohomology (specifically, $H^0$ vanishes). Intuitively, this result states that if a complex line bundle has negative curvature, then it has no nontrivial holomorphic sections; then, the Kodaira vanishing theorem states that under the assumption of positive curvature, all of the higher cohomology groups $H^q(M, E)$ on the manifold $M$ and line bundle $E$ vanish (i.e. $q > 1$).

With the Riemann-Roch theorem (which itself is a special case of an index theorem), we can reframe this as a criterion for nice Kähler manifold to be embedded in $\mathbb{CP}^N$ for some large $N$. This is quite a surprise, because in the complex case things are very different from $\mathbb{R}^n$: there are no partitions of unity and no compact complex submanifolds of $\mathbb{C}^n$ (other than a point), which ultimately follows from Liouville's theorem.

(3) In deformation theory, one asks how many differentiable complex structures there are on a given smooth manifold. This is a bit of a hopeless question, but we can ask the infinitesimal version: given a complex structure, how any infinitesimal deformations are there? These end up being once again governed by another cohomology group, and in particular, complex structures come in finite-dimensional families, unlike, say, Riemannian structures. However, not all of these first-order variations are integrable (which yields global structures from local ones), but higher-order cohomology groups dictate whether a global structure can be obtained.

A lot of questions concerning complex structure are difficult, and some are even open. For example, does $S^6$ admit a complex structure? We know it's not Kähler, but otherwise this is an open question. Note that $S^6$ is compact, but there's a possibility that it has a complex structure that can't be embeddied in $\mathbb{C}^n$.

**Holomorphic maps and complex manifolds.** Since everyone in the class has seen single-variable complex analysis, we can use it to move to several complex variables.

**Definition.** A *domain* will refer to an open, connected subset of $\mathbb{C}^n$.

For the time being, $U$ will always refer to a domain.

**Definition.** A function $f : U \to \mathbb{C}^m$ is *holomorphic* if it is *complex differentiable*, i.e. $\mathrm{d}f$ is complex linear.

We'll clarify what complex linear means in just a little bit.
This is equivalent to any of the following:

- $f$ satisfies the Cauchy-Riemann equations.
- $\overline{\partial} f = 0$.
- $f$ is analytic.
- $f$ has a continuous complex differential.

In the single-variable case, suppose $U \subset \mathbb{C}$ and $f : U \to \mathbb{C}$. Then, write $w = f(z)$ and $z = x + iy$, so $f$ can be thought of as a function into $\mathbb{R}^2$; similarly, write $w = u + iv$. This means we can make the notion of complex linearity precise: let $j : \mathbb{R}^2 \to \mathbb{R}^2$ correspond to multiplication by $i$ (well, just rotation by $90°$), so that $j \in \mathrm{End}(\mathbb{R}^2)$. As a matrix, we can write $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that $j^2 = -\mathrm{id}$. This is a way of specifying complex structure on $\mathbb{R}^2$, which we will see.

With this identification of $\mathbb{C}$ and $\mathbb{R}^2$, we can think of $f : \mathbb{R}^2 \to \mathbb{R}^2$ (or from a subset of $\mathbb{R}^2$, and so on). As a real transformation, $f$ has a real Jacobian

$$f_* = \mathcal{J}_{\mathbb{R}} f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix},$$

corresponding to the differential $\mathrm{d}f$.

Then, $\mathrm{d}f$ is said to be complex linear if it commutes with $j$, i.e. $\mathrm{d}f \circ j = j \circ \mathrm{d}f$, and when the matrices are expanded out, this is equivalent to requiring that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This is an elliptic differential equation, and therefore its solutions have very strong regularity properties. This is part of the reason they're so well-behaved.

Now, we can return to the case of several complex variables, and clarify what exactly it means to be holomorphic.

The idea is that at each point, the linearization of a holomorphic function is complex linear, and these vary continuously.

The standard complex structure on $\mathbb{C}^n$ is given by multiplying by $i$ in each coordinate. We have coordinates $z = (z_1, \ldots, z_n)$, and a Hermitian metric $|z| = |z_1|^2 + \cdots + |z_n|^2$. Each $z_k = x_k + iy_k$, so we can write $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, as $z = (x, y)$. Then, we once again get $j \in \mathrm{End}(\mathbb{R}^{2n})$ which multiplies by $i$ in every coordinate. Specifically,

$$j = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

which is a nice analogue of the single-variable case. Then, once again, complex linearity is equivalent to commuting with $j$.

Holomorphic functions have some nice regularity properties. Let $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}^n$. Then, $\mathrm{d}f$ can be regarded as a $\mathbb{C}$-valued 1-form

$$\omega = \sum_{k=1}^{n} (\alpha_k \,\mathrm{d}x_k + \beta_k \,\mathrm{d}y_k).$$

But for the purposes of complex geometry, there's a better basis; instead of using $\mathrm{d}x_k$ and $\mathrm{d}y_k$, it's better to use $\mathrm{d}z_k$ and $\mathrm{d}\overline{z}_k$, where $z_k = x_k + iy_k$, so $\mathrm{d}z_k = \mathrm{d}x_k + i\,\mathrm{d}y_k$, and $\overline{z}_k = x_k - iy_k$, so that $\mathrm{d}\overline{z}_k = \mathrm{d}x_k - i\,\mathrm{d}y_k$.

In the specific case $\omega = \mathrm{d}f$, the coefficients are familiar:

$$\mathrm{d}f = \sum_k \left( \frac{\partial f}{\partial x_k} \,\mathrm{d}x_k + \frac{\partial f}{\partial y_k} \,\mathrm{d}y_k \right).$$

One can (and should) verify that the dual basis to $\mathrm{d}z_k$ and $\mathrm{d}\overline{z}_k$ is

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left( \frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right)$$
$$\frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\left( \frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right).$$

Mind the sign change.

Then, we have operators $\partial$ and $\overline{\partial}$, defined as

$$\partial f = \sum_k \frac{\partial f}{\partial z_k}\,\mathrm{d}z_k$$
$$\overline{\partial} f = \sum_k \frac{\partial f}{\partial \overline{z}_k}\,\mathrm{d}\overline{z}_k.$$

Hence, $\mathrm{d}f = \partial f + \overline{\partial} f$, and $f$ is holomorphic iff $\overline{\partial} f = 0$.

In this new basis, $j$ is considerably easier to write down.

Here are some useful properties of holomorphic functions. Most come from one variable, and many but not all extend to several variables; there are a few properties which are only true in several complex variables.

**Theorem 1.1** (Cauchy integral formula). *Let $U \subsetneq \mathbb{C}$ be a ball (this can be made more general), $f : U \to \mathbb{C}$ be holomorphic on $U$ so that it extends continuously to $\overline{U}$. Then, for any $z_0 \in U$,*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(z)\,\mathrm{d}z}{z - z_0}.$$

From a differential-geometric perspective, this states that

$$\alpha = \frac{f(z)\,\mathrm{d}z}{z - z_0}$$

is a closed one-form on $U \setminus \{z_0\}$ (which you can check by computing $\mathrm{d}\alpha = 0$), so this is essentially a corollary of Stokes' theorem. This also relies on the calculation that

$$\frac{1}{2\pi i}\int_{|z-z_0|=\varepsilon} \frac{f(z)\,\mathrm{d}z}{z - z_0} \xrightarrow{\varepsilon\to 0} f(z_0).$$

This generalizes to several complex variables thanks to Fubini's theorem. Let $U$ be a *polydisc* (i.e. a product of discs), and $f : U \to \mathbb{C}^n$ be holomorphic on $U$ and extend continuously to $\overline{U}$. Then, the iterated integral satisfies

$$f(w) = \frac{1}{(2\pi i)^n}\int \frac{f(w)\,\mathrm{d}w}{(w_1 - z_1)\cdots(w_n - z_n)},$$

where $\mathrm{d}w = \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n$, where the integral is taken over the product of the boundaries.

Regarding the integrand as a power series, we can see the following.

**Theorem 1.2** (Osgood). *A holomorphic function of several variables is complex analytic (i.e. given by a complex power series).*

## 2. Complex Manifolds from Two Different Perspectives: 1/8/15

The two perspectives will be thinking of complex manifolds as integrable almost complex manifolds (linear algebraic) and a more complex-analytic perspective.

A complex manifold is just like a real manifold, except that the charts are subsets of $\mathbb{C}^n$ and the transition maps are holomorphic.

**Definition.** A *$n$-dimensional complex manifold $M$* is a smooth manifold on which there exists a collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ covering $M$, where each $U_\alpha \subset M$ is open and $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ has open image. Furthermore, if $\alpha$ and $\beta$ give two charts, then the transition maps $\varphi_\beta^{-1} \circ \varphi_\alpha$ are holomorphic on the overlaps.

The charts are called *holomorphic charts*, and the coordinates they induce are called *holomorphic coordinates.*

*Note.* In this class, we will require all manifolds to be Hausdorff, and have a countable topological basis.

*Note.* Just as the same topological manifold may admit many smooth structures, a smooth manifold can admit different complex structures; this will be useful in deformation theory.

**Definition.** $f : M \to \mathbb{C}$ is *holomorphic* if $f \circ \varphi_\alpha^{-1}$ is holomorphic for all holomorphic charts. Similarly, $f : M \to N$ is holomorphic if it is holomorphic as a map $\mathbb{C}^m \to \mathbb{C}^n$ on each chart.

One can check that these are well-defined notions, no matter which set of charts one uses.

*Note.* There are no nonconstant holomorphic functions on a compact, connected, complex manifold, which is just a corollary of the maximum modulus principle (in several variables, which follows from the single-variable statement, by checking each coordinate). This is one of the huge differences between complex geometry and real geometry.

**Definition.** $f : M \to N$ is a *biholomorphism* if it is a holomorphic bijection and $f^{-1}$ is also holomorphic.

This is the notion of equivalence in complex geometry, akin to a diffeomorphism in differential topology. If $M$ and $N$ are biholomorphic, one writes $M \simeq N$.

Another interesting fact is that the size of the charts matters; Liouville's theorem (in one variable) implies that $\mathbb{C} \not\simeq \mathbb{D}$.

*Note.* More worryingly, when $n \geq 2$, the unit ball ($\|z\| < 1$) is not holomorphic to the polydisc (the product of $n$ discs, or $|z_1| < 1, \ldots, |z_n| < 1$). The proof is not that easy, but ultimately it's because they have different automorphism groups; the automorphism group of the unit ball contains U(n), but that of the polydisc is abelian (once the origin is fixed). In particular, the Riemann mapping theorem does not extend to several variables (though the maximum modulus principle and analytic continuation both extend).

This is why we will prefer polydiscs to balls for bases; it allows us more naturally to do complex geometry in each variable.

**Example 2.1.**

(1) $\mathbb{C}^n$ is a complex manifold, with only one chart. More generally, any finite-dimensional complex vector space $V$ is a complex manifold (which will be useful when there's no natural basis). In complex geometry, this is called *affine space*.

(2) Similarly, any open subset of $\mathbb{C}^n$ is also a manifold.

(3) More nontrivially, we can make *tori*: consider a lattice $\Lambda$ in $\mathbb{C}^n$, e.g. $\Lambda = \mathbb{Z}^{2n}$, which can be thought of as $n$ linearly independent vectors (over $\mathbb{R}$) and then integer linear combinations of them. Then, $\mathbb{C}^n/\Lambda$ is a complex manifold, and is diffeomorphic to $T^{2n} = \underbrace{S^1 \times \cdots \times S^1}_{2n \text{ times}}$.

For example, if $n = 1$ and $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, for $\tau \in \mathbb{C} \setminus \mathbb{R}$, the quotient is an elliptic curve. Not all elliptic curves are biholomorphic, and therefore not all tori of a given dimension are. $E_\tau \simeq E_{\tau'}$ iff $\tau' = A\tau$ for an $A \in \mathrm{SL}(2, \mathbb{Z})$, i.e. the two lattices can be taken into each other by a linear transformation (which is where the special linear group comes from). Nonetheless, all tori of the same dimension are diffeomorphic.

(4) We also have complex projective space $\mathbb{CP}^n$, which is geometrically the space of complex lines in $\mathbb{C}^{n+1}$. It's hard to put a complex structure on it from this perspective, so it's easier to think of it algebraically, as $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and a $\lambda \in \mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by $(z_0, \ldots, z_n) \mapsto (\lambda z_0, \ldots, \lambda z_n)$. Thus, points on the same line through the origin are identified, so this is really the same thing.

Once we quotient out, keep the same coordinates: $[z] = [z_0, \ldots, z_n]$, where scaling factors are ignored (and brackets are used so we don't forget this). These are sometimes known as *homogeneous coordinates*.

To get charts, take

$$U_i = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$$

and $\varphi_i : U_i \to \mathbb{C}^n$ given by

$$\varphi_i([z_0, \ldots, z_n]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right).$$

$\mathbb{CP}^n \setminus U_i$ is $H_i = \{[z_0, \ldots, z_n] \mid z_i = 0\}$, which is called the *hyperplane at infinity in the $i^{\text{th}}$ coordinate*. $H_i \simeq \mathbb{CP}^{n-1}$, so $H_i$ is closed, and therefore $U_i$ is open.

To see that $\varphi_i$ is a bijection, its inverse is just $(w_0, \ldots, w_n) \mapsto [w_0, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_n]$. It's not too hard to check that the transition maps are holomorphic (because the product of holomorphic functions is holomorphic, and the quotient is too, whenever the denominator doesn't vanish, just as in the single-variable case).

A special case of this is the *Riemann sphere* $\mathbb{CP}^1 = S^2$. The two hyperplanes at infinity are the north and south poles.

The process of obtaining $\mathbb{CP}^n$ from $\mathbb{C}^{n+1}$ is called *projectivizing*; one can do this to any (finite-dimensional) complex vector space $V$ to get $\mathbb{P}(V) = (V \setminus \{0\})/\mathbb{C}^*$. This is of course biholomorphic to $\mathbb{CP}^n$, but there might not be a natural biholomorphism.

(5) The complex Grassmanian of $k$-dimensional complex planes in $\mathbb{C}^n$ is also a complex manifold; when $k = 1$ or $k = n - 1$, this is the same as $\mathbb{CP}^{n-1}$.

(6) The *Hopf manifold* is the quotient of $\mathbb{C}^n \setminus \{0\}$ by $\mathbb{Z}$ acting as follows: fix a $\lambda > 1$ (the topology is independent of the choice of $\lambda$) and for $k \in \mathbb{Z}$, $k * (z_1, \ldots, z_n) = (\lambda^k z_1, \ldots, \lambda^k z_n)$. The resulting quotient is a complex manifold diffeomorphic to $S^1 \times S^{2n-1}$, and will be important because when $n \geq 2$, it will provide an example of a complex manifold that cannot admit a Kähler structure (for topological reasons). When $n = 1$, one gets a torus which is an elliptic curve, and it's possible to explicitly write down which lattice it corresponds to.

(7) To produce more examples, one can talk about *complex submanifolds* in precisely the same way as one would talk about real submanifolds. $N$ is a $k$-dimensional complex submanifold of an $n$-dimensional manifold $M$ if there exists a collection of holomorphic charts on $M$ covering $N$ such that the image of $N$ under the chart maps is linear (specifically, $\mathbb{C}^{n-k}$).

This definition is motivated by the inverse function theorem; there are plenty of other definitions, but fortunately they're all equivalent.

Now, we can define lots of complex submanifolds of $\mathbb{C}^n$ or $\mathbb{CP}^n$; for example, if $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic, then the *zero locus* of $f$, i.e. $Z(f) = f^{-1}(0)$, is a complex manifold by the implicit function theorem. Useful examples of this include polynomials; if $f$ is cubic, then Sard's theorem is another way to show that for generic coefficients, the zero locus is smooth (i.e. 0 is a regular value) and therefore is a complex manifold.

If $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a *homogeneous polynomial of degree* $k$, i.e. $f(\lambda z_0, \ldots, \lambda z_n) = \lambda^k f(z_0, \ldots, z_n)$, e.g. $f(z_0, z_1) = z_0^3 + z_0 z_1^2 + z_1^3$, then $Z(f) \subset \mathbb{C}^{n+1}$, and if 0 is a regular value (which is not always true), then the homogeneity guarantees that $Z(f)$ descends as a submanifold of $\mathbb{CP}^n$. Now we can talk about varieties and all that.

(8) Another large class is the *complex Lie groups*, groups with a complex manifold structure such that multiplication and inversion (equivalently $(x, y) \mapsto x^{-1}y$) are holomorphic. For example $\mathrm{GL}(n, \mathbb{C})$ is a complex Lie group, but $\mathrm{U}(n)$ is not (for example, $\mathrm{U}(1) = S^1$ isn't even-dimensional!), ultimately because all compact, complex Lie groups are abelian (because the adjoint map is holomorphic and bounded, and therefore constant), and then (a bit harder to show) one can show they're all tori.

A lot of these examples were quotients; other examples exist, but they're sometimes harder to construct.

**Another Perspective.** For the purposes of deformation theory, there's a different, equivalent definition of a complex manifold, and it will be useful for some other cases, too. This will be more linear-algebraic, involving differential calculus or tensor calculus on complex manifolds. The idea is to take the linear-algebraic notion of an almost complex structure and then require integrability. In the process of defining this, we can also discuss the Dolbeault cohomology.

If $M$ is a complex manifold, one would expect $TM$ to be a complex vector bundle, and that's what we're going to see.

**Definition.** An *almost complex structure* on a real manifold $M$ is a vector bundle endomorphism $\mathcal{J} \in \mathrm{End}(TM)$ such that $\mathcal{J}^2 = -\,\mathrm{id}$.

*Note.* $\mathcal{J}$ makes $TM$ into a complex vector bundle, as $(a + ib) \cdot X = aX + b\mathcal{J}X$ for all $a, b \in \mathbb{R}$ and $X \in TM$. Thus, $\mathcal{J}$ is really just multiplication by $i$.

We'll define complex vector bundles later, but the general idea is that each fiber is a complex vector space and the trivialization functions can be chosen to be complex linear.

**Lemma 2.2.** *If $M$ is a complex manifold, then multiplication by $i$ in each holomorphic coordinate chart induces a well-defined almost complex structure on $TM$.*

*Proof.* We'll differentiate multiplication by $i$ and see what happens, and then prove its independence of choice of holomorphic chart.

Let's pick holomorphic coordinates $(z_1, \ldots, z_n)$, and decompose them into $z_k = x_k + iy_k$, so we have basis vectors $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial y_k}$ for the tangent space. Then, define $\mathcal{J}$ as last lecture, i.e. $\mathcal{J}\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}$ and $\mathcal{J}\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}$, and write $z = (x, y)$, so $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$. Then, $\mathcal{J}$ is the same matrix as last time, and $\mathcal{J}^2 = -\,\mathrm{id}$ (as a matrix or just by the action on the basis), and it's independent of a change of coordinates, because a change of coordinates $\psi$ is holomorphic, i.e. it commutes with $\mathcal{J}$. ⊠

For the rest of this lecture, assume $\mathcal{J}$ is an almost complex structure. A lot of the linear algebra we want still works for almost complex structures, though not all of it. Since $\mathcal{J}^2 = -\,\mathrm{id}$, then its eigenvalues are $\pm i$. After we complexify, it can even be diagonalized.

Consider the *complexified tangent space* $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$. Tensoring with $\mathbb{C}$ is what is meant by complexifying. $T_{\mathbb{C}}M$ is a complex vector space of dimension $n$.

**Lemma 2.3.** *Let $T^{1,0}(M)$ (resp. $T^{0,1}(M)$) denote the $i$ (resp. $-i$) eigenbundle (i.e. eigenspace at each fiber) of $\mathcal{J}$. Then:*

*(1)* $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$.
*(2)* $T^{1,0}M = \{X - i\mathcal{J}X \mid X \in TM\}$.
*(3)* $T^{0,1}M = \{X + i\mathcal{J}X \mid X \in TM\}$.

*Proof.* Regard $Z \in T_{\mathbb{C}}M$ as $Z = X + iY$ for $X, Y \in TM$. Then, define $T^{1,0}M$ and $T^{0,1}M$ as in (2) and (3), respectively; one can check that for all $Z \in T^{1,0}M$, $\mathcal{J}Z = iZ$, because $\mathcal{J}(X - i\mathcal{J}X) = \mathcal{J}X - i\mathcal{J}^2X = \mathcal{J}X + iX = i(X - i\mathcal{J}X)$. Then, the decomposition (1) is immediate, because any $X$ satisfies $2X = (X - i\mathcal{J}X) + (X + i\mathcal{J}X)$. ⊠

Next time, we'll talk about the integrability condition.

## 3. Tensor Calculus for Complex Manifolds: 1/13/15

Last time, we started talking about tensor calculus on the tangent bundle, but we can place it in the more general setting of complex vector spaces to make it useful in more places.

**Definition.** A *complex vector space* is a real vector space $V$ together with a real linear $\mathcal{J} : V \to V$ such that $\mathcal{J}^2 = -\operatorname{id}$.

This gives it the structure of a vector space over $\mathbb{C}$, but this alternate definition is useful as well.

**Definition.** The *complex conjugage* of a complex vector space $(V, \mathcal{J})$ is $\overline{V} = (V, -\mathcal{J})$.

We can *complexify* $V$ by taking $V \otimes_{\mathbb{R}} \mathbb{C}$; it's already complex, but now we have the structure induced by $\mathbb{C}$, so we can take its $i$-eigenspace $V^{1,0}$ and its $-i$-eigenspace $V^{0,1}$; thus, $V \otimes_R \mathbb{C} = V^{1,0} \oplus V^{0,1}$. As complex vector spaces, $V^{1,0} = \overline{V^{0,1}} \cong (V, \mathcal{J})$ (where $V^{1,0}$ has complex structure given by multiplication by $i$).

This extends to the dual $V^*$ and exterior powers in the same way.

*Note.* If $V$ is already a complex vector space, then $V \otimes_R \mathbb{C} = V \oplus \overline{V}$. (It also has a quaternionic structure, but that's less important.)

Now, we can apply this linear algebra to manifolds. Assume $(M, \mathcal{J})$ is an almost complex manifold; then, we can apply the above to its tangent space: $TM \otimes_R \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. To use forms, we'll look at the cotangent bundle $T^*M = \Lambda^1 M$. We can complexify it, to obtain $\Lambda^1_C M = \Lambda^{1,0} \oplus \Lambda^{0,1}$.

**Definition.** If $M$ is a complex manifold with local holomorphic coordinates $(z_1, \dots, z_n)$. $T^{1,0}M$ is also denoted $\tau M$, and is called the *holomorphic tangent space*; its local basis is $\left\{\frac{\partial}{\partial z_k}\right\}$. Similarly, $T^{0,1}M = \overline{\tau}M$ is called the *anti-holomorphic tangent space*, and has a local basis $\left\{\frac{\partial}{\partial \overline{z}_k}\right\}$.

We want the dual structure to factor through the direct sum, so let

$$\Lambda^{1,0} = \{\omega \in \Lambda^1_{\mathbb{C}} M \mid \omega(X) = 0 \text{ for all } X \in T^{0,1}M\},$$

and $\Lambda^{0,1}$ is defined similarly (with $T^{1,0}$ in place of $T^{0,1}$). Thus, $\Lambda^{1,0} = \{\eta - i\eta \circ \mathcal{J} \mid \eta \in \Lambda^1 M\}$, and $\Lambda^{0,1} = \{\eta + i\eta \circ \mathcal{J} \mid \eta \in \Lambda^1 M\}$.

We can extend this to $(p, q)$-forms by using the fact that

$$\Lambda^k(E \oplus F) = \bigoplus_{j=1}^{k} \Lambda^j E \otimes \Lambda^{k-j} F.$$

Thus, we just wedge $p$ things in $\Lambda^{1,0}$ and $q$ things in $\Lambda^{0,1}$ together to get a basis, i.e. $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$. Then, $\Lambda^k_{\mathbb{C}} M = \bigoplus_{p+q=k} \Lambda^{p,q} M$.

**Definition.** Denote by $A^{p,q}$ the space of smooth $(p, q)$-forms on $M$. That is, this is the space of sections $\Gamma(\Lambda^{p,q})$.

In differential geometry, this space of sections is usually denoted $\Omega$, but this will be used for holomorphic sections later.

Again, if $M$ is a complex manifold, we have some nice bases.

- A basis for $\Lambda^{1,0}$ is given by $\mathrm{d}z_k = \mathrm{d}x_k + i\,\mathrm{d}y_k$.
- Similarly, a basis for $\Lambda^{0,1}$ is given by $\mathrm{d}\overline{z}_k = \mathrm{d}x_k - i\,\mathrm{d}y_k$.
- If $\eta \in A^{p,q}$ (i.e. $\eta$ is a $(p, q)$-form), then in local coordinates

  $$\eta = \sum_{I,J} \eta_{IJ}\, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J,$$

  where $\mathrm{d}z_i = \mathrm{d}z_{i_1} \wedge \dots \wedge \mathrm{d}z_{i_p}$ and $\mathrm{d}\overline{z}_J = \mathrm{d}\overline{z}_{j_1} \wedge \dots \wedge \mathrm{d}\overline{z}_{j_q}$, where $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$.[1] The coefficients $\eta_{IJ}$ are smooth functions.

[1] Sometimes, the notation $\eta_{I\overline{J}}$ is used. This has no content, but is an interesting reminder, and sometimes is useful for keeping track of things.

All of this linear algebra works for almost complex manifolds, so let's talk about integrability, which will make the difference.

**Definition.** Assume $(M, \mathcal{J})$ is an almost complex structure. Then, the Nijenhuis[2] tensor is

$$N_{\mathcal{J}}(X, Y) = [X, Y] + \mathcal{J}[\mathcal{J}X, Y] + \mathcal{J}[X, \mathcal{J}Y] - [\mathcal{J}X, \mathcal{J}Y].$$

Here, $X, Y \in TM$, though we can complexify.

**Exercise 3.1.** Check that $N_{\mathcal{J}}$ is actually a tensor.

The Nijenhuis tensor's signs aren't arbitrary, and we'll see how to derive them,

**Proposition 3.2.** *If $\mathcal{J}$ comes from a complex structure on $M$, then $N_{\mathcal{J}} = 0$.*

This can be proven by checking for $X, Y \in \left\{ \frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k} \right\}$.
The converse is also true, which is the harder direction.

**Theorem 3.3** (Newlander-Nirenberg Integrability Theorem)**.** *If $N_{\mathcal{J}} = 0$, then $\mathcal{J}$ comes from a complex structure on $M$.*

*Proof sketch.* If $N_{\mathcal{J}} = 0$ and $\mathcal{J}$ is real analytic, then the Frobenius integrability theorem tells us that $\mathcal{J}$ is integrable. But the hard part is the regularity result: it's beautiful, but hard to show that $N_{\mathcal{J}} = 0$ induces a differential equation which leads $\mathcal{J}$ to be real analytic.

**Proposition 3.4.** *Assume $\mathcal{J}$ is an almost complex structure on a smooth manifold $M$. Then, the following are equivalent:*

*(1)* $\mathcal{J}$ *comes from a complex structure on $M$.*
*(2)* $N_{\mathcal{J}} = 0$.
*(3)* $T^{1,0}M$ *is* formally integrable, *i.e. for any $Z, W \in T^{1,0}M$, $[Z, W] \in T^{1,0}M$.*
*(4)* $T^{0,1}$ *is formally integrable.*
*(5)* $\mathrm{d}(A^{1,0}) \subseteq A^{2,0} \oplus A^{1,1}$ *(i.e. there's no $(0, 2)$-component).*
*(6)* $\mathrm{d}(A^{p,q}) \subseteq A^{p+1,q} \oplus A^{p,q+1}$.

When we define $\overline{\partial}$, we will see that this is also equivalent to $\overline{\partial}^2 = 0$.

*Proof sketch.* We're only going to show the easier parts, as we saw in the proof sketch of Theorem 3.3 it can get quite difficult.

To see why (2) $\iff$ (3), suppose $Z \in T^{1,0}M$, so that $Z = X - i\mathcal{J}X$ for $X \in TM$. If we take $X, Y$ to be local vector fields on $M$, then we can regard them as vector fields in $T^{1,0}M$; then, let $Z = X - i\mathcal{J}X$ and $W = Y - i\mathcal{J}X$, so one can check that $[Z, W] = N_{\mathcal{J}}(X, Y) + i\mathcal{J}N_{\mathcal{J}}(X, Y)$.

To see why (3) $\iff$ (5), take a $(1, 0)$-form $\omega$, so that $\omega(Z) = 0$ for $Z \in T^{0,1}M$. Thus, $\mathrm{d}\omega$ has no $(0, 2)$-part. This is equivalent to showing that $\mathrm{d}\omega(Z, W) = 0$ for all $Z, W \in T^{0,1}M$. But we can expand this out to

$$\mathrm{d}\omega(Z, W) = Z \cdot \omega(W) - W \cdot \omega(Z) - \omega[Z, W],$$

but the condition given shows $\omega(Z) = \omega(W) = 0$, so therefore $\mathrm{d}\omega$ has no $(0, 2)$-part iff the Lie bracket $[Z, W] \in T^{0,1}M$ for all $Z, W \in T^{0,1}M$, which is (4), and by complex conjugation, we get (3). Then, by the Leibniz rule, this can be extended to (6), and the other direction is immediate.

What's left is the equivalences of (1), (2), and (3), which is more or less the content of Theorem 3.3 and uses the Frobenius integrability theorem. ⊠

Now, let's use this.

**Proposition 3.5.** *Any almost complex structure on a real, orientable 2-manifold is integrable.*

*Proof.* This is by dimensionality: $N_{\mathcal{J}}(X, X) = 0$ and $N_{\mathcal{J}}(X, \mathcal{J}X) = 0$, so that's all we need for a basis. ⊠

*Note.* An almost complex structure is a topological condition (which we'll see because it involves cohomology), and therefore the only spheres that admit almost complex structures are $S^2$ and $S^6$.

On $S^6$, one can explicitly create a nonintegrable almost complex structure by considering it to be the unit sphere in $\mathbb{R}^7$, which can be regarded as the imaginary part of the *octonions* $\mathbb{O}$ (sometimes this is called the *Cayley numbers*). Then, there is a definition of a cross product $p \times v \perp p$ if $v \perp p$, and then $\mathcal{J}_p(v) = p \times v$. Calabi showed this isn't integrable; there may be others which could be integrable, but this is still open.

[2] Prononced "nigh-house."

Condition (2) of Proposition 3.4 implies that if $G$ is a complex Lie group, then its Lie algebra $T_eG$ is also a complex Lie algebra.

One would expect (and it's true) that if $M$ is a compact complex manifold, then $\mathrm{Aut}(M)$, the set of biholomorphic functions $M \to M$, is a Lie group. Its Lie algebra is the space of infinitesimal automorphisms of $M$, i.e. flows of "holomorphic" vector fields. We haven't defined these precisely; that's all right.

**Definition.** If $M$ is a complex manifold, then a *holomorphic vector field* on $M$ is a vector field $= X - i\mathcal{J}X \in T^{1,0}M$ (i.e. $X$ is real) such that the flow of $X$ consists of holomorphic maps, i.e. $\mathcal{L}_X\mathcal{J} = 0$.

Later, we'll see that these are equivalent to holomorphic sections of $\tau M$, the holomorphic vector bundle. For now, if $(z_1, \dots, z_k)$ are holomorphic coordinates on $M$, then

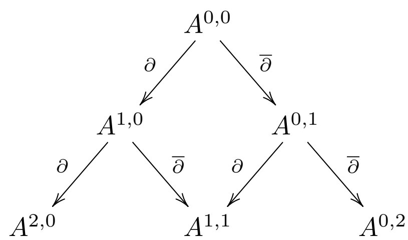
$$Z = \sum_k a_k(z_1, \dots, z_n)\frac{\partial}{\partial z_k},$$

where the $a_k$ are holomorphic functions. Again, to show all of these things, there's a lot of linear algebra.

**Dolbeault cohomology.** Suppose $M$ is a complex manifold, so that condition (6) of Proposition 3.4 implies that we have a map $\mathrm{d} : A^{p,q} \to A^{p+1,q} \oplus A^{p,q+1}$. Thus, we can define $\partial = \pi_{p-1,q} \circ \mathrm{d}$ and $\overline{\partial} = \pi_{p,q+1} \circ \mathrm{d}$. Thus, $\partial : A^{p,1} \to A^{p+1,q}$ and $\overline{\partial} : A^{p,q} \to A^{p,q+1}$.

These are both differential operators (i.e. $\mathbb{C}$-linear and satisfying the Leibniz rule, which one can check); furthermore $\partial^2 = \overline{\partial}^2 = 0$, and they anti-commute, as $\partial\overline{\partial} + \overline{\partial}\partial = 0$. This can be checked because $\mathrm{d}^2 = 0$, and then expanding out.

Now, we have something called the *Hodge complex*:




and so on.

**Definition.** The $(p, q)$ *Dolbeault cohomology* group $H^{p,q}(M)$ is the (co)homology of $A^{p,\bullet} \xrightarrow{\overline{\partial}} A^{p,\bullet+1}$. That is, it is the $\overline{\partial}$-closed $(p, q)$-forms on $M$ mod the $\overline{\partial}$-exact $(p, q)$-forms on $M$.

## 4. Dolbeault Cohomology and Holomorphic Vector Bundles: 1/15/15

*"I don't know why, but I always forget the $1/(2\pi i)$ term in the Cauchy integral formula!"*

Recall that we had a chain complex $A^{p,0} \xrightarrow{\overline{\partial}} A^{p,1} \xrightarrow{\overline{\partial}} \cdots$, and the Dolbealut cohomology group $H^{p,q}(M)$ is the $q^{\text{th}}$ homology group of the complex.

**Definition.** The *Hodge numbers* $h^{p,q}(M)$ of the manifold are given by the ranks of the respective Dolbeault cohomology groups.

It's hard to calculate this from the definition, like any cohomology.

**Example 4.1.** However, one simple case is $H^{0,0}$, because $\overline{\partial}$-closed $(0, 0)$-forms are holomorphic functions, and there are locally lots of them. In general, this could be infinite-dimensional, but if $M$ is compact and connected, they're globally only constant, so $H^{0,0}(M) = \mathbb{C}$, which is nice.

Why do we care about this construction? One useful case is the existence of an invariant of the complex structure. Another useful thing is that it's contravariant functorial: if $f : M \to N$ is a holomorphic map of complex manifolds, then there's an induced $f^* : H^{p,q}(N) \to H^{p,q}(M)$, and it is a group homomorphism. But Dolbeault cohomology is also useful in local deformation or obstruction theory, which can be useful for keeping track of obstructions to the existence of a complex structure on a manifold.

Suppose one wants to solve the equation $\overline{\partial}\beta = \alpha$ for a $(p, q)$-form $\alpha$. If this were to have a solution, $\overline{\partial}\alpha = 0$ (i.e. $\alpha$ is $\overline{\partial}$-*closed*), which can be checked locally. If there is no obstruction, then the equation has local solutions, but we may not be able to patch them together to get a global solution on $M$. The global obstruction is governed by Hodge theory, which says that if the cohomology class $[\alpha] \in H^{p,q}(M)$ vanishes, then we can do this.

There are two powerful tools to deal with this.

- Sheaf theory adopts the approach of looking at locally holomorphic functions and forms, and spends time worrying about how they patch together, which sheaves help with.
- A more analytic approach is to use Sobolev spaces, which relax the holomoprhic condition, but require one to worry about regularity.

Ideally, one could use both, but one may be more useful than the other in a given situation.

This lemma is really due to Dolbeault, but has Poincaré's name on it for some reason.

**Lemma 4.2** ($\overline{\partial}$-Poincaré)**.** *Any $\overline{\partial}$-closed $(p,q)$-form is locally $\overline{\partial}$-exact.*

As a corollary, if $P$ is the polydisc, then $H^{p,q}(P) = 0$ if $q > 0$.

*Proof sketch.* For the complete proof, see Huybrechts' book, §1.3.

We'll use the polydisc for working locally, since analysis is easier in compact spaces.

In one variable, assume $g$ is smooth on the closed unit disc $\overline{\mathbb{D}}$; then, we want an $f$ such that $\overline{\partial} f = g\,\mathrm{d}\overline{z}$. We can explicitly construct a solution:

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{g(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\overline{w},$$

which is smooth on $\mathbb{D}$ and satisfies $\overline{\partial} f = g\,\mathrm{d}\overline{z}$. This was a version of the Cauchy integral formula.

Now, let's talk about $(0,1)$-forms in $n$ variables; let

$$\alpha = \sum_{k=1}^{K} \alpha_j\,\mathrm{d}\overline{z}_j,$$

where $K \le n$. We'll induct on $K$; since $\alpha$ is $\overline{\partial}$-closed, then each $\alpha_j$ is holomorphic in $z_{K+1},\dots,z_n$.

One can integrate (made more precise in the book) the coefficient $\alpha_K$ of $\mathrm{d}z_K$ to get a $\beta$ such that $\alpha - \overline{\partial}\beta$ is a linear combination of $\mathrm{d}\overline{z}_1,\dots,\mathrm{d}\overline{z}_{K-1}$, which is the necessary inductive step.

Then, we can generalize to $(0,q)$-forms from $(0,1)$-forms. Let $A_K^{p,q}$ be space of $(p,q)$-forms that only involve $\mathrm{d}\overline{z}_1,\dots,\mathrm{d}\overline{z}_K$. Then, filter $A^{0,q}$ by $A_K^{0,q}$; one can show that if $\alpha \equiv 0 \bmod A_K^{0,q}$ and $\alpha$ is $\overline{\partial}$-closed, then one can find a $\beta$ such that $\alpha - \overline{\partial}\beta \equiv 0 \bmod A_{K-1}^{0,q}$.

Finally, generalizing yet further to $(p,q)$-forms is immediate from $(0,q)$, since we're merely adding $\mathrm{d}z$ terms. ⊠

For some purposes, it may be easier to have cohomology vanish on the open polydisc, which can be analyzed by exhausting it by closed polydiscs that approach it in the limit. Thus, there's an extension problem, of taking a solution on one polydisc and extending it to a new solution on a larger polydisc which agrees (or mostly agrees) on the smaller one.

Often in PDE theory, one considers a differential equation $Df(x) = g(x)$ (where $D$ is a differential operator). One way to solve this (intuitively, though it can be made precise) is to try to find a *kernel* $k(x,y)$ which satisfies $D_x k(x,y) = \delta_{x,y}$.[3] Then, general principles tell us that

$$f(x) = \int k(x,y)g(y)\,\mathrm{d}y$$

is a solution of the differential equation. There are lots of important issues of convergence and making sense of this, but in the context where this works,

$$D_x f = \int D_x k(x,y) g(y)\,\mathrm{d}y = \int \delta_{xy} g(y)\,\mathrm{d}y = g(x).$$

This applies to complex geometry: for us, $D = \overline{\partial}$, and we want to solve $\overline{\partial} f = g$. Our kernel was the *Cauchy kernel*

$$k(z,w) = \frac{\mathrm{d}w}{2\pi i(z-w)}.$$

Well, this is a one-form and not a function, and we're secretly using a metric, and so on; this isn't rigorous yet, but it can be and will be; this is the intuition for later on.

**Lemma 4.3** (Local $\partial\overline{\partial}$ Lemma)**.** *Let $M$ be a complex manifold and $\omega$ be a real $(1,1)$-form. Then, $\mathrm{d}\omega = 0$ iff $\omega$ is locally of the form $\omega = i\partial\overline{\partial}u$ for a (locally defined) real-valued function $u$.*

In this case, $u$ is called the *potential*.

[3] The $\delta$ function is not a function; it's a distribution, of course, but the equation can be reformulated such that this is all right.

*Proof.* Suppose we know that $\omega = i\partial\overline{\partial}u$, and we want to show that it's closed. Well, since $\omega$ is a $(1,1)$-form,

$$\mathrm{d}\omega = (\partial + \overline{\partial})(\partial\overline{\partial}u) = \partial^2\overline{\partial}u + \overline{\partial}\partial\overline{\partial}u = -(\overline{\partial})^2\partial u = 0,$$

because they anticommute.

The other direction uses both the real and complex versions of the Poincaré lemma; the real one is similar to the complex one, but follows from Stokes' theorem. Assume $\omega$ is d-closed, so that $\mathrm{d}\omega = 0$. Then. by the real Poincaré lemma, $\omega = \mathrm{d}\tau$, for some real-valued 1-form $\tau$. Then, thinking of $\tau$ as complex, we can uniquely decompose it as $\tau = \tau^{1,0} + \tau^{0,1}$; since $\tau$ is real, then $\tau^{0,1} = \overline{\tau^{1,0}}$ (which follows from the formula). Thus,

$$\omega = \mathrm{d}\tau = \underbrace{\partial\tau^{1,0}}_{(2,0)} + \underbrace{\overline{\partial}\tau^{1,0} + \partial\tau^{0,1}}_{(1,1)} + \underbrace{\overline{\partial}\tau^{0,1}}_{(0,2)}.$$

But since this is a unique decomposition and $\omega$ is a $(1,1)$-form, then $\partial\tau^{1,0} = 0$, and similarly $\overline{\partial}\tau^{0,1} = 0$ too; thus, $\omega = \overline{\partial}\tau^{1,0} + \partial\tau^{0,1}$.

By Lemma 4.2, $\tau^{0,1} = \overline{\partial}f$ for a (local) complex-valued function $f$, and similarly the complex conjugate $\tau^{0,1} = \partial\overline{f}$. Thus,

$$\begin{aligned}\omega &= \overline{\partial}\tau^{1,0} + \partial\tau^{0,1} = \overline{\partial}\partial\overline{f} + \partial\overline{\partial}f \\ &= 2\partial\overline{\partial}(i\,\mathrm{Im}(f)),\end{aligned}$$

so let $u = \mathrm{Im}(f)$. ⊠

This proof follows a common approach of doing one thing from analysis followed by a bunch of linear algebra.

We've talked about holomorphic functions, so it may also be useful to discuss holomorphic forms as well.

**Definition.** A *holomorphic p-form* is a $(p,0)$-form $\omega$ such that $\overline{\partial}\omega = 0$.

Equivalently, $\omega$ is a holomorphic $p$-form iff it can be written as

$$\omega = \sum \omega_I \,\mathrm{d}z_I,$$

where the $\omega_I$ are holomorphic functions. It turns out this will also be equivalent to $\omega$ being a section in a holomorphic vector bundle.

Note that if $q \neq 0$, a $(p,q)$-form $\alpha$ such that $\overline{\partial}\alpha = 0$ isn't called holomorphic; we'l see why in a bit.

**Holomorphic Vector Bundles.** Intuitively, a holomorphic vector bundle is a complex vector bundle such that the transition functions are holomorphic. Equivalently, it is a complex vector bundle with a $\overline{\partial}$ operator such that $\overline{\partial}^2 = 0$. This last condition means that Dolbeault cohomology only makes sense over holomorphic vector bundles.

While one can define real and complex vector bundles over any topological space, holomorphic vector bundles can only exist over complex manifolds.

**Definition.** Let $M$ be a complex manifold. Then, a *holomorphic vector bundle* of rank $r$ is a complex vector bundle $\pi : E \to M$ such that the following hold.

- There exist holomorphic charts $U_\alpha$ covering $M$ and holomorphic local trivializations, i.e. the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times \mathbb{C}^r \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_1} & \\ & U_\alpha & \end{array}$$

- The transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{C})$ are holomorphic. These are given as follows: going from $\beta \to \alpha$, we have for $x \in M$ and $e \in \mathbb{C}^r$, $\psi_\alpha \circ \psi_\beta^{-1}(x,e) = (x, g_{\alpha\beta}(x)e)$.

**Example 4.4.** Let $M$ be a complex manifold.

- The *trivial bundle* $M \times \mathbb{C}^r$ is of course a holomorphic vector bundle.
- $\tau M = T^{1,0}M$ (i.e. involving just $\frac{\partial}{\partial z}$) is a holomorphic vector bundle (but $T^{0,1}M$ isn't).
- Its dual $(\tau M)^* = \Lambda^{1,0}M$ is holomorphic as well (which follows from its functorial properties); in complex and algebraic geometry, this is often denoted $\Omega_M$, called the *bundle of holomorphic one-forms.*
- $\Omega^p = \Lambda^{p,0}M$, the space of $(p,0)$-holomorphic forms.[4]

[4] These forms are sections of the bundle, so we should speak carefully, but they are frequently talked about as the same spaces. A bundle can also be thought of as a sheaf of sections.

- The top power is a holomorphic line bundle, called the *canonical line bundle*. It has the special notation $\Omega_M^n = \Lambda^{n,0}M$. These are the analogues of volume forms, but in complex geometry.

Here's why $\tau M$ is holomorphic: choose local holomorphic coordinates $(z_1, \ldots, z_n)$ on $M$, so the local trivialization of $\tau M$ has as a basis $\{\partial z_k\}$. We need to check that the change-of-coordinates and transition functions are holomorphic.

If $z = z(w)$ is a change of holomorphic coordinates, then

$$\frac{\partial}{\partial w_\ell} = \sum_k \frac{\partial z_k}{\partial w_\ell}\frac{\partial}{\partial z_k},$$

with no extra terms because $w$ is also holomorphic; thus, these coordinate changes are holomorphic.

Similarly, a trivialization of $\Omega^p M$ has basis $\{\mathrm{d}z_I\}_I$, where $|I| = p$ for multi-indices $I$; one has to check that

$$\mathrm{d}w_K = \sum_{|I|=p} \frac{\partial w_K}{\partial z_I}\,\mathrm{d}z_I$$

has holomorphic functions (which relates to the Jacobian of the change of coordinates $z \to w$).

## 5. Holomorphic Vector Bundles and More Dolbeault Cohomology: 1/20/15

*"You may have seen something like this [tautological line bundle] at IKEA."*

Recall that if $M$ is a complex manifold, we have the *holomorphic vector bundles*, sections $\Omega_M = \tau M^* = \Lambda^{1,0}M$, where $\tau M = T^{1,0}M$. Then, $\Omega_M^p = \Lambda^{p,0}M$. Be careful, though; these are generally not $\Lambda^{p,q}$ for $q > 0$.

A general principle here is tht most natural constructions on real vector bundles still work in the complex or holomorphic case. Here are some examples which stay holomorphic, where $E$ and $F$ are holomorphic vector bundles.

- The direct sum, $E \oplus F$.
- The tensor product $E \otimes F$.
- The dual $E^*$.
- The exterior powers $\Lambda^k E$, and as a special case, the *determinant* line bundle $\Lambda^r E$, where $r$ is the rank of $E$.
- The *pullback*: if $f : V \to M$ is holomorphic and $E \to M$ is a vector bundle, then $f^* : V \to N$ is a holomorphic vector bundle. One useful example is that if $V \subset N$ is a complex submanifold, the *restriction* of $E|_V$ is defined as the pullback under inclusion $i : V \hookrightarrow M$, and is holomorphic.[5]

Another useful fact about holomorphic vector bundles is that the total space is a complex manifold, with the projection holomorphic.

**Definition.** Suppose $V \subset M$ is a complex submanifold. Let $TM|_V$ denote the restriction of $TM$ to $V$; then, we have a short exact sequence

$$0 \longrightarrow TV \xrightarrow{\varphi} TM|_V \longrightarrow N_V \longrightarrow 0,$$

for a holomorphic bundle $N_V = \mathrm{coker}(\varphi)$ called the *normal bundle* of $V$.

Keep in mind that while short exact sequences of complex vector bundles always split (i.e. if $0 \to E \to F \to G \to 0$ is short exact, then $F \cong E \oplus G$), this is not true for holomorphic bundles. For example, there's a rank 2 bundle $E$ over an elliptic curve which fits into a short exact sequence $0 \to \mathbb{C} \to E \to \mathbb{C} \to 0$ (where $\mathbb{C}$ denotes the trivial line bundle), but does not split. Since this sequence splits as a sequence of complex line bundles, we end up seeing that the same complex manifold may have nonisomorphic holomorphic structures.

Nonetheless, the determinant behaves well in short exact sequences: if $0 \to E \to F \to G \to 0$ is short exact, then $\det F \cong \det E \otimes \det G$ as holomorphic line bundles. In particular, we have the *adjunction formula* $\det(TM|_V) \cong \det TV \otimes \det N_V$, which is used all the time.

**Example 5.1.** Consider the *tautological line bundle* $\tau$ over $\mathbb{CP}^n$. Geometrically, we want the fiber of $\tau$ at a line $\ell \in \mathbb{CP}^n$ to be the line $\ell$. Algebraically, consider $\tau$ to be the total space $\tau = \{(\ell, u) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid u \in \ell\}$, which comes with the projection $\pi$ onto the first factor.

This is a holomorphic line bundle: consider coordinate charts $U_i$ on $\mathbb{CP}^n$, so $z_i \neq 0$, which comes with the projection $\pi$ onto the first factor.

This is a holomorphic line bundle: consider coordinate charts $U_i$ on $\mathbb{CP}^n$, so $z_i \neq 0$. The trivialization will be $\psi_i([\mathbf{z}], u) = ([\mathbf{z}], u_i)$, which (you can check) plays well with transition functions $g_{ij}([\mathbf{z}])t = z_i/z_j$, which is clearly holomorphic.

When $n = 1$, $\mathbb{CP}^1$ is the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Thus, we have two charts, $z \in \mathbb{C}$ and $w = 1/z$, whose overlap is $\mathbb{C}^\times$ (one ignores 0, the other ignores $\infty$). Thus, the transition functions are $g(w)(t) = w^{-1}t$, for $w \in \mathbb{C}^\times$.

[5] One can also pull back under the constant map, but the result is the trivial bundle.

This seems a little silly, but more interesting constructions, e.g. the canonical bundle or the tangent bundle, can be described in terms of $\tau$ (e.g. $T^*M \cong \tau^{\otimes 2}$: the transitions of $T^*M$ are between $\mathrm{d}z$ and $\mathrm{d}w$, so $\mathrm{d}w = \mathrm{d}(1/z) = -1/z^2\,\mathrm{d}z$).

*Remark.* A complex or holomorphic vector bundle is determined by the transitions functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$ (where $r$ is the rank of the vector bundle), such that $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, for all charts $\alpha$, $\beta$, and $\gamma$. This is known as the *Čech cocycle condition*, and relates to *Čech cohomology*, which we'll discuss later.

**Dolbeault Cohomology with Values in a Holomorphic Vector Bundle.** Let $E \to M$ be a rank-$r$ holomorphic vector bundle over an $n$-dimensional manifold. Choose a holomorphic trivialization (a local basis for the fiber) $\{e_k\}_{k=1,\dots,r}$, and let $\Lambda^{p,q}(E) = \Lambda^{p,q}_M \otimes E$. Then, we can define

$$\overline{\partial}_E \left( \sum_{k=1}^{n} \omega_k \times e_k \right) = \sum_{k=1}^{k} (\overline{\partial}\omega_k) \otimes e_k,$$

where $\omega_k$ are $(p, q)$-forms, i.e. in holomorphic coordinates $(z_1, \dots, z_n)$,

$$\omega_k = \sum_{I,J} \omega_k^{I\overline{J}}\, \mathrm{d}z_I\, \mathrm{d}z_J.$$

Since $\overline{\partial}^2 = 0$, then $\overline{\partial}_E^2 = 0$ as well, of course. And we can do some of the same stuff as before, taking smooth sections $A^{p,q}(E)$ of $\Lambda^{p,q}(E)$ (since we can only differentiate sections, such as vector fields, rather than vectors themselves), and we get maps

$$A^{p,0}(E) \xrightarrow{\overline{\partial}_E} A^{p,1}(E) \xrightarrow{\overline{\partial}_E} A^{p,2}(E) \xrightarrow{\overline{\partial}_E} \cdots \tag{1}$$

This should look very familiar, and if $E$ is the trivial line bundle we recover the Dolbeault cohomology from last time.

**Definition.** The *Dolbeault cohomology of $M$ with values in $E$* $H^{p,q}(M; E)$ is the homology of the complex (1), i.e. the $\overline{\partial}_E$-closed $(p, q)$-forms with values in $E$ modulo the $\overline{\partial}_E$-exact $(p, q)$-forms.

As a special case, $H^{0,0}(E)$ (sometimes $\Omega(E)$ or $H^0(M; E)$; the latter notation comes from Čech cohomology) is the sections $\sigma$ of $E$ which are holomorphic and satisfy $\overline{\partial}_E \sigma = 0$, i.e.

$$\sigma = \sum_k \sigma_k(z) \otimes e_k$$

in a holomorphic frame, were the coefficients $\sigma_k$ are holomorphic functions on $M$. Additionally, the tautological vector bundle $\tau \to \mathbb{CP}^n$ has no holomorphic sections (other than the zero section), which is a little bit of a chore to calculate.

There are more interesting examples, such as that of $\tau^*$; one appears in the homework.

*Remark.* By the $\overline{\partial}$-Poincaré lemma (which still works just as well in this setup), over a polydisc (i.e. locally), (1) has trivial homology except in degree 0, where one gets $\Omega(E) = H^{0,0}(M; E)$. This is called an *affine* or an *acyclic resolution*, and leads to the following theorem.

**Theorem 5.2** (Dolbeault)**.** *If $H^q(M, F)$ denotes the $q^{\text{th}}$ Čech cohomology with coefficients in $F$, then $H^{p,q}(M; E) = H^q(M, \Omega^p_M \otimes E)$.*

This will be a little less mysterious once we actually define Čech cohomology.

**Integrability of Complex Vector Bundles.** The question is, when is a complex vector bundle (over a complex manifold) a holomorphic vector bundle? The answer will be when there is an $\overline{\partial}_E$ operating on smooth sections of $E$, i.e. $\overline{\partial}_E : \Lambda^{0,0}(E) \to \Lambda^{0,1}(E)$ that acts as a differential operator. That is, it satisfies the Leibniz rule: when $\omega$ is a $(p, q)$-form and $\sigma$ is a section of $E$,

$$\overline{\partial}_E(\omega \otimes \sigma) = (\overline{\partial}\omega) \otimes \sigma + (-1)^q \omega \otimes \overline{\partial}_E \sigma.$$

Furthermore, we require $\overline{\partial}_E^2 = 0$.

Once this exists, it can be extended using complex linearity and the Leibniz rule to $\overline{\partial}_E : \Lambda^{p,q}(E) \to \Lambda^{p,q+1}(E)$.

**Theorem 5.3.** *Assume $E \to M$ is a complex vector bundle over a complex manifold $M$. Then, a holomorphic structure on $E$ is uniquely determine by a $\mathbb{C}$-linear differential operator $\overline{\partial}_E : \Lambda^{0,0}(E) \to \Lambda^{0,1}(E)$ such that $\overline{\partial}_E^2 = 0$.*

This should be thought of as a linear version of Theorem 3.3, but for vector bundles.

*Proof sketch.* All we have to do is find holomorphic trivializations to recover the holomorphic vector bundle, so let's do that.

Notice that $\overline{\partial}_E^2 = 0$ iff we can locally find a basis of $E$ consisting of holomorphic sections of $E$ (there's a little more to say here, but it's not as complicated as it looks). Now, we want to get local trivializations out of this. Going from a holomorphic bundle to this is relatively easy, so let's focus on the more difficult direction; suppose $E$ is a complex vector bundle with a local basis $\{\sigma_k\}$ of smooth sections of $E$.

Here are two things we know: that $\overline{\partial}_E^2 = 0$, and

$$\overline{\partial}_E \sigma_k = \sum_{j=1}^{n} \tau_{kj} \otimes \sigma_j, \tag{2}$$

where $\overline{\partial}_E \sigma = \tau \otimes \sigma$. Here, $\sigma = (\sigma_1, \ldots, \sigma_n)$ in this basis, and $\tau = (\tau_{ij})$ is a matrix of $(0,1)$-forms.

Applying the Leibniz rule to (2), we get that since $\overline{\partial}_E^2 \sigma_k = 0$, then

$$\overline{\partial}\tau + \tau \wedge \tau = 0.$$

Expanding out $\tau = (\tau_{jk})$, this also looks like

$$\overline{\partial}\tau_{k\ell} + \sum_j \tau_{kj} \wedge \tau_{j\ell} = 0.$$

Now, we want to find a change of basis $f : U \to \mathrm{GL}(r, \mathbb{C})$ from our basis $\{\sigma_k\}$ to a holomorphic basis $f = (f_{k\ell})$. Let $e_k = \sum f_{kj}\sigma_k$; we will want $\overline{\partial}_E e_k = 0$, i.e. $\overline{\partial} f + f \cdot \tau = 0$. This is a PDE, and locally the obstruction to solving it turns out to be precisely the requirement that $\overline{\partial}\tau + \tau \wedge \tau = 0$.[6]

This last part is a little confusing (what is everything exactly?), but when $M$ is one-dimensional, a lot of these just become functions, and the PDE has no obstruction; it's quite easy to integrate. In particular, $\overline{\partial}(\log f) + \tau = 0$, which is exactly the local Poincaré lemma. In higher dimensions, one looks for a local solution $f = \exp(F)$ (the Lie group exponential), where $F : U \to \mathfrak{gl}(r, \mathbb{C})$ (i.e. the Lie algebra), such that $\overline{\partial} F + \tau = 0$. This is again the Poincaré lemma, with values in the Lie algebra, since in some vague sense $\log(f) = F$.

## 6. Hermitian Bundles and Connections: 1/22/15

*"You shouldn't trust any of the signs I write... also any factors of $2\pi i$."*

Today, we'll add extra data in the form of a connection and a metric, and will try to make it compatible with the complex structure and a bundle. There's an obstruction, which vanishes when the bundle is holomorphic, leading to a unique connection called the *Chern connection*, which is analogous to the Levi-Civita connection (especially when the given bundle is the tangent bundle).

There will be a lot of different structures here, so be careful with exactly what goes where.

**Connections.** Let $E \to M$ be a real or complex vector bundle over a real manifold $M$.

**Definition.** A *connection* is a linear differential operator $\nabla : \Gamma(E) \to \Lambda^1(E)$, i.e. from smooth sections[7] to one-forms on $E$ (i.e. $\Lambda^1(E) = \Gamma(\Lambda^1 \otimes_M E)$), that satisfies the Leibniz rule

$$\nabla(f \cdot \sigma) = \mathrm{d}f \cdot \sigma + f\nabla\sigma.$$

Note that the linearity is real if $E$ is real, and complex if $E$ is complex.

The Leibniz rule means that if $v \in TM$, then the *directional or covariant* or *covariant derivative* $\nabla_v \sigma$ corresponds to the usual notion of derivative.

We can extend $\nabla$ to $p$-forms with values in $E$, i.e. in $\Lambda^p(E)$, again by the Leibniz rule: if $\eta \in \Lambda^p(M)$ and $\sigma \in \Gamma(E)$, then

$$\nabla(\eta \otimes \sigma) = \mathrm{d}\eta \otimes \sigma + (-1)^p \eta \otimes \nabla\sigma.$$

Define the *curvature operator* $R_\nabla = \nabla^2$, or more explicitly $R_\nabla \sigma = \nabla(\nabla\sigma)$. This is a map from sections to sections, a 2-form with values in $\mathrm{End}(E)$. But we should do this in coordinates, so that we can actually calculate stuff.

[6] This is technically a slightly illegal shortcut, since it's not always true that $\tau \wedge \tau = 0$. This is true in many cases, e.g. Riemann surfaces, line bundles, etc. If this is not true, often $\tau$ is real analytic, so it can be solved by the Frobenius theorem, expanding power series, and so on. Alternatively, we seem to have a twisted $\overline{\partial}$ given by $(\overline{\partial} + \tau)f = 0$; one needs a gauge transformation to convert this to the standard $\overline{\partial}$.

[7] Eventually, we'll have to weaken this regularity, and it will be important how many times things are differentiated.

Consider a chart $U \subset M$, and consider the frame $\{e_k(x)\}$ in $E$. (If we haven't defined "frame" yet, it's a smoothly varying basis for each fiber.) Any section of $E$ can be locally written in this frame as

$$\sigma = \sum_k \sigma_k e_k,$$

where $\sigma_k$ is $\sigma$ on $U$. Then, look at the *connection 1-form* $(\tau_{ij})$ defined by

$$\nabla e_k = \sum_\ell \tau_{k\ell} \otimes \sigma_\ell.$$

This is a one-form on $M$.

Note: from now on we will sum over repeated indices. It will be helpful to be aware of this.

Now, we can calculate this on $\sigma = \sigma_k e_k$:

$$\nabla(\sigma_k e_k) = \mathrm{d}\sigma_k \otimes e_k + \sigma_k \tau_{kj} \otimes e_j = (\mathrm{d}\sigma + \sigma\tau)e_j.$$

Thus, we can say that $\nabla = \mathrm{d} + \tau$, which can be used to define the connection 1-form (matrix). This isn't defined globally, but is well-behaved in coordinate transforms.

Then, we have a *curvature matrix* $\Theta = (\Theta_{ij})$; if $r = \mathrm{rank}(E)$, then this is an $r \times r$ matrix. This is a matrix of 2-forms with values in $\mathrm{End}(E) = E \otimes E^*$, and obeys the *Cartan structure equation* $\Theta = \mathrm{d}\tau - \tau \wedge \tau$.[8]

What's really important is to calculate how a change of trivialization or of frame affects these objects. These are called *gauge transformations* in the bundle, if you do gauge theory (and, well, if you do gauge theory everything we just defined has different notation). If $\mathbf{e} = (e_1, \dots, e_r)$ and $\mathbf{e}'$ is a different frame related by a gauge transformation $\mathbf{e}' = g(\mathbf{e})$, then

$$\tau_{\mathbf{e}'} = g\tau_{\mathbf{e}} g^{-1} + \mathrm{d}g \cdot g^{-1}.$$

This will be on the homework. The curvature, however, pulls back without the extra term, and there are Bianchi identities floating around, and so forth.

This was all review. Right. . . ?

Anyways, this all works for complex vector bundles over a real manifold; one requires $\nabla$ to be complex linear, and satisfy the Leibniz rule for $f \in C^\infty(M; \mathbb{C})$ (complex-valued smooth functions).

**Metric.** Now, let's add some metrics. Assume $E \to M$ is a smooth complex vector bundle over a real manifold.

**Definition.** A *Hermitian metric* on $E$ is a Hermitian inner product on the fibers $E_x = \pi^{-1}(x)$ of $E$ which varies smoothly in $x \in M$. That is, $h_x : E_x \times E_x \to \mathbb{C}$ is positive definite and *complex sesquilinear*, i.e. it's complex linear in the first coordinate and $h(u, v) = \overline{h(v, u)}$. (These two conditions imply it's anti-complex-linear in the second variable.) That this varies smoothly means that in a smooth frame, the coefficients of $h$ are smooth functions, i.e. $h(u, v) \in C^\infty(M)$ for all $u, v \in \Gamma(E)$.

$E$ and $M$ together with the Hermitian metric is called a *Hermitian vector bundle.*

*Note.* One can regard $h$ as an anti-complex-linear isomorphism $h : E \to E^*$, or equivalently an isomorphism $\overline{E} \cong E^*$ preserving complex structure.

It turns out (surprise!) that the real part of a Hermitian metric is a Riemannian metric $g$, since it's positive definite, and has the property that $g(\mathcal{J}u, \mathcal{J}v) = g(u, v)$; that is, $\mathcal{J}$ (on the bundle!) is an isometry.

The imaginary part will show up later, but if we write $h(u, v) = g(u, v) - i\omega(u, v)$ (i.e. $\omega = -\mathrm{Im}(h)$), then $\omega$ comes out to be a 2-form. It turns out that any one of these determine the other three; this leads to the wonderfully confusing fact that any of $h$, $g$, or $\omega$ can be called the Hermitian metric by different authors (or even sometimes the same author). Sometimes, this is written as $h = g - i\omega$.

For example, on the bundle $\mathbb{C}^r$,

$$h = \sum_{k=1}^r \mathrm{d}z_k \otimes \mathrm{d}\overline{z}_k = \sum_{k=1}^r \big((\mathrm{d}x_k)^2 + (\mathrm{d}y_k)^2\big) - i \sum_{k=1}^r \mathrm{d}x_k \wedge \mathrm{d}y_k.$$

We can always put a metric and a connection on a vector bundle $E$, since we're still in the real case, and have partitions of unity. Metrics and connection behave well under operations such as dual, direct sum, pullback, etc.

For example, let $M$ be a complex manifold and $E = TM$, so we have the holomorphic tangent bundle $\tau M = T^{1,0}M$. Then, we get a Hermitian metric on $M$, which leads to a metric and a connection on $E^* = \Lambda^{1,0}M$ (and by dualizing, also on $\Lambda^{0,1}M$), and in the same way one can get a metric and connection on $\Lambda^{p,q}(E)$.

[8] Be particularly careful with the signs here; not only are they confusing and sometimes get accidentally dropped, but also they often differ between authors who are doing everything correctly! It seems that, just like our frame $\{e_k\}$, there are consistent local definitions of what the signs are, but we can't assemble them into a global definition.

**Adding Complex Structure.** For now, let $E \to M$ be a complex vector bundle, where $M$ is a real manifold (soon to be complex, but not yet), and let $h$ be a Hermitian metric on this bundle.

**Definition.** A connection $\nabla$ is called *compatible* with $h$ if $h$ is parallel (i.e. $\nabla h = 0$) as a real tensor.

Equivalently, $h$ is parallel iff it satisfies the Leibniz rule

$$\mathrm{d}_X h(u,v) = h(\nabla_X u, v) + h(u, \nabla_X v),$$

where $X \in TM$ and $u, v \in \Gamma(E)$.

Now, suppose $E \to M$ is a complex vector bundle and $M$ is a *complex* manifold; then, choose a complex linear $\nabla$ in $E$. It can be decomposed along $\Lambda^1 E = \Lambda^{1,0}(E) \oplus \Lambda^{0,1}(E)$ (i.e. compose with projections): $\nabla = \nabla^{1,0} + \nabla^{0,1}$. Then, $\nabla^{1,0} : \Gamma(E) \to \Lambda^{0,1}(E)$ satisfies the following Leibniz rule:

$$\nabla^{0,1}(f \cdot \sigma) = \overline{\partial} f \otimes \sigma + f \nabla^{0,1} \sigma.$$

There is a similar one for $\nabla^{1,0}$, but with $\partial$ instead of $\overline{\partial}$.

**Definition.** If $E$ is a holomorphic vector bundle, a connection $\nabla$ is said to be *compatible with the holomorphic structure* (sometimes *compatible with the integrable structure*) of $E$ if $\nabla^{0,1} = \overline{\partial}_E$.

Before we go on to the Chern connection, let's look at this a little more. $(\nabla^{0,1})^2 = (R_\nabla)^{0,2}$ (which can be computed by expanding $(\nabla^{0,1} + \nabla^{1,0})^2$), i.e. just the $(0,2)$-part of the curvature. In particular, this means that the $(0,2)$-part of the curvature is 0 iff $\nabla$ is compatible with the holomorphic structure. This should look like the proofs we did last time.

**Proposition 6.1.** *Assume $E \to M$ is a holomorphic vector bundle and $h$ is a Hermitian metric on $E$. Then, there exists a unique connection $\nabla$ on $E$, called the* Chern connection*, that is compatible with both the Hermitian structure of $h$ and the holomorphic structure of $E$.*

This plays a similar role to the Levi-Civita connection over real manifolds, but is compatible with more structure.

After we show this, the curvature of $\nabla$ is of type $(1,1)$; we saw above that the $(0,2)$-term is $\overline{\partial}_E^2 = 0$, and unsurprisingly the $(2,0)$-term comes from $\partial^2 = 0$.

*Proof of Proposition 6.1.* The formula for the connection is

$$\nabla = \overline{\partial}_E + h^{-1}(\overline{\partial}_E)_* \circ h.$$

Here, we use $h : E \to E^*$ (anti-complex-linear). Then, it suffices to check the compatibility conditions, e.g. in coordinates, though an intrinsic proof also works. This is the only possible one once the conditions are written down.

Since $h : E \to E^*$ is anti-complex-linear, then $h$ is $\nabla$-parallel (because it's compatible), i.e. $\nabla h = 0$. Thus, if one takes a section $u$ of $E$, then by the Leibniz rule

$$\nabla(h(u)) = (\nabla h)(u) + h(\nabla u) = h(\nabla u),$$

and therefore $h$ commutes with $\nabla$; thus, by anti-complex-linearity, the $(0,1)$-part has to be determined by $\overline{\partial}$, and the rest has to be determined by pulling back by $h$. Thus, this is unique. ⊠

If you don't like this, we can do it in coordinates; we can choose either a holomorphic frame or a Hermitian frame. In a local frame, though, $\nabla$ is determined by the connection 1-form $\tau$, i.e. $\nabla e_k = \tau_{kj} \otimes e_j$, so writing this as a column matrix, $\nabla \mathbf{e} = \tau \otimes \mathbf{e}$.

If $\{e_k\}$ is a holomorphic frame, then let $h_{ij}$ be the coefficients of $h$ in this basis. Then, the compatibility condition ensures that $\partial h = \tau \cdot h$ and $\overline{\partial} h = h^{\mathrm{T}} \overline{\tau}$, so $\overline{\tau} = \partial h \cdot h^{-1}$. Thus, in the unitary (Hermitian) frame, $\tau$ is skew-Hermitian.

## 7. Kähler Metrics: 1/27/15

Last time, we showed that if $E \to M$ is a holomorphic vector bundle and $h$ is a Hermitian metric, then there exists a unique connection, called the *Chern connection*, that is compatible with the metric. We'll use this next week in Hodge theory, which is the theory of the Dolbeault cohomology $H^{*,*}(M, E)$, where $M$ is a compact, complex manifold. This relates to an operator, known as the Hodge star, which provides a finite-dimensional version of Poincaré duality (called *Serre duality*). We can also fix a metric $h$ and consider the space of holomorphic structures given by vector bundles $E \to M$; this is known as deformation theory.

But this is all later; this week we are going to talk about Kähler metrics. Today, we'll take a special case of the discussion last time, where $M$ is a complex manifold and $E = TM$. Some Hermitian metrics on $E$ will have a special property and will be called *Kähler metrics*, though they don't always exist. When they do, they have very special properties.

There are many equivalent definitions of Kähler metrics, and it will be helpful to pass between them. Just as we showed the existence and uniqueness of the Chern connection last time, we'll spend this lecture proving these definitions are equivalent.

**Definition.** A Hermitian metric $h = g - i\omega$ is called a *Kähler metric* if rhe *Kähler form* $\omega = -\operatorname{Im} h$ is d-closed, i.e. $\mathrm{d}\omega = 0$.

**Theorem 7.1.** *The following are equivalent for a Hermitian metric $h = g - i\omega$.*

*(1) $h$ is Kähler.*
*(2) Locally, $\omega$ has a* potential*, i.e. $\omega = i\partial\overline{\partial}u$, called the* Kähler potential.
*(3) $h$* osculates to order 2 *to the standard metric, i.e. at each $z_0 \in M$ there exist holomorphic coordinates $z$ on $M$ such that*

$$h_{ij}(Z) = \delta_{ij} + O(|z|^2).$$

*(4) The Chern and Levi-Civita connections coincide.*

For (3), recall that any Riemannian metric looks like a Euclidean metric in some coordinates (specifically, the Riemann normal coordinates) to first order, but here it's possible to do better.

There are actually even more definitions and equivalences, but this seems like a reasonable place to start

**Example 7.2.**

- The *flat connection* or *Euclidean connection* on $\mathbb{C}^n$ is given by

$$h = i\sum_{k=1}^n \mathrm{d}z_k \otimes \mathrm{d}\overline{z}_k = \frac{1}{2}i\partial\overline{\partial}|z|^2,$$

so it has a global potential, and is thus Kähler.

- For an example with a local but not global potential, consider $\mathbb{CP}^n$ with the standard *Fubini-Study form*

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\overline{\partial}\log(1+|z|^2), \tag{3}$$

which is defined on $\mathbb{C}^n = \mathbb{CP}^n \setminus H$, where $H$ is the hyperplane at infinity. This can be shown to extend to $\mathbb{CP}^n$, though this require some calculation. Nonetheless, thi illustrates that it has a local, but not global, potential.

There's a better way to write it, though: we know about a projection $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$, quotienting by $\mathbb{C}^*$, so we can describe the pullback

$$(\pi^*\omega_{\mathrm{FS}}) = \frac{i}{2\pi}\partial\overline{\partial}\log|z|^2,$$

with $z \in \mathbb{C}^{n+1} \setminus 0$. Then, $\pi$ sends $(z_0, \ldots, z_n) \mapsto [z_0, \ldots, z_n]$. It's a simple but useful exercise to show that these are the same notion, and that this one descends to the quotient.

Now, why is this Kähler? Clearly we have a local potential, but we need to check that we even have a metric. So, why is it positive definite? This comes after a calculation on (3):

$$\frac{2\pi}{i}\omega_{\mathrm{FS}} = \frac{1}{1+|u|^2}\sum_k \mathrm{d}u_k \wedge \mathrm{d}\overline{u}_k + \frac{1}{(1+|u|^2)^2}\Big(\sum \overline{u}_k\,\mathrm{d}u_k\Big) \wedge \Big(\sum u_k\,\mathrm{d}\overline{u}_k\Big).$$

This is a brute-force calculation, but one can see why it's positive definite: when $u = 0$, lot of things cancel, and when $u \neq 0$, it's possible to use a unitary transformation in $\mathrm{U}(n+1)$ on $\mathbb{C}(n+1)$ to rotate to that particular point: this form is homogeneous.[9]

Another nice thing about the Fubini-Study form is that if $\ell$ is a line, then $\int_\ell \omega_{\mathrm{FS}} = 1$. Thus, it has a nonzero de Rham cohomology class $[\omega] \in H^2(M;\mathbb{Z})$, and is therefore Poincaré dual to the line $\ell \in H_2(M,\mathbb{Z})$. This can be generalized, but for $\mathbb{CP}^n$ it's particuarly nice.

*Note.* One can describe $\mathrm{Aut}(\mathbb{CP}^n)$, i.e. its biholomorphisms, as $\mathrm{Aut}(\mathbb{CP}^n) = \mathrm{PGL}(n+1,\mathbb{C})$. In single-variable complex analysis, this is just the description of Möbius transformations. Now, since the data of a Kähler form is the data of a certain Hermitian metric, then those biholomorphisms that fix $\omega_{\mathrm{FS}}$ are isometries, which come from $\mathrm{U}(n+1)$, and these act transitively on $\mathbb{CP}^n$. Hence, $\mathbb{CP}^n$ is homogeneous.

**Corollary 7.3.** *If $h$ is a Kähler form, then its restriction to a complex submanifold is still Kähler. In particular, every complex submanifold of $\mathbb{CP}^n$ is Kähler.*

[9] You can do the explicit calculation at a particular point, but do you really want to?

This follows directly from the definition: $\mathrm{d}\omega = 0$, so $\mathrm{d}\omega|_M = 0$.

For example, we mentioned earlier that *complete intersections* of hypersurfaces in $\mathbb{CP}^n$, i.e. transverse intersections of zero loci of homogeneous polynpmials are complex submanifolds.[10]

**Example 7.4.** On the unit ball in $\mathbb{C}^n$, there's

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log(1-|u|^2).$$

After we discuss curvature, we'll see that this metric has negative curvature.

**Example 7.5.** As a non-example, remember the Hopf manifold? When $n \geq 2$, it ha no Kähler structure, which is because of a cohomology condition.

*Fact.* If $M$ is a compact, Kähler manifold, then $H^{2j}(M,\mathbb{R}) \neq 0$ whenever $j \leq \dim_{\mathbb{C}} M$. This is because the Kähler metric $\omega$ is d-closed, so $[\omega] \in H^2(M)$ is nonzero, and it's nondegenerate, so $\Lambda^n\omega$ is a volume form on $M$, and so in the top dimension, $[\omega]^n \neq 0$ in $H^{2n}(M)$.

*Note.* Kähler metrics also induce nonzero Dolbeault cohomology, which we will show later: $H^{j,j}(M) \neq 0$ if $M$ is Kähler. This follows for a very similar reason: $\omega$ is $\overline{\partial}$-closed, so $[\omega] \in H^{1,1}(M)$.

In Hodge theory, if $\omega$ is a Hermitian metric (not necessarily even Kähler), then $\omega^n$ is still nondegenerate, and (this part requires some argument) gives a nonzero class in $H^{n,n}(M)$.

*Note.* A d-closed, nondegenerate 2-form on a real manifold is called *symplectic*; thus, all Kähler metrics are symplectics. In fact, symplectic is to Kähler as almost complex is to complex; both differ by an integrability condition, and any symplectic manifold has an almost complex geometry.

*Proof outline of Theorem 7.1.* Let $h$ be a Hermitian metric on a complex manifold $(M, \mathcal{J})$.

That (1) is equivalent to (2) is the statement of the Poincaré $\partial\overline{\partial}$ lemma, Lemma 4.3.

That (1) follows from (3) is just a calculation. In the other direction, it's a PDE: we want to find a holomorphic change of coordinates $\varphi$ to "kill" the 1-jet of $h$. It'll end up being sufficient for $\varphi$ to be holomorphic in $z$ and $\overline{z}$.

First of all, there's a linear cange of coordinates to make $h_{ij}(0) = \delta_{ij}$, i.e.

$$h_{ij}(z) = \delta_{ij} + \sum a_{ijk}z_k + \ell(\overline{z}) + O(|z|^2),$$

where $\ell(\overline{z})$ is some linear term that we don't care about. Since $h$ is Hermitian, it's determined by the one in $z$, and is in fact its complex conjugate. But since $\mathrm{d}\omega = 0$, then $a_{ijk} = a_{kji}$, which is an unpleasant, but not very difficult, calculation. Thus, just take

$$w_j = z_j + \frac{1}{2}\sum_{i,k} a_{ijk}z_iz_k,$$

and just check it, though we won't do this here. The idea is, since it's merely quadratic, one can just make a solution by hand.

Finally, that (4) $\iff$ (1) is a brute-force calculation, though it can be simplified somewhat; see Morianu's book. Here are the main ideas:

- Both the Chern and Levi-Civita connections are compatible with the metric.
- The Levi-Civita connection is torsion-free, though the Chern connection may not be. It turns out that the torsion of the Chern connection is 0 iff $\mathrm{d}\omega = 0$.
- $\mathcal{J}$ is compatible with respect to Chern (which is sometimes taken as a definition of the Chern connection), though the Levi-Civita connection may not be. It turns out that the Levi-Civita connection is parallel iff $N_{\mathcal{J}} = 0$ and $\mathrm{d}\omega = 0$.

We won't provide all of the ingredients of this calculation, but the ingredients will all be provided, so that checking the rest is simpler. Recall that the *torsion* of a metric is $T(X,Y) = \nabla_XY - \nabla_YX - [X,Y]$, and the Levi-Civita connection is the unique torsion-free connection compatible with a given Riemannian metric $g$ (i.e. $\nabla g = 0$); similarly, the Chern connection is the unique connection compatible with a Hermitian metric $h = g - i\omega$, i.e. $\nabla h = 0$ and $R_\nabla^{0,2} = 0$. Thus, $\nabla g = 0$, $\nabla\omega = 0$, since $\omega = g(J\cdot,\cdot)$.

Recall also that the curvature for a Riemannian manifold was $R_{XY} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, and if $\nabla$ is torsion-free and $\omega$ is a $k$-form, then

$$\mathrm{d}\omega(X_0,\dots,X_k) = \sum_k (-1)^k\nabla_{X_k}(\omega)(X_0,\dots,\widehat{X}_k,\dots,X_n).$$

[10] In differential geometry, the term "transverse intersections" is used, and in complex or algebraic geometry, one sees "complete intersections." You also might want to intersect more general holomorphic functions in this definition, but these end up only being polynomials, anyways!

(Here, $\widehat{X}_k$ means the absence of index $k$.)

Thus, since $\nabla g = 0$ and $\nabla \omega = 0$ for the Chern connection, then $\nabla \mathcal{J} = 0$, so (as was on the homework), the $(1,1)$ part of the torsion $T$ vanishes. Since torsion basically measures the difference between $\mathrm{d}\omega$ and $\nabla\omega$, then the torsion is only $(0,2)$.

Thus, one can prove two lemmas. Assume $h$ is Hermitian on an almost complex manifold and $\nabla$ is the Levi-Civita connection.

**Lemma 7.6.** *There exists a connection $\widetilde{\nabla}_X Y = \nabla_X Y - (1/2)\mathcal{J}(\nabla_X \mathcal{J})Y$, which is a connection preserving $g$ and $\mathcal{J}$, but has torsion $T = -(1/4)N_{\mathcal{J}}$.*

**Lemma 7.7.** *$\nabla \mathcal{J} = 0$ iff $N_{\mathcal{J}} = 0$ and $\mathrm{d}\omega = 0$.*

*Proof sketch of Lemma 7.6.* This is yet another calculation, using

- the Leibniz rule,
- that $g(\mathcal{J}\cdot, \mathcal{J}\cdot) = g(\cdot,\cdot)$, and
- $\mathcal{J}^2 = -\mathrm{id}$, so $\nabla\mathcal{J}$ and $\mathcal{J}$ anticommute.

Once we address Lemma 7.7, then all of the required equivalences are proven. ⊠

## 8. Curvature on Kähler Manifolds: 1/29/15

*"I've forgotten how to do logic."*

Recall that last time, we showed that the Levi-Civita connection is equal to the Chern connection iff $M$ is Kähler. We provided much of the proof, but not all of it.

**Lemma 8.1.** *Let $M$ be a complex manifold, $h$ a Hermitian metric, and $\nabla$ the Levi-Civita connection. Then, $\overline{\partial}$ for $\nabla$, regarded as $\overline{\partial}^{\nabla} : \Gamma(TM) \to \Lambda^{0,1}(TM)$ is*

$$\overline{\partial}^{\nabla}_V X = \frac{1}{2}(\nabla_V X + \mathcal{J}\nabla_{\mathcal{J}V} X - \mathcal{J}(\nabla_V \mathcal{J})(X)). \tag{4}$$

**Corollary 8.2.** *In particular, if $M$ is Kähler, then the Levi-Civita and Chern connection coincide.*

This is because in this case, $\mathcal{J}$ is parallel to $\nabla$, and therefore terms drop out and the same expression holds.

*Proof of Lemma 8.1.* Though one could calculate this by brute force, there's a cleverer way to do this, involving the linearization of the *holomorphic map equation* at the identity. This is defined in the following way: if $\varphi : M \to M$ is holomorphic, then

$$0 = \overline{\partial}\varphi = \frac{1}{2}(\mathrm{d}\varphi + \mathcal{J}\circ \mathrm{d}\varphi \circ \mathcal{J}),$$

which is equivalent to $\varphi^*\mathcal{J} = \mathcal{J}$.

Thus, the following one-parameter family of variations is a vector field:

$$X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t.$$

But be careful with this linearization: $\mathcal{J}$ may not be parallel, so we have to keep track of its variation. Thus, when we linearize, it ends up being $\overline{\partial}^{\nabla} X = 0$, where $\overline{\partial}^{\nabla}$ is exactly a in (4).

If one linearizes the condition $\varphi^*\mathcal{J} = \mathcal{J}$ instead, then it's just the Lie derivative, by definition:

$$0 = \mathcal{L}_X \mathcal{J} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi^*\mathcal{J}).$$

Thus, $X$ is a holomorphic vector field, i.e. an infinitesimal automorphism.

Recall that $\mathrm{Aut}(M)$ is a Lie group, though it may be infinite-dimensional. Certainly $\mathrm{id} \in \mathrm{Aut}(M)$, and we can consider its tangent space $T_{\mathrm{id}}\,\mathrm{Aut}(M)$. Recall back in differential topology, one did something similar: using the flow equation, $T_{\mathrm{id}}\,\mathrm{Diff}(M) = \mathcal{X}(M) = \Gamma(TM)$ (smooth vector fields). This is a little weird, because this Lie group is definitely infinite-dimensional (in our case, it'll be finite-dimensional). In any case, this tangent space is a Lie algebra, which will be useful; it'll be important to make sense of this precisely, but it is useful motivation.

**Exercise 8.3.** Calculate the infinitesimal automorphisms (i.e. the biholomorphic ones) of $\mathbb{CP}^n$.

Back in the world of complex geometry, if $M$ is a complex manifold, $T_{\mathrm{id}}\,\mathrm{Aut}(M) = \mathcal{X}_{\mathrm{holo}}(M) = H^0(M; TM)$, i.e. holomorphic vector fields. (Here, as always, $TM$ is the holomorphic bundle, the $(1,0)$-part).

*Note.* $X$ is a *Killing vector field* if it is an infinitesimal isometry, i.e. in $T_{\mathrm{id}}\,\mathrm{Isom}(M)$; and isometries are defined as $g(\varphi(\cdot), \varphi(\cdot)) = g(\cdot,\cdot)$, or, equivalently, $\varphi^*g = g$.

**Exercise 8.4.** Calculate the linearizations of both limits; this should agree with the standard definition of a Killing vector field.

Now, back to the proof. Remember that the Levi-Civita connection is defined on the real tangent bundle, and the Chern connection is defined on the holomorphic tangent bundle, so one must pull them back and identify them somehow to identify them. That is, we need $T^{1,0}M \simeq TM$, as an isometry of complex vector bundles.

Consider $\psi : TM \to T^{1,0}M$ given by $X \mapsto (1/2)(X - i\mathcal{J}X) = Z$. Then, $Z \mapsto X$ is called taking the real part. Then, one can pull back the Levi-Civita connection by $\psi^{-1}$ to get $\nabla$ on $T^{1,0}M$. Then, with $\overline{\partial}^{\nabla}$ as in (4), one just checks that it satisfies the Leibniz rule for $\overline{\partial}$. Some care should be taken with $\overline{\partial}$ versus d, but in the end, one can show that $(\overline{\partial} f)(X) = \mathrm{d}_Z f$ for all $f : M \to \mathbb{C}$.

Finally, we need to check that this is $\overline{\partial}$. This is done by looking at the holomorphic tangent bundle, and showing that $\overline{\partial}^{\nabla} Z = 0$ is the same as the real part of $X$ satisfying $\mathcal{L}_X\mathcal{J} = 0$. ⊠

**Curvature.** Curvature is very special for Kähler manifolds. Recall that in the real case, where $E = TM$, there's the *Riemann curvature tensor* $R(X, Y, Z, W) = g(R(X, Y)Z; W)$, with coefficients $R_{ijk\ell} = g^i R_{ijk}$. There's also the *Ricci tensor*

$$\mathrm{Ric}(X, Y) = \mathrm{Tr}(v \mapsto R(v, X)Y) = \sum_k R(e_k, X, Y, e_k).$$

In a frame, this has coefficients $R_{ij} = R^k_{ikj}$. Finally, there's the *scalar curvature* $= \mathrm{Tr}(\mathrm{Ric}) = R_{ij}g^{ij}$. There are other curvatures, but we won't worry about this; similarly, one can define these on almost complex or complex manifolds, but we'll go directly to the Kähler case.

Now, let $M$ be a Kähler manifold, and $\nabla$ denote the Levi-Civita (and therefore also Chern) connection.

**Definition.** The *Ricci form* is $\rho(X, Y) = \mathrm{Ric}(\mathcal{J}X, Y)$.

**Proposition 8.5.** *$\rho$ is a closed 2-form, and agrees with $i \cdot \mathrm{Tr}_{\mathbb{C}}(R_\nabla)$.*

Therefore it has a geometric interpretation: in cohomology, $[\rho] = 2\pi c_1(M)$ (that is, it represents the first Chern class), or equivalently,

$$c_1(M) = \frac{i}{2\pi}[\mathrm{Tr}\, R_\nabla] = \frac{1}{2\pi}[\rho].$$

This follows from Chern-Weil theory, which we'll discuss later on; one could also take it as a definition. That this class is independent of the choice of Kähler metric, which is a good exercise.

This is also related to $\mathrm{U}(n)$ holonomy and parallel transport, which we'll expand on later.

Another application of curvature is the notion of *Kähler-Einstein metrics*, which show up in physics. In the real case, the *Einstein equation* is $\mathrm{Ric} = \lambda g$ for a constant $\lambda$, and the analogous equation for Kähler manifolds is the *Kähler-Einstein equation* $\rho = \lambda\omega$ (with the Ricci and Kähler forms). Since the right-hand side is positive definite, then the left-hand side is required to be as well. This relates to the special notion of a *Ricci-flat* metric, where $\rho = 0$ everywhere, and the Calabi-Yau theorem.

*Note.* If $E \to M$ is a complex vector bundle and $\nabla$ is a complex connection, then $R_\nabla \in \Lambda^2(M, \mathrm{End}_{\mathbb{C}}(E))$ (that is, it's a two-form), so one can take its complex trace $\mathrm{Tr}_{\mathbb{C}} : \mathrm{End}_{\mathbb{C}}(E) \to \mathbb{C}$, yielding a two-form on $M$ (with values in the trivial bundle), $\mathrm{Tr}\, R_\nabla \in \Lambda^2(M)$.

Here are some useful properties, even if they may be a little simple.

**Proposition 8.6.** *Let $M$ be a Kähler manifold.*

*(1)* $\mathrm{Ric}(\mathcal{J}X, \mathcal{J}Y) = \mathrm{Ric}(X, Y)$.

*(2)*
$$\mathrm{Ric}(X, Y) = \frac{1}{2}\mathrm{Tr}_{\mathbb{R}}(R(X, Y) \circ \mathcal{J}).$$

*(3) $\rho$ is real $(1, 1)$ and closed; $\omega$ is also closed $(1, 1)$.*

*(4) $\rho = i\,\mathrm{Tr}_{\mathbb{C}}(R_\nabla)$ for the holomorphic vector bundle $E = TM$.*[11]

*(5) In holomorphic coordinates, if $H$ is the matrix of $h$, then*
$$\rho = -i\partial\overline{\partial}\log\det(H).$$

*(6) Intrinsically, $\rho$ represents $c_1(TM) = -c_1(K_M)$; specifically,*
$$c_1(TM) = c_1(\det_{\mathbb{C}} TM) = \frac{i}{2\pi}[\mathrm{Tr}_{\mathbb{C}}(R_\nabla)] = \frac{1}{2\pi}[\rho].$$

[11] Here, $R_\nabla \in \Lambda^{1,1}(\mathrm{End}(TM))$, so taking the trace gives something in $\Lambda^{1,1}(M)$, as before.

*Proof.* (1) follows because $\mathcal{J}$ is parallel, and then follows from properties of $R$, so it can be calculated in an orthonormal basis $\{e_k\}$ (or even better a unitary basis).

Specifically, since $R_\nabla$ is complex linear, then

$$R(X, Y, \mathcal{J}Z, \mathcal{J}W) = R(X, Y, Z, W) = R(\mathcal{J}X, \mathcal{J}Y, Z, W).$$

Then, substitute in a basis and take the trace.

(2) requires more work. Use the first Bianchi identity,[12] which says that cyclic permutations of the terms in $R(X, Y, Z, \cdot)$ don't affect the result. Then,

$$\begin{aligned}\mathrm{Ric}(X, Y) &= \sum_k R(e_k, X, \mathcal{J}Y, \mathcal{J}e_k)\\ &= \text{mess} - R(\mathcal{J}Y, e_k, X, \mathcal{J}e_k),\end{aligned}$$

and when the mess of terms simplifies, it ends up being the formula we wanted.

For (3), use (1), which implies that $\rho$ is $(1,1)$; then, by the second Bianchi identity, $\nabla R_\nabla = 0$, and $\mathrm{d}\rho$ can be written as a combination of terms in $\nabla\rho$, so therefore $\mathrm{d}\rho = 0$.

(4) is (2) plus some linear algebra: for a skew-Hermitian matrix $A$, one can write $A = \begin{pmatrix} 0 & -A_\mathbb{R} \\ A_\mathbb{R} & 0 \end{pmatrix}$; then, $A = A_\mathbb{R}\mathcal{J}$, so $i\,\mathrm{Tr}_\mathbb{C}(A) = \mathrm{Tr}_\mathbb{R}(A_\mathbb{R} \circ \mathcal{J})$. Then, apply the spectral theorem. This follows because since $A$ is skew-Hermitian, then $A = -\overline{A}^{\mathrm{T}}$, i.e. $A \in \mathfrak{u}(n)$, which is equivalent to $iA$ being Hermitian.

Another way to think of this is that $\mathrm{Tr} : \mathfrak{u}(n) \to i\mathbb{R}$. Recall that in a unitary frame, the matrix of one-forms $\tau$ and the curvature matrix $\Theta$ are both skew-Hermitian, i.e. $\tau, \Theta \in \mathfrak{u}(n)$. This has connections to representations of Lie groups! $\tau \in \Lambda^1(\mathrm{Ad}(E))$, and $\Theta \in \Lambda^2(\mathrm{Ad}(E))$.

There are sometimes sign errors or missing factors of $i$, relating to the fact that the natural Lie algebra of $S^1$ is $i\mathbb{R}$, but people often write it as $\mathbb{R}$, so things can get muddled.

Anyways, we get $\exp : \mathfrak{u}(n) \to \mathrm{U}(n)$, and the associated log, and so $\log\det(A) = \mathrm{Tr}(\log A)$. This is the insight behind Chern-Weil theory, and in particular when one does a brute-force calculation in a basis, then (5) falls out. This involves pushing around Christoffel symbols.

$$\Gamma^i_{ki} = \frac{1}{2} g^{im} \frac{\partial g_{im}}{\partial x_k} = \frac{1}{2} g^{-1} \frac{\partial g}{\partial x_k} = \mathrm{d}\log\sqrt{\det g}.$$

For the Kähler case, of course we'll compute in a unitary frame, and let $h_{i\bar{j}}$ be the coefficient of $\mathrm{d}z_i \otimes \mathrm{d}\overline{z}_j$, so that $\overline{h}_{i\bar{j}} = h_{\bar{i}j}$. Then, because $T^{1,0}M$ is $\nabla$-parallel, all mixed Christoffel symbols are 0. There's more calculation to be done, e.g. $R_\nabla$ is of type $(2,2)$, and so on, so a lot of terms drop out. ⊠

## 9. Hodge Theory: 2/3/15

This will seem like a considerable digression from Kähler metrics, but the two come together and interplay in interesting ways. It would be nice to talk about them simultaneosly, but this is as close as one can get.

Hodge theory is a way of analyzing various cohomology theories on complex manifolds. We've already seen several different complexes in this context:

- When $M$ is a real manifold, there's the de Rham cohomology $\mathrm{d} : A^k(M) \to A^{k+1}(M)$.
- When $M$ is a complex manifold, there's the Dolbeault cohomology $\overline{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$.
- The Dolbeault cohomology could also be contstructed as $\partial : A^{p,q} \to A^{p+1,q}$.
- Finally, if $E \to M$ is a holomorphic vector bundle, then Dolbeault cohomology with coefficients in $E$ i given by maps $\overline{\partial}_E : A^{p,q}(M; E) \to A^{p,q+1}(M; E)$.

The commomalities are an operator $D$ ($\mathrm{d}$ or $\overline{\partial}$ or $\partial$), which is a first-order linear differential operator, such that $D^2 = 0$. Thus, one can take the cohomology of this complex, which is $D$-closed forms modulo $D$-exact forms.

Then, for every $\alpha \in A^{p,q}$ (or more generally the objects in the complex), there's an induced $[\alpha] \in H^{p,q}(M)$ (or more generally, the homology group), given by $[\alpha] = \left\{\alpha + \overline{\partial}\varphi \mid \varphi \in A^{p,q-1}(M)\right\}$. Today's question is: is there a best representation for this cohomology class? In general, no, but with the choice of a metric there does end up being a good choice.

Our goal is to prove that if $M$ is compact, $\dim H^{p,q}(M)$ is finite, and there will be a version of Poincaré duality, called Serre duality. This involves the pairing $A^k \times A^{n-k} \to \mathbb{C}$, given by

$$(\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta.$$

[12]. . . which was discovered by Ricci, not Bianchi, but whatever.

In order for these integrals to make sense, we'll assume $M$ is compact (otherwise, one could take compactly supported sections, I suppose).

This is completely intrinsic, and descends first of all to a pairing $A^{p,q} \times A^{n-p,n-q} \to \mathbb{C}$, with the same formula, and then to cohomology, e.g. $H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}$ for de Rham cohomology and $H^{p,q}(M) \times H^{n-p,n-q}(M) \to \mathbb{C}$ for Dolbeault cohomology. We will prove this is nondegenerate; then, the canonical representatives will (in the presence of a metric) lead to a way of passing between the two cohomology classes.

Thus, fix a metric (Riemannian for de Rham; Hermitian for Dolbeault). Then, there's another pairing

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha(z), \beta(z) \rangle \, \mathrm{d}V, \tag{5}$$

where $\mathrm{d}V$ is a volume form. This just takes the pointwise inner product and integrates it, and is called the $L^2$ *inner product on* $A^k$. This means there's a norm $\|\alpha\| = \langle \alpha, \alpha \rangle$; is there a representation of $[\alpha]$ that minimizes the norm (or equivalently, the norm squared)? The goal is to find

$$\min_{\varphi} \|\alpha + D\varphi\|^2,$$

which is basically a variational problem. It's basically an orthogonal projection of $\alpha$ onto the space of exact forms.

**Lemma 9.1.** *Assume $\alpha$ minimizes the norm in $[\alpha]$. Then, $\langle \alpha, D\psi \rangle = 0$ for all smooth $\psi$.*

In some sense, these are the Euler-Lagrange equations for this problem.

*Proof.* If $\alpha$ is the minimum, then for $t \in \mathbb{R}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \|\alpha + tD\psi\|^2 = \langle \alpha, D\psi \rangle + \langle D\psi, \alpha \rangle.$$

This is sufficient for a Riemannian inner product, but if $\langle \cdot, \cdot \rangle$ is Hermitian, then all we know is that the real part is zero. For the imaginary part, one should minimize over complex lines, i.e. $t \in \mathbb{C}$ (or just using $\|\alpha + itD\varphi\|^2$), after which the same calculation shows the imaginary part is zero. ⊠

**Definition.** Let $E, F \to M$ be vector bundles over a manifold $M$, and $D : \Gamma(E) \to \Gamma(F)$. Then, fix metrics on $E$, $F$, and $M$, which creates an $L^2$ inner product on $\Gamma(E)$ and $\Gamma(F)$, once again given by (5). Then, the *($L^2$) formal adjoint* of $D$ is $D^* : \Gamma(F) \to \Gamma(E)$ defined as

$$\langle D\alpha, \beta \rangle = \langle \alpha, D^*\beta \rangle,$$

for all compactly supported smooth section $\alpha \in \Gamma(E)$ and $\beta \in \Gamma(F)$.

It will also be useful that the adjoint of $D^*$ is equal to $D$.

*Note.* It's easy to show that if it exists, then it's unique, by some formal nonsense, but in our case, it's possible to calculate $D^*$, using integration by parts, Stokes' theorem, and the Hodge star operator.

It will also end up that $D^*$ is a first-order differential operator; $D$ as an operator on $L^2$ is unbounded, which means futzing around with Sobolev spaces, and so on.

In summary, here's what we've bought so far.

- Finding the "best" representation for $[\alpha]$, when $\alpha$ is closed, boils down to solving $D^*\alpha = 0$ and $D^\alpha = 0$.
- Thus, it boils down to solving $\Delta\alpha = 0$, where $\Delta = DD^* + D^*D$ is the *Laplacian.*

These are equivalent by some formal nonsense.

$$\begin{aligned} \langle \Delta\alpha, \alpha \rangle &= \langle DD^*\alpha, \alpha \rangle + \langle D^*D\alpha, \alpha \rangle \\ &= \|D^*\alpha\|^2 + \|D\alpha\|^2. \end{aligned}$$

Thus, $\Delta\alpha = 0$ iff $D\alpha = 0$ and $D^*\alpha = 0$.

**Definition.** A solution of $\Delta\alpha = 0$ is called a *harmonic.*

The space of harmonics depends on what cohomology theory we're talking about: for Dolbeault cohomology, the $(p,q)$-harmonics are $\mathcal{H}^{p,q} = \left\{x \in A^{p,q} \mid \Delta_{\overline{\partial}}\alpha = 0\right\}$, i.e. solutions to $\overline{\partial}\alpha = 0$ and $\partial\alpha = 0$.

This is a little messy, since there are multiple cohomology theories floating around, and in general they're differet. But in the case of Kähler manifolds, the harmonics are all the same!

**Proposition 9.2.**

- *The minimum norm of $[\alpha]$ is achieved for a harmonic $\alpha$.*
- *$\Delta$ is formally self-adjoint, and semipositive definite.*

This boils down to solving the equation $\Delta\alpha = 0$.

*Note.* If a minimal $\alpha$ exists, then it is unique: such an $\alpha$ satisfies

$$\|\alpha + D\varphi\|^2 = \|\alpha\|^2 + \|D\alpha\|^2 + \langle \alpha, D\varphi\rangle + \langle D\varphi, \alpha\rangle \geq \|\alpha^2\|,$$

and they're equal iff $D\varphi = 0$.

Thus, all of the work goes into showing the existence of a minimal element, i.e. solving $\Delta\alpha = 0$. Here's an outline.

- First, one can show the existence of a weak solution.
- Then, regularity implies that a weak solution is smooth and a strong solution. This crucially uses the fact that $\Delta$ is an elliptic operator (which has to be checked separately for each cohomology theory) is an elliptic operator, and therefore one can use the spectral theorem for self-adjoint, elliptic operators.

The second step is less PDE theory and more linear algebra: given that $\Delta$ is self-adjoint and elliptic, one can consider its eigenspaces, satisfying $\Delta\alpha = \lambda\alpha$. These are finite-dimensional eigenspace, and the eigenvectors are smooth; thus, the spectrum is discrete, and there's an $L^2$-orthogonal decomposition into eigenspaces, which helps greatly in finding a solution.

Here, we require $M$ to be compact; otherwise, $\Delta$ wouldn't be elliptic.

**Theorem 9.3** (Hodge)**.** *Assume $M$ is a* compact*, complex manifold, and fix a Hermitian metric. Conisider the Laplacian $\Delta : A^{p,q}(M) \to A^{p,q}(M)$ from the Dolbeault cohomology and the harmonics $\mathcal{H}^{p,q} = \ker(\Delta)$. Then:*

*(1) $\mathcal{H}^{p,q}$ is finite-dimensional.*
*(2) There is a well-defined orthogonal projection $h : A^{p,q} \to \mathcal{H}^{p,q}$, inducing an isomorphism $H^{p,q} \cong \mathcal{H}^{p,q}$.*
*(3) There exists a unique operator $G : A^{p,q} \to A^{p,q}$ such that:*
  - $G|_{\mathcal{H}^{p,q}} = 0$,
  - $G\overline{\partial} = \overline{\partial}G$,
  - $G\overline{\partial}^* = \overline{\partial}^* G$, *and*
  - $I = h + \Delta \circ G$.

  *The last condition gives a sort of orthogonal decomposition.*
*(4) There is an orthogonal decomposition*

$$\begin{aligned} A^{p,q} &= \mathcal{H}^{p,q} \oplus \overline{\partial} A^{p,q-1} \oplus \overline{\partial}^* A^{p,q+1} \\ \alpha &= h(\alpha) + \overline{\partial}(\overline{\partial}^* G\alpha) + \overline{\partial}^*(\overline{\partial} G\alpha). \end{aligned}$$

The operator $G$ in (3) is called *Green's operator*.

*Note.* The same theorem works with minor, predictable modifications for de Rham theory and the other cohomology theories listed at the beginning of the class.

In order to dig into the proof, it's necessary to understand what $D^*$ (in this case, $\overline{\partial}^*$) is doing. Here's where the Hodge star comes in; it's some more linear algebra.

Given a real manifold $M$, fix a metric, so that there's the *rescaled $L^2$ inner product* on $A^k$ given by

$$\langle \alpha, \beta\rangle = \frac{1}{k!}\int_M \langle \alpha(x), \beta(x)\rangle \,\mathrm{d}\,Vol,$$

where $\mathrm{d}\,Vol = \sqrt{\det g_{ij}}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$ is a volume form.

If $\{e_i\}$ is a basis of a vector space $V$, then $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ is a basis of $\Lambda^k V$.

**Definition.** The *Hodge star operator* $\star : A^k \to A^{n-k}$ is defined by $\langle \alpha, \beta\rangle = (\alpha, \star\beta)$, i.e.

$$\frac{1}{k!}\int_M \langle \alpha, \beta\rangle \,\mathrm{d}\,Vol = \int_M \alpha \wedge (\star\beta).$$

This comes out sending basis vectors to basis vectors: let $e_{I^c}$ denote wedging together all of the $e_j$ that aren't contained in the multi-index $e_I$. Then, $*e_I = \pm e_{I^c}$, where the sign is the sign of the permutation $(I, I^c)$ (in $S_n$)).

The Hodge star has some nice properties.

- $\star(\mathrm{d}\,Vol) = 1$.
- $\star$ is an ieometry, i.e. $\langle \star\alpha, \star\beta\rangle = \langle \alpha, \beta\rangle$.
- $\star^2 = (-1)^{k(n-k)}$ on $A^k$, i.e. $\star$ is self-adjoint up to sign.
- If $M$ is complex, $\star : A^{p,q} \to A^{n-q,n-p}$.

This means the following diagram commutes.

$$\begin{array}{ccc} A^k & \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} & A^{k+1} \\ \updownarrow \star & & \updownarrow \star \\ A^{n-k} & \overset{\mathrm{d}}{\longleftarrow} & A^{n-k-1} \end{array}$$

**Lemma 9.4.** $\mathrm{d}^* : A^{k+1} \to A^k$ *is given by the formula* $\mathrm{d}^* = -(-1)^{nk} \star \mathrm{d}\star$. *Similarly,* $\overline{\partial}^* = - \star \partial \star$.

Thus, there's a similar commutative diagram for the complex case. Thee both follow from Stokes' theorem.

## 10. The Hard Lefschetz Theorem: 2/5/15

## 11. Line Bundles and Chern Classes: 2/10/15

Today, we'll finish the proof of the Hard Lefchetz theorem, and discuss line bundles and Chern classes.

Recall that if $M$ is a Kähler manifold and $\omega$ is a Kähler form, then the *Lefschetz operator* $\Lambda\cdot = \omega\Lambda\cdot$, so $L : A^{p,q} \to A^{p+1,q+1}$. Then, $\operatorname{adj} \Lambda = L^*$.

Let $H$ be the *counting operator* $H(\omega) = (K - n)\eta$ when $\eta \in A^K(M)$ and $n = \dim_{\mathbb{C}} M$. This shifts the dimension down.

Recall that the Lie algebra $\mathfrak{sl}_2$ consists of trace-free $2 \times 2$ matrices, with a generating set

$$X = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \qquad Y = X^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying certain commutativity relations $[X, h] = 2X$, $[Y, h] = -2Y$, and $[X, Y] = h$ (which is easy to check).

**Lemma 11.1.** *If $M$ is a Kähler manifold, then:*

*(1) $[L, H] = 2L$, $[\Lambda, H] = -2\Lambda$, and $[L, \Lambda] = H$; thus, these form an $\mathfrak{sl}_2$-representation.*
*(2) This descends to cohomology: $[L, \Delta] = [\Lambda, \Delta] = 0$.*

Both parts are fairly easy to directly check.

**Theorem 11.2** (Hard Lefschetz theorem)**.** *If $M$ is a compact Kähler manifold and $n = \dim_{\mathbb{C}} M$, then $L^k : H^{n-k}(M) \to H^{n+k}(M)$ is an isometry, and both the de Rham and Dolbeault cohomnologies are $\mathfrak{sl}_2(\mathbb{Z})$-representations.*

**Example 11.3.** For $H^*(\mathbb{CP}^1)$ (which is on the homework... *cough*), both $H^{0,0}$ and $H^{1,1}$ are one-dimensional, and the rest are zero. Something similar happens in $H^*(\mathbb{CP}^n)$.

**Corollary 11.4.** *If $M$ is a complex submanifold of $\mathbb{CP}^N$ and $h \subseteq \mathbb{CP}^N$ are hyperplanes, then $\Lambda h^k : H_{n+k}(M) \to H_{n-k}(M)$ is an isometry (where $h^k$ is the intersection of the $k$ planes in homology), and the Poincaré dual to the Fubini-Study form is $\mathrm{PD}(\omega_{\mathrm{FS}}) = h$.*

This is a homological restriction on the existence of a Kähler structure of a manifold, and is useful for proving that some manifolds don't have a Kähler structure. There are several other, similar Lefschetz theorems, which also involve intersection theory.

*Note.* The irreducible representations of $\mathfrak{sl}_2$ are classified by dimension, with exactly one in each dimenion: the trivial one $\mathbb{C}$, the *fundamental representation* $V(1) = H^*(\mathbb{CP}^1)$, and more generally the $n^{\text{th}}$ one is $n+1$-dimensional and given by $V(n) = H^*(\mathbb{CP}^n) = \operatorname{Sym}^n(\mathbb{C}^2)$.

Since $H$ is just a shifting operator, then these decompose into eigenspaces for $H$:

$$V(n) = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n.$$

These are the eigenspaces for $H$; each is a copy of $\mathbb{C}$, so $L$ and $\Lambda$ commute with them, and shift from one to the next. Thus, at some point, it terminates with $\ker(L^{n+1})$. Similarly, $V_{-n}$ is $\ker(\Lambda)$ intersected with the eigenspace of $H$. These subspaces are called primitive.

**Definition.** The *primitive cohomology* $P^* \subseteq H^*$ is $P^* = \ker(\Lambda) \cap H^*$.

Thus, there is a decomposition

$$H^k = \bigoplus_p L^p P^{k-2p}.$$

One can also use this to realize the *Riemann bilinear pairing* (for a Kähler manifold): instead of a skew-symmetric bilinear form $Q(\alpha, \beta) = \int_M \alpha \wedge \beta$, one can instead create one on $H^k$ by

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \omega^{n-k}.$$

This is Hermitian; $H(\alpha, \beta) = i^k Q(\alpha, \overline{\beta})$. This is a very useful technique for studying Kähler manifolds further, though we won't use it any more.

**Line Bundles.** The goal for today is to understand complex or holomorphic line bundles up to isomorphism, including when they're isomorphisms or classifying them. Line bundles are easier to deal with than higher-dimensional cases, but it's nice to have them too.

There are several approaches to line bundles; they're all useful in different contexts.

- One way is via transition functions, which ends up in a 1-Čech cocycle.[13]
- Another way is as zeros of open sections, though in the holomorphic case it'll be necessary to consider meromorphic functions, and tracking zeros and poles with multiplicity. This leads to a notion of divisors.
- One very good way is via the first Chern class, which is the Euler class, and relates to both of the previous approaches, the first via an exponential function, and the second thanks to Poincaré duality, By Chern-Weil theory, it also relates to curvature.

We will spend the most time working on the third approach.

One important question: is there a difference between isomorphism as complex line bundles and as holomorphic line bundles? Relatedly, how many different holomorphic structures can one place on the same complex line bundle? Typically, they'll come in families. This leads to deformation theory: deformations of holomorphic structures, moduli problems, and Kodaira-Spencer theory. But this is just motivation.

**Definition.** Let $M$ be a complex manifold. Then, the *Picard group* $\mathrm{Pic}(M)$ is the set of holomorphic line bundles, up to isomorphism; here, multiplication is given by $L_1, L_2 \mapsto L_1 \otimes L_2$ and inversion by $L_1 \mapsto L_1^*$. Then, $\mathrm{Pic}^0(M)$ or $\mathrm{Jac}(M)$, called the *Jacobian*, is the subgroup of topologically trivial ones.

Before presenting some massive theory, it's good to have some concrete examples in mind.

**Example 11.5.** We can show that $\mathrm{Pic}(\mathbb{CP}^n) = \mathbb{Z}$, generated by the tautological line bundle (from Dolbeault cohomology), and therefore $\mathrm{Pic}^0(\mathbb{CP}^n) = 0$. Thus, isomorphism as holomorphic line bundles over $\mathbb{CP}^n$ is the same as isomorphism of smooth, complex line bundles over $\mathbb{CP}^n$.

However, on an elliptic curve $E$, that's no longer the case; there exists a one-parameter family of holomorphic line bundles which are all topologically trivial (and therefore isomorphic as complex line bundles). Another way to say that is that the moduli space is one-dimensional. In particular, $\mathrm{Pic}^0(E) \cong E$, with the isomorphism canonical as soon as the origin is chosen. This will be as topological spaces (where $\mathrm{Pic}^0(E)$ happens to have a natural topology and becomes a one-dimensional complex manifold), but we don't know that yet.

Topologically, think of an elliptic curve as a torus; then, given a point $p \in E$, one can product a holomorphic line bundle $O(p)$ which has a holomorphic section that has a simple zero at $p$ and nowhere else. Then, $O(p-q) = O(p) \otimes O(q)^*$ (because of where the simple zeroes are, and such); this is topologically trivial, but not holomorphically so when $p \neq q$; if it were, then there would exist a meromorphic $f$ on $E$ with a simple zero and a simple pole, which isn't possible, since it gives a degree-1 map $f : E \to \mathbb{CP}^1$.

This is just like in complex analysis: on the sphere there is one unique complex structure, and on the torus there's a one-parameter family of them. The reasoning is basically the same (Dolbeault cohomology).

Recall that for a line bundle the transition functions take the form $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$.[14] These maps can be required to be smooth or holomorphic, depending on the kind of bundle. Note that we want this to be covariant, so $g_{\alpha\beta}$ is a transition from $\beta$ to $\alpha$.

On the intersection of $U_\alpha$, $U_\beta$, and $U_\gamma$, they satisfy a 1-Čech cocycle condition, that $g_{\alpha\gamma} = g_{\alpha\beta} \circ g_{\beta\gamma}$.

In particular, $g_{\alpha\alpha}$ is the identity and $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$. Thus, the above cocycle condition can be rewritten in the equivalent form $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = 1$. Thus, the transition functions give a class in the one-dimensional Čech cohomology of (the cover of) the manifold, with values in $\mathcal{O}^*$, the nowhere vanishing holomorphic functions. That is, $(g_{\alpha\beta}) \in \check{H}^1(M, \mathcal{O}^*)$.

**Theorem 11.6.** *There is a natural group isomorphism* $\mathrm{Pic}(M) \cong \check{H}^1(M, \mathcal{O}^*)$.

$\mathcal{O}^*$ is a sheaf, a generalization of a bundle, and whenever there's a sheaf, there's the *exponential sequence*

$$1 \longrightarrow \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 1$$

where $\exp : f \mapsto \exp(2\pi i f)$. This is a short exact sequence, and therefore there's a long exact sequence in cohomology:

$$H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{\delta^*} H^2(M, \mathbb{Z}) \longrightarrow \cdots$$

[13]If you haven't seen Čech cohomology before, it's worth reading up on sheaves and cohomology in Griffith and Harris, §3, or in Huybrechts.

[14]"Linear algebra is just the representation theory of $\mathrm{GL}(n, \mathbb{C})$; we'll do the representation theory of $\mathrm{GL}(1, \mathbb{C})$."

Then, $(g_{\alpha\beta}) \in H^1(M, \mathcal{O}^*)$, and, as we'll see, the first Chern class is in $H^2(M, \mathbb{Z})$. This sequence, and the related technique of *Chern cohomology*, can be used to more efficiently prove things such as the results in Example 11.5.

A section $s$ in a line bundle is a collection of $s_\alpha : U_\alpha \to \mathbb{C}$, required to be smooth or holomorphic depending on context, so that $s_\alpha = g_{\alpha\beta} s_\beta$ (which is where it's nice to have the signs right!). A nowhere vanishing section corresponds to a global trivialization of the bundle, and also corresponds to Čech-exact functions $s_\alpha : U_\alpha \to \mathbb{C}^*$. In particular, the space of trivializations corresponds to $H^0(M, \mathcal{O}^*)$ (which we'll say more explicitly next time).

A good way to think of $H^0$ is as holomorphic automorphisms of the line bundle; in particular, $H^0(M, \mathcal{O}^*)$ corresponds to holomorphic, nowhere vanishing global function on $M$; if $M$ is compact, all such functions are constant, and in this case $H^0(M, \mathcal{O}^*) = \mathbb{C}^*$.

**Chern classes.** There are many different definitions of Chern classes; they satisfy a universal property, so they can be axiomatized; there's also a definition in terms of classifying spaces, which is possibly the best, but definitely the scariest; and in this class, we'll use Chern-Weil theory, which involves curvature.

In any case, they have universal properties, are topologically invariant under (continuous, but we only care about) smooth isomorphism of complex vector bundles, and come from obstruction theory. Thus, they're useful for understanding things such as whether two bundles are isomorphic, or whether one is trivial, and so on. Most interestingly, though, is whether a bundle has a nowhere vanishing section (given by the top Chern class), or $k$ such (globally) linearly independent sections?

Chern classes are defined for any continuous complex vector bundle over a topological space, though we'll use smooth bundles over complex manifolds.

**Definition.** Let $E \to M$ be a smooth bundle over a complex manifold $M$; then, the *first Chern class* $c_1(E) \in H^2(M, \mathbb{Z})$ is the unique element that satisfies the following axioms.

(1) Naturality: $c_1(f^*E) = f^*(c_1(E))$.
(2) Sums: $c_1(E \oplus F) = c_(E) \oplus c_1(F)$.
(3) Normalization: $c_1(\tau) = -h$, where $\tau \to \mathbb{CP}^1$ is a tautological line bundle, and $h \in H^2(\mathbb{CP}^1, \mathbb{Z})$ is the generator, Poincaré-dual to the point, i.e.
$$\int_{\mathbb{CP}^1} c_1(\tau) = -1.$$

These will also imply how $c_1$ behaves with products, which isn't an axiom: $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Surprisingly, it's also additive.

We don't want to think too much about category theory, but the assignment $c_1(E)$ from a complex vector bundle $E$ is functorial. Here are some more properties:

- Normalization also implies that $c_1(\tau) = -h$ for $\tau \to \mathbb{CP}^n$ the tautological line bundle, which is somewhat immediate from restriction to $\mathbb{CP}^1$.
- But more interestingly, recall that $\det(E) = \Lambda^{\mathrm{top}}(E)$; then, $c_1(\det E) = c_1(E)$, which is a property we will end up using.

All of these properties follow from something called the *splitting principle* and the other axioms. This has nothing to do with complex analysis, but is good to know; a good reference is Bott and Tu. While it's not true that every complex vector bundle splits as a direct sum of line bundles, but after a pullback to something called a *splitting manifold* $M(E)$, it does so, and this splitting manifold is a canonical construction (that does depend on the bundle), and then one can pushforward.

The basic idea of the splitting manifold is that if one pulls $E$ back to itself, it has a canonical section $\sigma(x, e; e) = (x, e)$ (this is a map $E \to \pi^*E$), and this descends to the projectivization $\mathbb{P}(E)$ of $E$, and thus splits off a line bundle; then, repeat this process.

There are two more useful properties worth mentioning; we might actually prove them later.

- $c_1$ is a complete topological invariant for smooth line bundles, i.e. $E \cong F$ iff $c_1(E) = c_1(F)$.
- $c_1(L) = \chi(L)$ is the Euler class, which is the Poincaré dual of the generating section.

Next time, we'll construct all of the Chern classes using curvature.

## 12. 2/12/15

## 13. Sheaves and Čech Cohomology: 2/17/15

*"Algebra... whatever."*

Next time, we will discuss divisors and line bundles, albeit not in a whole lot of detail, since it relies on commplex analysis we haven't talked about. To read more about this, read the appendix to Moroianu and §2.3 of Huybrechts.

But today, we'll talk about sheaves, sheaf cohomology, and applications. Sheaves should be thought of as generalizations of bundles, especially the notion of things patched together like sections. We'll define Čech cohomology on sheaves, and show Dolbeault's theorem, that $\check{H}(M;\Omega^p) \simeq H^{p,q}(M)$. It will also be helpful that a short exact sequence of these leads to a long exact sequence on cohomology.

These will be applied to classify holomorphic or complex line bundles, which relates to the 1-Čheck cocycle and the 1st Chern class. This relates to obstructions and determining whether a given complex line bundle has a holomorphic structure. This relates to somehow yet two more Lefschetz theorems.

**Sheaves.** Let $M$ be a complex manifold. Recall that locally there are lots of holomorphic functions, but globally maybe there are fewer. For example, if $M$ is compact, they must be constant.

This motivates the *sheaf of holomorphic functions* on $M$. On each open $U \subseteq M$, let $\mathcal{O}_M(U)$ be the space of holomorphic functions on $U$. The usual example is that $U = U_\alpha$ is a chart, so that $\mathcal{O}_M(U_\alpha) \cong \mathcal{O}_{\mathbb{C}^n}(\varphi_\alpha(U_\alpha))$.

In many arguments, it was helpful to shrink the charts, which leads to the notion of *germs of holomorphic functions.* This is the *stalk* of the sheaf at any point $x \in M$, i.e. the germ at $x$ is $\mathcal{O}_{M,x} = \varinjlim_{U:x\in U} \mathcal{O}_M(U)$; that is, one considers the holomorphic functions in any neighborhood of $x$, with two functions considered the same if they agree on any open neighborhood of $U$.

Note that this is an extremely informal argument; it would take perhaps six lectures to do everything properly and rigorously!

In the same way, one can consider $\mathcal{O}_M^*$, the *sheaf of nowhere vanishing holomorphic functions* on $M$; $\Gamma(E) = \mathcal{A}(E)$, the *sheaf of smooth sections* on a smooth vector bundle $E$; and $\Omega(E) = \mathcal{O}(E)$, the *sheaf of holomorphic sections* of a holomorphic vector bundle $E$. There are more examples of sheaves, e.g. skyscraper sheaves, but we won't worry about them right now.

We've seen this before in Dolbeault cohomology; for example, $\mathcal{A}^{p,q}(M;E)$ consists of smooth section of $\Lambda_M^{p,q} \otimes E$.

**Definition.** A *sheaf* $\mathcal{F}$ on a second-countable, Hausdorff topological space $M$ is an association:

- To each open $U \subseteq M$, associate a group (or vector space, or some algebraic gadget) $\mathcal{F}(U) = \Gamma(U,\mathcal{F})$, called the *section of* $\mathcal{F}$ *over* $U$.
- To each inclusion $U \subseteq V$ of open sets, associate the *restriction map* $r_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$, which is required to be a homomorphism.

These are required to satisfy the following compatibility conditions.

(1) $r_{U,V} \circ r_{V,W} = r_{U,W}$ for all $U \subseteq V \subseteq W$ open, and $r_{U,U} = \mathrm{id}$.
(2) The gluing axiom: if $\sigma_i \in \mathcal{F}(U_i)$ for $i = 1,2$ agree on an overlap $U_1 \cap U_2$, then there is a $\sigma \in \mathcal{F}(U_1 \cup U_2)$ such that $\sigma|_{U_i} = \sigma_i$ for each $i$.
(3) The uniqueness axiom: if $\sigma \in \mathcal{F}(U_1 \cup U_2)$ and $\sigma|_{U_i} = 0$ for each $i$, then $\sigma = 0$.

Even though the geometric intuition is reasonably clear, the algebra takes some time to write down.

The topological condition is satisfied by all smooth manifolds, which will be enough for this class.

*Note.* In the definition of a sheaf, if only axiom (1) is satisfied, the resulting structure is called a *presheaf.* This is equivalent to a contravariant functor from the open sets of $M$ (with inclusions a morphisms) to the category of groups (or whichever algebraic structure we're looking at).

**Definition.** The *stalk* or *fiber* at $x$ is the generalization of germs: $\mathcal{F}_x = \varinjlim_{U:x\in U} \mathcal{F}(U)$.

One can define morphisms of sheaves and presheaves, and with a little care, exact sequences. For example, consider the *exponential sequence* below:

$$0 \longrightarrow \mathbb{Z} \xhookrightarrow{\ i\ } \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0.$$

Here, $\exp : f \mapsto \exp(2\pi i f)$. $i$ is injective for all $U$, but is surjective only if $U$ is "small enough," e.g. if $U = \mathbb{C}^*$, then it's not possible to take the logarithm for all $z \in \mathcal{O}^*(U)$. Thus, the naïve definition of an image of cokernel of a sheaf is only a presheaf; be careful! However, the naïve notion of a kernel is fine.

*Note.* To each presheaf $\mathcal{F}$, one can associate a sheaf $\mathcal{F}^+$, called the *sheafification* (the process is called *sheafifying*, which is fun to say), which is a completely algebraic procedure; if $\mathcal{F}$ is a sheaf, then $\mathcal{F}^+ = \mathcal{F}$, and this manages to fix the exactness problem raised just above.

**Sheaf Cohomology.** Technically, sheaf cohomology and Čech cohomology are two different things, but they will be presented as one thing; specifically, we'll be talking about the latter. These record the failure of local sections to glue globally.

Fix a (pre)sheaf $\mathcal{F}$ over a smooth manifold $M$, and choose a locally finite cover $\mathcal{U} = \{U_\alpha\}$ of $M$. For each finite index set $I$, let $U_I = \bigcap_{\alpha\in I} U_\alpha$.

Then, define the *Čech (co)chain complex*

$$\check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{\delta} \check{C}^2(\mathcal{U},\mathcal{F}) \longrightarrow \cdots$$

in the following way: $\check{C}^k$ is the set of sections on $(k+1)$-overlaps: $\check{C}^0 = \prod_\alpha \mathcal{F}(U_\alpha)$, and in general,

$$\check{C}^k = \prod_{|I|=k+1} \mathcal{F}(U_I).$$

Then, the *(co)boundary* operator is

$$(\delta\sigma)_I = \sum_{j=0}^{k+1} (-1)^j \left(\sigma_{I\setminus j}\right)\big|_{U_i}.$$

It's the alternating sum of looking at one less overlap. Of course, there are two different sign conventions, e.g. Huybrechts uses the opposite one.

So, you can check (or Čech) that $\delta^2 = 0$, and when one takes the (co)homology, one obtains the *Čech cohomology* $\check{H}^q(\mathcal{U},\mathcal{F})$ of the cover $\mathcal{U}$ with coefficients in $\mathcal{F}$. Then, one can refine the cover $\mathcal{U}$, which induces a chain homotopy, so one can take the direct limit to obtain the Čech cohomology of $M$.

$$\check{H}^q(M,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U};\mathcal{F}).$$

This is really exactly the same idea as constructing refinements of partitions to obtain the Riemann integral!

Don't Google Čech cohomology, because the first result is a bunch of really confusing category-theoretic words. Unless you're all about category theory, I guess.

For example, $\check{H}^0(M,\mathcal{O})$ is the group of globally holomorphic functions, and $\check{H}^0(M,\mathbb{Z})$ is the set of locally constant integer-valued functions.

Short exact sequences (which one has to be careful with) induce long exact sequences as before. And the inclusion from holomorphic sections to smooth ones induces the following commutative diagram.

Short exact sequences (which one has to be careful with) induce long exact sequences as before. And the inclusion from holomorphic sections to smooth ones induces the following commutative diagram.

$$\begin{array}{ccccccccc}
H^1(M,\mathbb{Z}) & \longrightarrow & H^1(M,\mathcal{O}) & \longrightarrow & H^1(M,\mathcal{O}^*) & \xrightarrow{\delta^*} & H^2(M,\mathbb{Z}) & \longrightarrow & H^2(M,\mathcal{O}) \\
\| & & \downarrow & & \downarrow & & \| & & \downarrow \\
H^1(M,\mathbb{Z}) & \longrightarrow & H^1(M,\mathcal{A}) & \longrightarrow & H^1(M,\mathcal{A}^*) & \xrightarrow{\delta^*} & H^2(M,\mathbb{Z}) & \longrightarrow & H^2(M,\mathcal{A})
\end{array} \tag{6}$$

Here are some results we'll use, but not prove.

**Proposition 13.1.** *If $\mathcal{F}$ is the sheaf of smooth sections in some bundle, then one can patch local sections, so if $q > 0$, then $\check{H}^q(M;\mathcal{F}) = 0$.*

Thus, in particular, $\check{H}^q(M; A^{*,*}) = 0$.

**Theorem 13.2** (Leray)**.** *If the cover is* acyclic*, then $\check{H}^*(\mathcal{U};\mathcal{F}) = \check{H}^*(M;\mathcal{F})$, i.e. $\check{H}^q(\mathcal{U}_I,\mathcal{F}) = 0$ for all $q > 0$ and $I$ finite.*

**Proposition 13.3.** *The Čech and "usual" (e.g. singular) cohomologies on a manifold coincide: $\check{H}^*(M;\mathbb{Z}) = H^*(M;\mathbb{Z})$.*

If one takes an *acyclic resolution*

$$0 \longrightarrow \mathbb{R} \hookrightarrow A^0 \xrightarrow{\mathrm{d}} A^1 \xrightarrow{\mathrm{d}} A^2 \longrightarrow \cdots$$

it's possible to use the Poincaré lemma to prove the following.

**Theorem 13.4** (de Rham)**.** *$\check{H}^q(M;\mathbb{R}) = H^q_{\mathrm{dR}}(M)$ (that is, the de Rham cohomology).*

**Theorem 13.5** (Dolbeault)**.** *Using the $\overline{\partial}$-Poincaré lemma, one can use the resolution*

$$0 \longrightarrow \Omega^p \hookrightarrow A^{p,0} \xrightarrow{\overline{\partial}} A^{p,1} \xrightarrow{\overline{\partial}} A^{p,2} \xrightarrow{\overline{\partial}} \cdots$$

*to show that $\check{H}^q(M,\Omega^) = H^{p,q}(M)$.*

Basically, on a manifold, all cohomology theories are very similar.

Now, we can actually use this to do some complex geometry. If this was bewildering, keep in mind that one cares about the Čech cohomology to keep track of how local sections do or don't patch into global ones.

**Theorem 13.6** (Kodaira-Spencer)**.** *Complex line bundles on a smooth manifold $M$ are classified up to isomorphism by $[g_{\alpha\beta}] \in \check{H}^1(M;\mathcal{A}^*) \simeq H^2(M;\mathbb{Z})$, with the isomorphism given by $\delta^*(L) = [c_1(L;\nabla)] = c_1(L)$, the first Chern class.*

*Holomorphic line bundles on a complex manifold $M$ are classified by $[g_{\alpha\beta}] \in \check{H}^1(M;\mathcal{O}^*)$. Thus, there is a natural isomorphism* $\mathrm{Pic}(M) \simeq H^1(M;\mathcal{O}^*)$.

*Proof.* First, look back at (6), using the exponential sequence for smooth or holomorphic functions. Since $\mathcal{A}$ has a partition of unity, so $H^q(M,\mathcal{A}) = 0$ when $q > 0$. Thus, $\delta^*$ is an isomorphism.

It's important to check that a holomorphic (or smooth) line bundle and a choice of trivialization gives a 1-Čech cocycle, and its class is independent of trivialization; then, that any cocycle in $\check{H}^1(\mathcal{U})$ represent a line bundle trivialized over $\mathcal{U}$; and if a cocycle $[g]$ is exact, then the bundle is trivial, i.e. if $g = \delta(\varphi)$, then $g_{\alpha\beta} = \varphi_\beta/\varphi_\alpha$.

Last lecture, we showed this is equivalent to showing that $\varphi$ is a nowhere zero section of $L$, and therefore is equivalent to a trivialization.

These can all be walked through, but it's still unclear why $\delta^*(L) = [c_1(L;\nabla)]$. These follow from formulas we proved last time! This is a big diagram chase, with an appropriately big diagram I wasn't able to write down in time. We have that $c_1(L,\nabla) - (i/2\pi)\Theta$ and $\Theta_\alpha = \mathrm{d}\tau_\alpha$, and also that $\tau_\beta - \tau_\alpha = -\mathrm{d}\log(g_{\alpha\beta})$; see the last lecture for more details. It will also be helpful to have the calculation

$$\delta^* g = \frac{1}{2\pi i}(\log g_{\alpha\beta} + \log g_{\beta\gamma} + \log g_{\gamma\alpha}).$$

But $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$.

One ends up taking the double Čech-de Rham complex $\check{C}^q(\mathcal{U}, A^p)$ with values in $A^p$. ⊠

**Corollary 13.7.** *If $H^{0,2}(M) = 0$, then any complex line bundle has a holomorphic structure; if $H^{0,1}(M) = 0$, then this structure is unique.*

*Proof.* From the exponential sequence and Dolbeault's theorem, $H^q(M,\mathcal{O}) = H^{0,q}(M)$ for all $q$, so plug the assumptions into the sequence

$$H^1(M,\mathcal{O}) \longrightarrow H^1(M,\mathcal{O}^*) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathcal{O}).$$ ⊠

**Theorem 13.8** (Lefschetz theorem on $(1,1)$-classes)**.** *Assume $M$ is a compact, Kähler manifold. Then,* $\mathrm{Pic}(M) \to H^{1,1}(M) \cap H^2(M,\mathbb{Z})$ *surjects, i.e. for all $\gamma \in H^{1,1}(M,\mathbb{Z})$, there exists a holomorphic line bundle structure such that $c_1(L) = \gamma$; and moreover, for any real, closed $(1,1)$-form $\omega$ such that $[\omega] = c_1(L)$ there exists a Hermitian metric such that the Chern connection has $c_1(L,\nabla) = 0$.*

The proof is once again a big diagram chase, using the de Rham and Dolbeault cohomologies, as well as the $\partial\overline{\partial}$-Poincaré lemma.

## 14. 2/19/15

## 15. Special Holonomy and Kähler-Einstein Metrics: 2/24/15

Today, we'll return to the differential-geometric world, rather than the algebraic geometry from the last few lectures. The goal is to provide a holonomy condition for Kähler metrics and the Kähler-Ricci flat condition, which relates to the notion of a Calabi-Yau manifold. More generally, there are bundle versions of these, known as the Hermitian-Einstein or Hermitian-Yang-Mills condition.

There will be a trichotomy of 0, positive definite, and negative definite made in the curvature; there will be a clear relation between this and other properties, though we don't have time to cover the relationship between the curvature and stability.

**Holonomy.** If $E \to M$ is a bundle over a (real or complex) manifold $M$ and $\nabla$ is a connection on it, then there's a parallel transport $P_\gamma : E_x \to E_y$, where $\gamma : [0,1] \to M$ is a path from $x$ to $y$. $P_\gamma$ is a linear isomorphism, and any $\sigma_0 \in E_x$ can be uniquely extended to a *parallel section* $\sigma(t)$ along $\gamma$, i.e. a solution to the ODE $\nabla_{\dot\gamma(t)} = 0$ and initial condition $\sigma(0) = \sigma_0$.

That's holonomy in complete generality; if $\gamma$ is a loop ($x = y$), then $P_\gamma$ is an automorphism, and it might not be the identity; that is, $P_\gamma \in \mathrm{GL}(E_x)$. If one takes all possible loops $\gamma$ based at $x$, the result is the *holonomy group* $\mathrm{Hol}_x(E,\nabla) = \{P_\gamma\}$ (for loops $\gamma$ based on $x$). $\mathrm{Hol}_x(E,\nabla) \subseteq \mathrm{GL}(E_x)$. One can also define the *restricted holonomy subgroup*, which is generated only by contractible loops, in case it's useful to have something with vanishing $\pi_1$.

Holonomy is conjugate along paths; that is, if $\gamma$ is a path from $x$ to $y$, then $\mathrm{Hol}_y(E,\nabla) = P_\gamma \circ \mathrm{Hol}_x(E,\nabla) \circ P_\gamma^{-1}$. In particular, if $M$ is connected, there is a well-defined (up to conjugacy) holonomy subgroup, called $\mathrm{Hol}(E,\nabla)$.

**Flat Bundles.** Like everything else in this class, we'll barely cover this, and it would be nice to spend more time on them.

A bundle is flat if its curvature is, which strictly speaking depends on the connection. In differential geometry, the curvature is called $R^\nabla$, but sometimes, this is called $F$, and flatness just means that $R^\nabla = 0$.

By Stokes' theorem, on a flat bundle, holonomy is invariant under homotopy of paths relative to endpoints, which is to say that it descends to the fundamental group $\pi_1(M)$; that is, this gives the *holonomy representation* (sometimes called *monodromy*, though technically that's a different word) $\pi_1(M,x) \to \mathrm{GL}(E_x)$.

*Note.* As a corollary, the bundle is flat iff the restricted holonomy is trivial (which is why assuming a manifold is simply connected is nice). On a complex manifold, this also implies that all Chern classes are torsion, i.e. identically zero in $H^*_{\mathrm{dR}}(M)$.

*Note.* If $M$ is simply connected, so that holonomy and restricted holonomy are identical, then any flat (principal) $G$-bundle is trivial, i.e. it has a nonzero, parallel section. This is proven by existence and uniqueness of a solution to an ODE.

In general, if $M$ isn't simply connected, local parallel sections always exist, though they may not patch globally. Equivalently, a nice solution will exist on the universal cover, but it might not be well-defined on $M$ itself.

*Note.* Holonomy identifies the moduli space of flat, principal $G$-bundles (up to isomorphism) with representations $\pi_1(M) \to G$ up to conjugacy; there is a one-to-one correspondence. In other words, given a representation, there is some sort of line bundle. The best reference for this is the paper by Atiyah and Bott.

For example, if $G$ is a finite group, then a principal $G$-bundle (i.e. with a free $G$-action) is a regular cover, and this is just a restatement of deck transformations and their relations to the fundamental group.

As another example, the moduli space of flat, holomorphic line bundles on a compact Riemann surface $\Sigma$ is Jacobian $\mathrm{Pic}^0(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), S^1)$ (where $S^1$ is interpreted as an abelian group), since in this case, $G = \mathrm{U}(1) = S^1$, so it's abelian, and there is no conjugacy. And since this descends to $H_1$ as the abelianization, this is also $H_1(\Sigma, S^1)$.

All of these facts are nice and important and pretty, though strictly speaking we won't depend on them in this class.

**Holonomy and Kähler Manifolds.** Let $M$ be a $2n$-dimensional Riemannian manifold, $E = TM$, and $\nabla$ be the Levi-Civita connection. Now, this means parallel transport is an isometry, because $\nabla$ is metric-compatible, so $\mathrm{Hol}_x(M,g)$ is conjugate to a subgroup of $\mathrm{O}(2n)$.

Berger's theorem is hopefully review from differential geometry; it classifies all possible holonomies on a simply connected, irreducible, Riemannian manifold.

**Definition.** A Riemannian manifold is *irreducible* if it isn't a product of lower-dimensional Riemannian manifolds; equivalently, it gives an irreducible representation.

It's possible to do the same thing for an $n$-dimensional complex manifold $M$ with a Hermitian metric. Out of Berger's list, here are a couple possible holonomies.

- If the holonomy is $\mathrm{U}(n)$, the result is a Kähler manifold.
- If the holonomy is $\mathrm{SU}(n)$, the result is a Ricci-flat Kähler manifold, i.e. a *Calabi-Yau manifold.*
- If $n$ is even, $\mathrm{Sp}(n/2)$ gives rise to a *hyper-Kähler manifold* (also known as a *holomorphic symplectic manifold*).

There are a few more, but these three are the most interesting to us, and we'll focus in particular on the first two. The third case, though, is very special and very rare; it has rich properties, since it has so much symmetry, and in fact it can be regarded as the quaternionic analogue of the Kähler condition! This theory is much more rigid, and it's a special case of the second one, since $\mathrm{Sp}(n/2) \subseteq \mathrm{SU}(n) \subseteq \mathrm{U}(n)$, and indeed $\mathrm{Sp}(n/2) = \mathrm{U}(n/2, \mathbb{H})$, which is a subgroup of $\mathrm{U}(n, \mathbb{C})$. For example, if $n = 2$ (since 1 isn't divisible by 2), then $\mathrm{Sp}(1) = \mathrm{SU}(2)$, so K3 surfaces and tori are hyper-Kähler.

For the rest of the class, we'll focus on the first two.

**Proposition 15.1** (Holonomy principle)**.** *Any holonomy-invariant (pointwise) tensor at $T_xM$ extends uniquely to a parallel tensor field on $TM$.*

This follows, like so many before it, from the existence and uniqueness of a solution to a certain ODE.

**Lemma 15.2.** *Let $(M,g)$ be a $2n$-dimensional real, connected manifold; then, it is Kähler iff* $\mathrm{Hol}(M,g) \subseteq \mathrm{U}(n)$*. Moreover, if the holonomy is precisely* $\mathrm{U}(n)$*, this structure is unique: there exists a unique integrable almost complex structure $\mathcal{J}$ such that $(M, \mathcal{J}, g)$ is Kähler.*

*Proof.* Suppose $M$ is Kähler. Then, $\nabla\mathcal{J} = 0$ (where $\nabla$ is the Levi-Civita connection), and holonomy is complex linear, which implies that $\mathrm{Hol}(M, x) \subseteq \mathrm{O}(2n)$ and $\mathrm{Hol}(M, x) \subseteq \mathrm{GL}(n, \mathbb{C})$, but the intersection of these two is $\mathrm{U}(n)$ (i.e. transformations which preserve the metric and are complex linear must be unitary). This was used implicitly earlier in the class, during Chern-Weil theory.

The other direction is harder; suppose $\mathrm{Hol}(M) \subseteq \mathrm{U}(n)$. Fix $\mathcal{J}_x$, a complex structure on the fiber $T_xM$. Then, since the holonomy is unitary, then $\mathcal{J}_x$ is holonomy invariant, so by the holonomy principle, there's a unique extension to a parallel tensor $\mathcal{J} \in \mathrm{End}(TM)$. By the chain rule, this means $\mathcal{J}^2$ is parallel, but at $x$, $\mathcal{J}^2 = -\mathrm{id}$, and $-\mathrm{id}$ is also parallel. But by existence and uniqueness, this means $\mathcal{J}^2 = -\mathrm{id}$ everywhere, so $\mathcal{J}$ is an almost complex structure parallel with respect to the Levi-Civita connection; that is, $\nabla\mathcal{J} = 0$.

Now, $g$ is parallel, so $\omega(\cdot, \cdot) = g(\mathcal{J}\cdot, \cdot)$ is parallel, too (the composition of parallel tensors is parallel), and therefore it is also closed, and therefore $M$ is Kähler.

Uniqueness is left as an exercise. ⊠

Note that this uses that the Levi-Civita connection is torsion-free, which is embedded within the implication $\nabla\omega = 0 \implies \mathrm{d}\omega = 0$.

Before talking about the Ricci-flat condition, there will be a brief intermission, to discuss Kähler-Einstein metrics. Assume $M$ is a Kähler manifold; then, there is the *Ricci form* $\rho(X, Y) \overset{(1)}{=} \mathrm{Ric}(\mathcal{J}X, Y) \overset{(2)}{=} i\,\mathrm{Tr}_{\mathbb{C}}(R^\nabla)$. In local, holomorphic coordinates, let $H = H_{i\bar{\jmath}}$; then, $\rho = -i\partial\overline{\partial}\log\det(H)$, where $\nabla$ is the Chern connection on $TM$, not the Levi-Civita connection.

(2) implies that $i\rho$ is the curvature of the Chern connection on the canonical line bundle $K_M = \Lambda^{n,0}$, and is the determinant bundle of the holomorphic cotangent bundle. Using this, one can prove that the *Chern form* $c_1(K_M, \nabla) = (1/2\pi)\rho = -c_1(TM, \nabla)$, ultimately following from Chern-Weil theory; we stated it a month ago, but now there is a proof (along with a proof for the expression for $\rho$ in coordinates).

When $(M, g)$ is Kähler, then (1) implies that $g$ is Einstein (i.e. $\mathrm{Ric}(g) = \lambda g$) iff $g$ is Kähler-Einstein (i.e. $\rho = \lambda\omega$; in both cases, $\lambda \in \mathbb{R}$, and it's called the *Einstein constant*). Up to a factor of $2\pi i$, $\rho$ represents the first Chern class, so if $g$ is Kähler-Einstein, then $c_1(M) = (\lambda/2\pi)[\omega]$.

The Kähler-Ricci-flat condition, i.e. $\lambda = 0$, implies that $c_1(M) = 0$ in $H^2_{\mathrm{dR}}(M)$. The Calabi-Yau theorem will imply that the converse is true.

This gives a restriction on $c_1(M)$ if $M$ admits a Kähler-Einstein metric: it has to be 0, positive, or negative, in the following sense.

*Note.* A real, $(1, 1)$-form $\alpha$ is *positive* (written, of course $\alpha > 0$), if $\alpha(\cdot.\mathcal{J}\cdot)$ is positive definite; negative definiteness is defined similarly.

For example, $\alpha = -\operatorname{Im} h$, if $h$ is a Hermitian metric, is positive. Note that if $\alpha$ is closed, then it's a Kähler form.

*Note.* Much of this discussion extends to special metrics on holomorphic vector bundles over Kähler manifolds. There are connections (no pun intended) to a variational problem and to applications within physics.

If $E$ is a holomorphic vector bundle, $h$ a Hermitian metric on $E$, and $\nabla$ the Chern connection, then $R^\nabla \in \Lambda^2(M; \mathrm{End}(E))$, so, tracing $R^\nabla$ in the $M$-direction using the Kähler metric, $\mathrm{Tr}_g(R^\nabla) \in \mathrm{End}(E)$, so if $\sigma_\alpha$ is a basis of $E$, then $(\mathrm{Tr}_g R^\nabla)^\beta_\alpha = R^\beta_{\alpha i\bar{\jmath}} g^{i\bar{\jmath}}$.

A *Hermitian-Einstein* or *Hermitian-Yang-Mills* metric on $E$ is a solution to $\mathrm{Tr}_g(R^\nabla) = \lambda\,\mathrm{id}$, for a constant $\lambda \in \mathbb{R}$. This is a nonlinear analogue to the Laplacian equation from Hodge theory.

Next time, we'll talk about the Calabi-Yau theorem and its extensions, and talk about $\mathrm{SU}(n)$ holonomy.