Varieties of Reverse Constructive Mathematics

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Four kinds of "reverse mathematics"

From "Subsystems of Second-Order Arithmetic" (S. Simpson):
The goal of **reverse mathematics** is to determine which set existence axioms are needed to prove a particular theorem of "ordinary" (classical) mathematics [**CLASS**].

From a recent lecture by D. Bridges:
Constructive reverse mathematics has two aims:

1. Determining which constructive principles are needed to prove particular theorems of constructive mathematics in the style of Bishop [**BISH**].
2. Determining which nonconstructive principles need to be added to constructive mathematics in order to prove particular classical theorems.

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Russian reverse mathematics, started by Markov and his followers more than fifty years ago, is part of "Russian recursive mathematics" [**RUSS**]. Based on intuitionistic logic, it accepts Church's Thesis CT_0 ("Every constructive function is recursive") and Markov's Principle MP_0 ("If a recursive algorithm cannot fail to converge, then it converges").

Wim Veldman, who does intuitionistic mathematics [**INT**] in the style of Brouwer, says **intuitionistic reverse mathematics** should determine which intuitionistic axioms (countable choice AC, continuous choice CC, bar induction BI, Brouwer's fan theorem FT) are needed to prove the theorems of intuitionistic analysis.

For analysis, $\mathrm{BISH} \subseteq \mathrm{CLASS} \cap \mathrm{INT} \cap \mathrm{RUSS}$ so they overlap.

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What are the essential parts of a reverse mathematics program?

- Decide on the *basic objects* of the mathematics to be studied.
- Decide on a *representation* of those objects.
- Decide on the *logic* to be studied (and the logic to be used).
- Fix a *minimal* or *basic* and a (possibly temporary) *maximal* formal or informal axiomatic theory to study.
- Identify *interesting intermediate theories* and use these to classify (over the basic theory) results which hold in the maximal theory.

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Choosing and representing the basic objects

- CLASS, BISH, RUSS and INT all study
  - natural numbers
  - rational numbers (which can be coded by natural numbers)
- CLASS studies sets of numbers.
- BISH studies (constructive) real numbers presented as regular sequences of rationals, and constructive functions. A sequence $\{x_n\}$ of rationals is *regular* if

$$\forall n \forall m [\, |x_n - x_m| \leq n^{-1} + m^{-1} \,] .$$

- RUSS studies (recursive) real numbers presented as recursively Cauchy sequences of rationals, and recursive operators.

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INT studies

- *infinitely proceeding sequences* or *choice sequences* $n_0, n_1, n_2, \ldots$ of numbers
- real numbers presented as *real-number generators*, either
  - converging sequences of (integer codes of) overlapping closed intervals with rational endpoints (Brouwer, Veldman) or
  - Cauchy sequences of rationals (Heyting, Kleene-Vesley)
- continuous functions on *spreads* or structured sets, such as
  - the *binary fan* 2^ω (Cantor space)
  - the *universal spread* ω^ω (Baire space)
  - the spread of *canonical* real number generators, infinitely proceeding sequences of rationals $\{x_n\}$ satisfying

$$\forall n \, |x_n - x_{n+1}| \leq 2^{-(n+1)}.$$

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Choosing the Logic

CLASS: classical second-order logic with extensional set equality.

BISH and INT: intuitionistic two-sorted logic with decidable equality for numbers and extensional equality (which is *not* decidable) for number-theoretic functions:

$$(\alpha = \beta) \ \equiv \ \forall x[\alpha(x) = \beta(x)].$$

In the intended interpretation α, β range over *all constructive sequences* (BISH) or *all choice sequences* (INT).

RUSS: just intuitionistic first-order logic with decidable equality for numbers, since gödel numbers can be used to represent the (recursive) functions. (Because $\{e\}(m) \simeq n$ is only r.e., RUSS needs extra care in dealing with number-theoretic functions.)

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Why are functions preferred to sets in constructive mathematics?

- Only *definable* sets (species) play a significant role in CM.
- Set membership need not be constructively decidable, e.g.

$$x \in A \;\Leftrightarrow\; [x = 0 \vee \forall y \neg T(x, x, y)]$$

  defines a property of numbers which is satisfiable (by 0) but not recursively decidable. In contrast,

$$\forall m \forall n [\alpha(m) = n \vee \neg(\alpha(m) = n)]$$

  holds in BISH and INT, as in CLASS.
- Constructive Zermelo-Fraenkel set theory CZF, which BISH accepts, uses A^B instead of $\mathcal{P}(A)$.

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Choosing Basic and Maximal Axioms for Analysis

CLASS works within second order arithmetic $\mathbf{Z}_2$:

- Basic arithmetic of natural numbers.
- Induction axiom
  $\forall X[0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X) \rightarrow \forall n(n \in X)]$.
- Comprehension schema $\exists X \forall n[n \in X \leftrightarrow \varphi(n)]$ where $\varphi(n)$ may be any formula without X free.

The minimal theory for CLASS is $\mathbf{RCA}_0$:

- Restrict induction to Σ^0_1 sets (with parameters).
- Restrict $\varphi(n)$ to have only bounded quantifiers. (This gives the recursive comprehension schema Δ^0_1-CA.)

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For **BISH**, "all constructive mathematics" lies within the common part of CLASS, RUSS and INT, possibly extended to Aczel's CZF.

Troelstra and van Dalen provide a minimal theory **EL** for BISH:

- Two-sorted intuitionistic (Heyting) arithmetic with the induction schema

  $$A(0) \wedge \forall s(A(x) \rightarrow A(x+1)) \rightarrow A(x)$$

  where $A(x)$ may be any formula.
- λ-conversion: $(\lambda x.t)(s) = t[x/s]$.
- Primitive recursion.
- Quantifier-free countable choice QF-AC$_{00}$:

  $$\forall x \exists y A(x,y) \rightarrow \exists \alpha \forall x A(x, \alpha(x))$$

  where $A(x,y)$ is quantifier-free and has no free α.

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For **RUSS** the minimal theory is intuitionistic first-order arithmetic **HA**. Troelstra proposes **HA** + MP_0 + ECT_0 as a maximal theory, where

- MP_0 is Markov's Principle

  $$\forall x[\neg\neg\exists y R(x,y) \rightarrow \exists y R(x,y)]$$

  where $R(x,y)$ is primitive recursive and hence decidable.
- ECT_0 is an extended version of the classically false form CT_0 of Church's Thesis used by RUSS. It says that *every* partial number-theoretic function whose domain can be defined from Σ_1^0 predicates without using $\vee$ or $\exists$ can be extended to a partial recursive function. $>$

The first minimal theory proposed for **INT** was the two-sorted theory **M** Kleene used in [1969] to formalize the theory of partial recursive functionals and his function-realizability interpretation for intuitionistic analysis. **M** consists of

- Two-sorted intuitionistic arithmetic, formalized using finitely many primitive recursive function constants and Church's λ, with unrestricted induction schema.
- Countable comprehension for functions ("non-choice") $\mathrm{AC}_{00}!$:

$$\forall x \exists ! y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)),$$

from which follows also by intuitionistic logic (with the decidability of first-order equality) the schema $\mathrm{AC}_{01}!$:

$$\forall x \exists ! \alpha A(x, \alpha) \rightarrow \exists \beta \forall x A(x, (\beta)^x)$$

where $(\beta)^x$ is $\lambda y \beta(\langle x, y \rangle)$ (the xth section of β). The only restrictions are the obvious ones on the variables.

The maximal theory for INT is Kleene and Vesley's **FIM**, which can be axiomatized as $\mathbf{M} + \mathrm{AC}_{01} + \mathrm{BI}_1 + \mathrm{CC}_{11}$ where

- AC_{01} is countable choice

  $$\forall x \exists \alpha A(x, \alpha) \to \exists \beta \forall x A(x, (\beta)^x),$$

  from which $\mathrm{AC}_{00}!$ is derivable.

- BI_1 is "bar induction," a version of Brouwer's Bar Theorem, enabling backward induction on the universal spread:

  $$\forall \alpha \exists x \rho(\overline{\alpha}(x)) = 0 \wedge \forall w[\rho(w) = 0 \to A(w)] \\ \wedge \forall w[\forall s A(w * \langle s \rangle) \to A(w)] \to A(\langle\, \rangle).$$

  (Notation: $*$ concatenates codes for finite sequences, $\overline{\alpha}(x)$ is the code for $\alpha(0), \ldots, \alpha(x-1)$, and so $\overline{\alpha}(0) = \langle\, \rangle$ codes the empty sequence.)

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- CC_{11} is *Brouwer's Principle of Continuous Choice*, abbreviated

$$\forall\alpha\exists\beta A(\alpha,\beta) \rightarrow \exists\sigma\forall\alpha[\exists\beta\ \{\sigma\}[\alpha] \simeq \beta \\ \wedge \forall\beta(\ \{\sigma\}[\alpha] \simeq \beta \rightarrow A(\alpha,\beta)\ )].$$

  Here σ codes a continuous functional with a modulus of continuity for it. Precisely, $\{\sigma\}[\alpha](x) \simeq y$ abbreviates
  $$\sigma(\langle x\rangle * \overline{\alpha}(\mu z.\sigma(\langle x\rangle * \overline{\alpha}(z)) > 0)) = y + 1.$$

There is no restriction on the logical complexity of any of the formulas $A(x, y)$, $A(w)$, $A(\alpha,\beta)$, which may contain free number and/or function variables.

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Intermediate theories and mathematical equivalents

CLASS works mostly with the theories

$$\mathrm{RCA}_0 \subsetneq \mathrm{WKL}_0 \subsetneq \mathrm{ACA}_0 \subsetneq \mathrm{ATR}_0 \subsetneq \Pi^1_1 - \mathrm{CA}.$$

Simpson's "Subsystems of Second Order Arithmetic" [1999] and "Reverse Mathematics 2001" (reviewed by U. Berger in the March 2007 *BSL*) give many details.

Weak König's Lemma WKL_0 ("Every infinite subtree of the binary tree has an infinite branch") is equivalent over RCA_0 to

- the Intermediate Value Theorem on [0,1]
- the Heine-Borel Theorem on $[0,1]^n$
- Brouwer's fixed point theorem for uniformly continuous operators on $[0,1]^n$.

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ACA_0 (arithmetical comprehension plus full second-order induction) is equivalent over RCA_0 to

- Every bounded sequence of real numbers has a least upper bound.
- the Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.
- König's Lemma: Every infinite, finitely branching tree has an infinite path.

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The main aim of reverse mathematics in **BISH** seems to be to classify, over **EL** or over **EL** + AC_{00}, the nonconstructive theorems of classical mathematics. Bridges and Richman's "Varieties of Constructive Mathematics" [1987] studies e.g.

- Π^0_1-LEM:
  $\forall n(\alpha(n) = 0 \vee \alpha(n) = 1) \rightarrow \forall n \alpha(n) = 0 \vee \neg \forall n \alpha(n) = 0$.
- LPO ("Limited Principle of Omniscience"):

$$\forall n(\alpha(n) = 0 \vee \alpha(n) = 1) \rightarrow \forall n \alpha(n) = 0 \vee \exists n \alpha(n) = 1.$$

- LLPO ("Lesser LPO"): Every binary sequence α with at most one 1 has the property that $\forall x \alpha(2x) = 0 \vee \forall x \alpha(2x+1) = 0$.

Each is in fact a function existence axiom; e.g. LPO asserts that the Σ^0_1 predicate $\exists n \alpha(n) \neq 0$ has a characteristic function.

Each is refutable in INT, by continuity.

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H. Ishihara, who initiated informal reverse mathematics in BISH, has introduced and studied other axioms including *Weak Markov's Principle*" WMP (cf. "Techniques of Constructive Analysis" [2006] by Bridges and Vita), and a *boundedness principle* BD-N:

$$\forall\alpha[\forall n A(\alpha(n)) \to \exists m \forall n[\alpha(m+n) < m+n]] \to \exists m \forall n[A(n) \to n < m]$$

which is equivalent over **EL** + AC_{00} to

- a form of the Banach Inverse Mapping Theorem
- Every sequentially continuous function from a separable metric space to a metric space is pointwise continuous.

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The constructive analogue of WKL_0 is Brouwer's Fan Theorem FT ("If a subtree of the binary tree has only finite branches, then there is a finite bound to the lengths of the branches") to which it is classically equivalent. FT holds in INT but contradicts RUSS, so is independent over BISH.
Over $\mathbf{EL} + \mathrm{AC}_{00}$, FT is equivalent to

- WKL_0!: Each infinite subtree T of the binary fan with the property

$$\forall\alpha\forall\beta[\exists n\neg(\alpha(n)=\beta(n)) \rightarrow \exists n[\neg T(\overline{\alpha}(n)) \vee \neg T(\overline{\beta}(n))]]$$

  has an infinite path. (J. Berger and H. Ishihara [2005])
- Dini's Theorem for compact (complete and totally bounded) metric spaces. (J. Berger and P. Schuster [2006])

RUSS considers intermediate theories obtained by adding to **HA** one or more of MP_0, CT_0, ECT_0.

Beeson [1975] showed that the Kreisel-Lacombe-Shoenfield and Myhill-Shepherdson Theorems are independent over **HA** $+ \mathrm{ECT}_0$ but provable in **HA** $+ \mathrm{MP}_0 + \mathrm{ECT}_0$. He found a version KLS^* of KLS which is equivalent to MP_0 over **HA**.

Number-realizability (Kleene [1945], D. Nelson [1947]) establishes the consistency of **HA** $+ \mathrm{MP}_0 + \mathrm{ECT}_0$.

Troelstra [1973] proved that if **T** is **HA** or **HA** $+ \mathrm{MP}_0$ then ECT_0 is equivalent over **T** to "number-realizability $=$ truth."

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INT considers three kinds of principles over **M**:

- Axioms of countable choice AC_{00}, AC_{01} and dependent choice DC_{00}, DC_{11}.
- Principles of fan induction (or "Brouwer's Fan Theorem" FT) and bar induction BI, in several versions.
- Continuity principles: continuous comprehension $CC_{10}!$, weak continuity WC_{10} and WC_{11}, continuous choice CC_{10} and CC_{11}, and Troelstra's 'generalized continuity' principles GC_{10} and GC_{11}, all conflicting with CLASS.

All but DC_{00}, DC_{11}, GC_{10} and GC_{11} are theorems of **FIM**. Most relative independence questions have been solved. The resulting lattice of theories forms a framework for intuitionistic reverse mathematics.

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Over **M** (or **EL** + AC_{00}) with intuitionistic logic, FT entails the Heine-Borel Theorem for $[0, 1]$ and is equivalent to each of the following:

- uniform continuity of continuous functions on a compact metric space,
- Riemann integrability of continuous real-valued functions on $[0, 1]$,
- boundedness of continuous real-valued functions on $[0, 1]$.

For detailed proofs see e.g. I. Loeb [2005].

FT does *not* entail the usual classical form of the Bolzano-Weierstrass or Intermediate Value Theorem.

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Some variants of INT accept at least the weak form KS^-

$$\exists\alpha[(\exists x\alpha(x) \neq 0 \rightarrow A) \wedge (\forall x\alpha(x) = 0 \rightarrow \neg A)]$$

(with α not free in A) of "Kripke's Schema," which was inspired by Brouwer's late philosophy but conflicts with CC_{11} (Myhill).

A few intuitionists see nothing wrong with Markov's Principle MP_0, or even with the form MP_1:

$$\forall\alpha[\neg\neg\exists x\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0],$$

because $\vdash_{\mathbf{M}} \forall x(\alpha(x) = 0 \vee \neg\alpha(x) = 0)$. A constructive theory can't have both, since $\mathbf{M} + \mathrm{MP}_1 + \mathrm{KS}^- = \mathbf{M} + (A \vee \neg A)$.

MP_1 is consistent with $\mathbf{FIM} + \mathrm{DC}_{11} + \mathrm{GC}$.

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Questions for constructive reverse mathematics

Beeson [1975] identified two fundamental questions in the foundations of constructive mathematics:

1. Are all extensional functions continuous?
2. What general principles for defining sets (or species) are constructively justifiable?

He found strong evidence for their independence over **HA** (and **HAS**, the intuitionistic arithmetic of species).

Reconsidering Simpson's question, we may ask:

3. What general principles for asserting the existence of (extensional) functions are constructively justifiable?
4. Which function existence axioms are needed to prove classical theorems consistent with **FIM**?

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Some things we know about arithmetic

HA and **PA** prove the same Π^0_2 sentences by **Markov's Rule**:

If $\vdash_{\mathbf{HA}} \forall x(R(x) \vee \neg R(x)) \wedge \neg\neg\exists x R(x)$ then $\vdash_{\mathbf{HA}} \exists x R(x)$.

So **HA** and **PA** have the same provably recursive functions, in the strong sense given by the

Church-Kleene Rule for arithmetic:

If $\vdash_{\mathbf{HA}} \forall x \exists y A(x, y)$ where $\forall x \exists y A(x, y)$ is closed, then for some gödel number e:

$$\vdash_{\mathbf{HA}} \forall x[\{\mathbf{e}\}(x)\downarrow \ \wedge \ A(x, \{\mathbf{e}\}(x))].$$

And of course **PA** can be interpreted in **HA** by the Gödel-Gentzen negative translation $E \mapsto E^g$. $>$

Arithmetic in the context of analysis

Observation. All of (first-order) **PA** is consistent with **FIM**, by Kleene's classical function-realizability interpretation, since every classically true arithmetical sentence can be realized by *some* function.

Goodman [1976]: **EL** $+$ AC_{01} is conservative over **HA** for arithmetical sentences.

Troelstra [1974]: **EL** $+$ AC_{01} $+$ FT $+$ CC_{11} is conservative over **EL** $+$ AC_{01} (hence also over **HA**) for arithmetical sentences.

So $\mathbf{FIM}^{-}$ (like **FIM** but with FT replacing BI_1) is conservative over **HA** for arithmetical sentences, so has the same consistency strength as **HA** and **PA**.

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Observations.

- **M** $+ \mathrm{BI}_1$ proves some arithmetical sentences which are not number-realizable and hence not provable in **HA**.
- **M** $+ \mathrm{BI}^c$ proves all of **PA**, where BI^c is

$$\forall\alpha\exists x R(\overline{\alpha}(x)) \wedge \forall w[R(w) \rightarrow A(w)] \\ \wedge \forall w[\forall s A(w * \langle s\rangle) \rightarrow A(w)] \rightarrow A(\langle\, \rangle)$$

  with $R(w), A(w)$ *strictly arithmetical*.

Note: BI^c with sequence parameters conflicts with **FIM** (Kleene [1965]).

Questions:

- Is **M** $+ \mathrm{BI}^c$ conservative over **PA** for arithmetical sentences?
- If not, what is its consistency strength?

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Some admissible rules for constructive theories

Kleene [1969]: If $\mathbf{T}$ is $\mathbf{M}$ + BI + MP or $\mathbf{FIM}$ + MP, and $\mathbf{T}^c$ is its classically correct part, then $\mathbf{T}$ satisfies the **Church-Kleene Rule** for analysis:

If $\vdash_{\mathbf{T}} \exists\alpha A(\alpha)$ where $\exists\alpha A(\alpha)$ is closed, there is a gödel number e such that $\vdash_{\mathbf{T}^c} \forall x\{\mathbf{e}\}(x)\downarrow$ and $\vdash_{\mathbf{T}} A(\{\mathbf{e}\})$.

Beeson [1980] claimed that "Every known constructive theory" satisfies Markov's Rule and the (closed) disjunction and existence rules. (A theory $\mathbf{T}$ satisfies the *disjunction rule* if, whenever $\vdash_{\mathbf{T}} A \vee B$ where A, B are closed, then $\vdash_{\mathbf{T}} A$ or $\vdash_{\mathbf{T}} B$.) The same holds for the appropriate Church-Kleene Rule, which implies the $\vee$ and $\exists$ rules.

Definition: A theory **T** extending **HA** or **M**, with intuitionistic logic, is *recursively acceptable* if and only if

- **T** satisfies the Church-Kleene Rule,
- **T** satisfies Markov's Rule (with parameters), and
- **T** is consistent with Markov's Principle.

Constructive theories are recursively acceptable, i.e.:

- The arithmetical theories **HA** $\pm$ $\mathrm{MP}_0 \pm \mathrm{CT}_0 \pm \mathrm{ECT}_0$ are recursively acceptable. Beeson [1975] showed that they are also closed under the Kreisel-Lacombe-Shoenfield rule.
- The analytical theories **M** $\pm$ $\mathrm{AC}_{00} \pm \mathrm{AC}_{01} \pm \mathrm{MP}_1 \pm \mathrm{BI}_1 \pm \mathrm{CC}_{10} \pm \mathrm{CC}_{11} \pm \mathrm{GC}_{10} \pm \mathrm{GC}_{11}$, hence in particular **EL** $\pm$ AC_{00} and **FIM**, are all recursively acceptable. Troelstra has checked that they satisfy the rules corresponding to GC_{10}, GC_{11}.

Observations:

- **HA** + MP_0 proves that every Δ^0_1 relation has a recursive characteristic function. (Only the converse, that every recursive relation is Δ^0_1, holds in **HA**.)
- **HA** + MP_0 + ECT_0 proves that the constructive arithmetical hierarchy collapses at Σ^0_3.
- **M** + BI_1 + MP_1 proves that every Δ^0_1 relation has a characteristic function recursive in the sequence parameters.
- **M** + BI_1 + MP_1 proves the constructive arithmetical hierarchy (with or without sequence parameters) is proper. (**FIM** can't prove this.)

So bar induction and Markov's principle save the arithmetical hierarchy. However,

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Veldman [1981]: In **FIM** the constructive analytical hierarchy collapses at Σ^1_2.

So the continuity principle (as an axiom schema) destroys the analytical hierarchy, just as Church's Thesis (as an axiom schema) destroys the arithmetical hierarchy.

There are other parallels between Church's Thesis in arithmetic and *Brouwer's Thesis* ("Every constructive function is continuous") in analysis. For example, $\mathbf{M} + \mathrm{GC}_{11}$ axiomatizes function-realizability over **M** (Troelstra) just as $\mathbf{HA} + \mathrm{ECT}_0$ axiomatizes number-realizability over **HA**.

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Since recursive functions are continuous, a recursively acceptable theory can only prove the existence of continuous functions. Maybe we should call *this* requirement, e.g.

- If $\vdash \forall\alpha\exists x A(\alpha, x)$ then $\vdash \forall\alpha\exists m\exists x\forall\beta[\overline{\beta}(m) = \overline{\alpha}(m) \rightarrow A(\beta, x)]$, or
- If $\vdash \forall\alpha\exists\beta A(\alpha, \beta)$ then $\vdash \exists\sigma\forall\alpha[\,\{\sigma\}[\alpha]\!\downarrow\, \wedge A(\alpha, \{\sigma\}[\alpha])\,]$

Brouwer's Rule and use it (rather than a continuity *axiom*) as a guide for formalizing Brouwer's intuitionistic mathematics.

Then we could look for *intuitionistically and classically* acceptable axioms (possibly in the theory of functionals) which, if added to **M** + $\mathbf{BI}_1$, would save the analytical hierarchy too.

Note: Myhill [1975] asked if his intuitionistic set theory obeyed a version of Brouwer's Rule. Beeson credited Kreisel with emphasizing derived rules for constructive theories.

Classical, as opposed to constructive, existence

Brouwer [1908] famously wrote that "the theorems which are usually considered as *proved* in mathematics, ought to be divided into those that are *true* and those that are *non-contradictory*."

Thus he distinguished between proving that a mathematical object *exists* and proving that it *cannot fail to exist*. This corresponds formally to the distinction between $\exists x$ or $\exists \alpha$, and $\neg\neg\exists x$ or $\neg\neg\exists\alpha$. The doubly negated forms are Krauss' *classical existential quantifiers*.

The corresponding *classical universal quantifiers* are $\forall x \neg\neg$ and $\forall \alpha \neg\neg$. This double negation breaks the algorithmic dependence, on x or on α, of the scope of the quantifier.

Can there be no non-recursive functions?

In a recursively acceptable theory, only recursive sequences can be proved to exist. But if "α is general recursive" is expressed by

$$GR(\alpha): \ \exists e \forall x [\{e\}(x) \simeq \alpha(x)],$$

then $\forall \alpha GR(\alpha)$ obviously contradicts continuity. Moreover, fan and bar induction fail on the recursive sequences. But

Vesley [1971] proposed a schema VS which entailed all of Brouwer's weak counterexamples and was consistent with **FIM** and (JRM [1973]) also with "Weak Church's Thesis" $\forall \alpha \neg\neg GR(\alpha)$.

So INT can't prove that nonrecursive sequences exist, not even in the classical sense of $\neg\neg\exists\alpha\neg GR(\alpha)$.

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How about Markov's Principle?

RUSS accepts MP_0; BISH works consistently with MP_1; in the context of **FIM**, MP_1 is recursively acceptable. While Brouwer didn't accept Markov's Principle, he gave no counterexample (Luckhardt [1976,1977]).

Solovay, JRM [2003]: $\mathbf{M} + BI_1 + MP_1$ proves that no arithmetical predicate $A(x)$ (with or without sequence parameters), in fact no classically Δ^1_1 predicate, can fail to have a characteristic function. So MP_1 is a *classical* function existence axiom for intuitionistic analysis.

Over $\mathbf{M}$, $BI_1 + MP_1$ is equivalent to the result BI^*_1 of replacing $\forall\alpha\exists x[\rho(\overline{\alpha}(x)) = 0]$ in BI_1 by $\forall\alpha\neg\neg\exists x[\rho(\overline{\alpha}(x)) = 0]$.

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Other recursively acceptable classical function existence axioms include the *classical axioms of countable choice* AC^*_{00}:

$$\forall x \neg\neg \exists y A(x, y) \rightarrow \neg\neg \exists \beta \forall x \neg\neg A(x, \beta(x))$$

and AC^*_{01}:

$$\forall x \neg\neg \exists \alpha A(x, \alpha) \rightarrow \neg\neg \exists \beta \forall x \neg\neg A(x, (\beta)^x).$$

Over **M**, AC^*_{00} is equivalent to a version of bar induction:

$$\forall \alpha \neg\neg \exists x R(\overline{\alpha}(x)) \ \wedge \forall w[(R(w) \rightarrow \neg A(w)) \ \wedge \\ (\forall s \neg A(w * \langle s \rangle) \rightarrow \neg A(w))] \rightarrow \neg A(\langle\, \rangle).$$

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JRM [2008]: **FIM** is consistent simultaneously with

- $A \vee \neg A$ for purely arithmetical A.
- AC^*_{00} for arithmetical $A(x, y)$ and hence
- $\neg\neg\forall x[A(x) \vee \neg A(x)]$ for arithmetical $A(x)$ (with parameters allowed), for example

$$\forall \rho \neg\neg\forall x[\exists y \rho(< x, y >) = 0 \vee \forall y \rho(< x, y >) \neq 0].$$

- "There are no functions which are not classically Σ^1_1":

$$\forall \alpha \neg\neg\exists e \forall x \forall y[\alpha(x) = y \leftrightarrow \neg\neg\exists \beta \forall z \neg T(e, x, y, \overline{\beta}(z))]$$

  (proof uses Spector-Gandy Theorem) and hence
- "There are no functions which are not classically Δ^1_1."

The theory obtained by adding all these axioms to **FIM** is not recursively acceptable (it contains **PA**) but it obeys Brouwer's Rule.

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