QUADRATIC FORMS AND PFISTER NEIGHBORS IN CHARACTERISTIC 2

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Abstract. We study Pfister neighbors and their characterization over fields of characteristic 2, where we include the case of singular forms. We give a somewhat simplified proof of a theorem of Fitzgerald which provides a criterion for when a nonsingular quadratic form q is similar to a Pfister form in terms of the hyperbolicity of this form over the function field of a form φ which is dominated by q. From this, we derive an analogue in characteristic 2 of a result by Knebusch saying that, in characteristic $\neq 2$, a form is a Pfister neighbor if its anisotropic part over its own function field is defined over the base field. Our result includes certain cases of singular forms, but we also give examples which show that Knebusch's result generally fails in characteristic 2 for singular forms. As an application, we characterize certain forms of height 1 in the sense of Knebusch whose quasi-linear parts are of small dimension. We also develop some of the basics of a theory of totally singular quadratic forms. This is used to give a new interpretation of the notion of the height of a standard splitting tower as introduced by the second author in [28].

Table of contents

1. Introduction
2. Witt cancellation and Witt decomposition
3. Subforms and dominated forms
4. Further notions and results needed in proofs
   4.1 Pfister forms and quasi-Pfister forms
   4.2 Function fields of quadratic forms and splitting towers
   4.3 The Arf-invariant and the Clifford invariant
5. Fitzgerald's theorem
6. Pfister neighbors and an analogue of a theorem by Knebusch in characteristic 2
7. Forms of nondefective height 1
8. Totally singular forms and quasi-Pfister neighbors

1991 *Mathematics Subject Classification.* Primary 11E04; Secondary 11E81.

Key words and phrases. Quadratic form, Pfister form, Pfister neighbor, quasi-Pfister form, quasi-Pfister neighbor, function field of a quadratic form, degree of a quadratic form, splitting tower, height of a quadratic form, singular quadratic form.

Both authors have been supported in part by the European research networks FMRX-CT97-0107 and HPRN-CT-2002-00287 "Algebraic K-Theory, Linear Algebraic Groups and Related Structures", and by the program INTAS 99-00817 "Linear Algebraic Groups and Related Linear and Homological Structures."

1. Introduction

An important aspect of the algebraic theory of quadratic forms is the study of function fields of quadrics and the isotropy behaviour of quadratic forms over such function fields. Questions of this type have already, sometimes implicitly, appeared in Pfister's seminal work on quadratic forms in the 1960s, which then inspired a large number of researchers in the early 1970s (e.g., Arason, Elman, Lam, Knebusch, Wadsworth, etc.) to investigate further problems in the algebraic theory of quadratic forms where the use of function fields of quadratic forms became an indispensable tool or even a subject of study per se.

Knebusch's papers [23], [24] constitute a major advance in that they attempted for the first time to provide a self-contained in-depth study of function fields of quadratic forms and the behaviour of quadratic forms over such function fields, with particular emphasis laid upon the "generic" properties of such function fields. These two papers lead to a wealth of new result, and opened many possible pathways for further research by asking important questions, some of which have only just recently been answered using new techniques developed partly in the wake of Voevodsky's work on the Milnor conjecture. Knebusch's papers were followed in the early 1980s by various articles on function fields most notably by Fitzgerald, Elman-Lam, Elman-Lam-Wadsworth, etc.

Starting in the 1990s, function field questions once again became of interest due to Merkurjev's construction of fields with u-invariant equal to any given even number (the u-invariant of a field F being the supremum of the dimensions of anisotropic forms over F which are torsion in the Witt ring of F). Of particular importance became the question of when an anisotropic form over F becomes isotropic over the function field $F(\psi)$ of another form ψ. This question has been studied extensively by the present authors, Izhboldin, Karpenko, Merkurjev, Vishik, etc.

Most of the above mentioned studies have one thing in common : they all deal initially only with quadratic forms over fields of characteristic $\neq 2$. Several of the results obtained have been extended to quadratic forms over fields of characteristic 2 (see, e.g., [1], [6], [7], [8], [10], [11], [12], [26], [31]). However, in most of these articles the emphasis has been on nonsingular quadratic forms (forms without radical) or regular forms (nonsingular forms or those with nonvanishing radical of dimension ≤ 1).

Only recently have there been efforts to develop a theory of function fields of quadratic forms in characteristic 2 which systematically includes the case of singular forms, following the initial ideas developed in [23], [24]. In [25], Knebusch embarks on a study of the generic aspects of such a theory (places, specializations, etc.) which in its scope goes far beyond previous attempts. The aspect of the isotropy of quadratic forms over function fields of quadratic forms (singular or not) and the behaviour of forms over their own function field has been treated systematically in [27], [28], [30]. It is this latter aspect which will constitute the centerpiece of the present article.

The central topic of this paper is the extension to characteristic 2 of two important theorems concerning Pfister forms and Pfister neighbors, and whose characteristic $\neq 2$ versions are due to Fitzgerald [15] and Knebusch [24]. In characteristic $\neq 2$, Pfister forms are tensor products of a finite number of binary (regular) forms and they play a central role in the algebraic theory of quadratic forms. A Pfister neighbor of a Pfister form is a form similar to an orthogonal summand of this

Pfister form and of dimension greater than half the dimension of this Pfister form. It turns out that much of the information about a Pfister form can already be retrieved from information about a neighbor of this Pfister form and vice versa. Now anisotropic Pfister forms and neighbors can intrinsically be characterized by how they behave over their own function field. In fact, an anisotropic form is similar to a Pfister form iff the form is hyperbolic over its own function field ([23], [37]), and an anisotropic form φ over F is a Pfister neighbor if its anisotropic part over its own function field is defined over the ground field F, i.e. there exists a form ψ over F such that $(\varphi_{F(\varphi)})_{\mathrm{an}} \cong \psi_{F(\varphi)}$. This latter statement is due to Knebusch [24, Th. 7.13] and also follows from a more general result due to Fitzgerald [15, Th. 1.6] which essentially can be stated as saying that if an anisotropic form φ over F becomes hyperbolic over $F(q)$ for some form q over F, and if q is in a certain sense "big enough" compared with φ, then φ is similar to a Pfister form.

The plan of the papers is as follows. The next three sections provide an account of some basic facts on quadratic forms in characteristic 2 and list some of the deeper results which we will use in our proofs. Since general references to facts about singular forms are relatively sparse in the literature, we decided to include, sometimes in quite some detail, rather basic and elementary results which might hitherto have gone unnoticed or which might belong to mathematical folklore and for which we did not find a suitable reference.

Section 2 deals with Witt cancellation and Witt decomposition which yield a notion of Witt equivalence based on comparing anisotropic parts of (possibly singular) forms.

Section 3 deals with the notions of subform (i.e. an orthogonal summand of another form) and of dominated form (i.e., the restriction of a form to a sub-vector space of the underlying vector space of another form), as well as the notion of a nonsingular completion of a singular form (i.e. a nonsingular form of minimal dimension dominating a given singular form). These notions of domination and nonsingular completion plus our notion of Witt equivalence provide very effective tools (in particular, the "completion lemma" 3.9) in order to circumnavigate the use of Witt cancellation in the original proofs of Knebusch's and Fitzgerald's theorems in characteristic $\neq 2$. The main result of section 3 is Theorem 3.11 which is the most general version in characteristic 2 of determining the maximal dimension of a common dominated form of two forms in terms of the dimensions of totally isotropic subspaces of these two forms and of their orthogonal difference.

In section 4, we give definitions of various notions and objects which will appear in the statements and proofs of our results (Pfister form, function field, splitting tower, degree, Arf-invariant and Clifford algebra, etc.), and we will state several deep results used in the proofs, some of them rather recent (e.g. [6], [7], [18]).

Section 5 is almost entirely devoted to the proof of Fitzgerald's theorem in characteristic 2. Our version of the theorem was formulated so that it also includes the case of singular forms (dominated by nonsingular forms) and their function fields.

Fitzgerald's theorem is then applied in section 6 to prove a characteristic 2 version of Knebusch's theorem. It should be remarked that Knebusch's theorem carries over to characteristic 2 for nonsingular forms without much difficulty, but we construct various types of counterexamples showing that it fails in general for singular forms. However, we conjecture that if the dimension s of the radical of an anisotropic form q over F is less than $\dim q/2$, then Knebusch's theorem holds, and

we prove this conjecture for $s \leq 4$ and for a particular case when $s = 5$ (Theorem 6.6).

As an application, we classify in section 7 forms of nondefective height 1 (i.e., forms which are not totally singular but whose anisotropic parts over their own function fields are totally singular) provided they satisfy the same hypotheses on the radical as the forms in our version of Knebusch's theorem (Theorem 7.5).

The final section 8 stands somewhat apart as it deals almost exclusively with totally singular forms. This section can be read independently of the others (perhaps with the exception of Theorem 8.16), but two or three of its results will be used in earlier sections (and might be taken for granted at that moment). It turns out that one can develop a meaningful theory of totally singular quadratic forms in characteristic 2 by almost altogether ignoring the theory of nonsingular resp. non-totally singular forms. In particular, one has notions of forms which we call quasi-Pfister forms and quasi-Pfister neighbors with properties which resemble in a surprising way many of the better known properties of "classical" Pfister forms and Pfister neighbors. Since Pfister forms and neighbors are part of the main theme of the present paper, it is for this reason too that we decided to include this section on totally singular forms. It should be noted that some of the results in this last section or variations of these results have first been shown by the second author in [29] where also other aspects of totally singular forms have been investigated. Our approach here, however, is different and based on the notion of what we call the norm field of a totally singular form. This norm field is a totally inseparable multiquadratic extension of the field of squares of the base field which is in a natural way associated to a totally singular form and which encodes various properties of that form. As an application of these techniques, we give in Theorem 8.16 (resp. Theorem 4.5) an interpretation of the standard height (as defined in [28]) of an anisotropic form (not necessarily totally singular) as the maximal height of a field tower such that the dimensions of the anisotropic parts of this form over the fields along this tower are strictly decreasing.

For all basic results on forms in characteristic 2 which we mention without reference, we refer the reader to [10], [11], [12], and sections 1.4 and 2.4 in [34].

2. Witt cancellation and Witt decomposition

Let F be a field of characteristic 2. A quadratic form over F is a pair (V, q) of an F-vector space V and a function $q : V \to F$ such that

(1) $q(ax) = a^2 q(x)$ for all $a \in F$ and all $x \in V$;
(2) There exists a symmetric bilinear form $b_q : V \times V \to F$ such that $q(x + y) - q(x) - q(y) = b_q(x, y)$ for all $x, y \in V$.

It is well-known that there exists a basis e_i, f_i, g_j and elements $a_i, b_i, c_j \in F$, $1 \leq i \leq r$, $1 \leq j \leq s$ of V with $\dim V = 2r + s$ such that for all $v = \sum_{i=1}^{r}(x_i e_i + y_i f_i) + \sum_{j=1}^{s} z_j g_j$ one has

$$q(v) = \sum_{i=1}^{r}(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^{s} c_j z_j^2 \ .$$

For a quadratic form given by such a polynomial we will write

$$q = [a_1, b_1] \perp [a_2, b_2] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \cdots, c_s \rangle$$

for short. If q is written in this way, we will call it a *normalized* form. We will often and implicitly use certain isometries which are rather obvious or easy to prove. We will refer to them as *standard relations* and they are the following :

$$\begin{gathered}\langle a\rangle \cong \langle x^2 a\rangle,\ [a,b] \cong [ax^2, bx^{-2}] \quad \text{for all } a,b \in F,\ x \in F^*;\\ \langle a,b\rangle \cong \langle a, a+b\rangle \cong \langle b,a\rangle,\ [a,b] \cong [a,a+b+1] \cong [b,a] \quad \text{for all } a,b \in F;\\ [a,b] \perp \langle c\rangle \cong [a+c,b] \perp \langle c\rangle \quad \text{for all } a,b,c \in F;\\ [a,b] \perp [c,d] \cong [a+c,b] \perp [c,b+d] \quad \text{for all } a,b,c,d \in F.\end{gathered}$$

Remark 2.1. In many books on forms in characteristic 2, the notation $\langle c_1, \cdots, c_s\rangle$ is used for the *bilinear form* given by $b(X,Y) = \sum_{i=1}^{s} c_i x_i y_i$ for $X = (x_1, \cdots, x_s)$, $Y = (y_1, \cdots, y_s)$. Since we consider almost exclusively quadratic forms and in order to simplify notations, we use the notation $\langle c_1, \cdots, c_s\rangle$ for the diagonal quadratic form $q(X) = b(X,X)$, and we denote the bilinear form by $\langle c_1, \cdots, c_s\rangle_b$ instead.

Let K/F be a field extension. If φ is a form over F, then we will denote by φ_K the form obtained after scalar extension from F to K. We write $D_K(\varphi)$ for the elements represented nontrivially by φ over K, $D_K^*(\varphi) = D_K(\varphi) \cap K^*$, and $G_K(\varphi)$ for the group of similarity factors of φ over K, i.e. $G_K(\varphi) = \{a \in K^* \mid a\varphi_K \cong \varphi_K\}$.

The subspace of V spanned by the g_i as above is nothing but the *radical* $\mathrm{rad}(q) = V^\perp$ of q, i.e. $\mathrm{span}_{V/F}\{g_1, \cdots, g_s\} = \{x \in V \mid b_q(x,y) = 0\ \forall y \in V\}$. If $s > 0$, then $q_r = [a_1,b_1] \perp [a_2,b_2] \perp \cdots \perp [a_r,b_r]$ is generally not determined uniquely up to isometry. However, $q_s = \langle c_1, \cdots, c_s\rangle$ is always determined uniquely up to isometry as the radical $V^\perp$ is determined uniquely, and q_s in the above representation will be called the *quasi-linear part* of q and denoted by $\mathrm{ql}(q)$. In particular, r and s are determined uniquely and we say that q is of type (r,s). We note that the type of q does not change after passing to an extension field.

q is said to be *nonsingular* if $s = 0$. If $s \neq 0$ (and $r = 0$) then q is said to be *singular* (*totally singular*). q will be called *nondefective* if q is nonsingular or if $\mathrm{ql}(q)$ is anisotropic.

Consider another totally singular form $q'_s = \langle c'_1, \cdots, c'_s\rangle$. Then it is an easy exercise to show that $q_s \cong q'_s$ (i.e. the two quadratic forms are isometric) if and only if $\mathrm{span}_{F/F^2}\{c_1, \cdots, c_s\} = \mathrm{span}_{F/F^2}\{c'_1, \cdots, c'_s\}$, i.e. the elements c_i generate the same sub-vector space of the vector space F over the field F^2 as the elements c'_i. In particular, if $\{d_1, \cdots, d_t\}$, $t \leq s$, is a basis of $\mathrm{span}_{F/F^2}\{c_1, \cdots, c_s\}$, then $q_s \cong \langle d_1, \cdots, d_t, 0, \cdots, 0\rangle$, with $\langle d_1, \cdots, d_t\rangle$ anisotropic. We call $\langle d_1, \cdots, d_t\rangle$ the *anisotropic part* of q_s. This anisotropic part is, by the above, uniquely determined up to isometry, and in fact we can choose the elements d_i among the c_i. We thus get :

Lemma 2.2. *Let q be a quadratic form over F of type (r,s). Let E be any field extension of F. Then the anisotropic part of* $\mathrm{ql}(q_E)$ *is defined over F.*

Proof. Let $\mathrm{ql}(q) = \langle c_1, \cdots, c_s\rangle$, $c_i \in F$. After reindexation, let $\{c_1, \cdots, c_t\}$, $t \leq s$, be a basis of $\mathrm{span}_{E/E^2}\{c_1, \cdots, c_s\}$. Then the anisotropic part of $\mathrm{ql}(q_E)$ is isometric to $\langle c_1, \cdots, c_t\rangle_E$ and thus defined over F. □

Remark 2.3. The above observation, as elementary as it is, seems to have gone to some extent unnoticed. This is evidenced for example in [8, 1.3], where a proof is found for a rather special case of the above lemma, namely for E an inseparable quadratic extension and q totally singular.

A hyperbolic plane $\mathbb{H}$ is a nonsingular binary form isometric to $[0,0]$, and a nonsingular form is said to be hyperbolic if it is isometric to an orthogonal sum of hyperbolic planes. One verifies readily that if $q = q_r \perp q_s$ is a quadratic form with q_r nonsingular and q_s totally singular and anisotropic, and if q is isotropic, then $q \cong [0,0] \perp q'_r \perp q_s$ with q'_r nonsingular.

We thus get the following result which is an analogue in characteristic 2 to the usual Witt decomposition in characteristic $\neq 2$.

Proposition 2.4. *Let q be a quadratic form over F. Then $q \cong i \times \mathbb{H} \perp \tilde{q}_r \perp \tilde{q}_s \perp j \times \langle 0 \rangle$ with $\tilde{q}_r$ nonsingular, $\tilde{q}_s$ totally singular and $\tilde{q}_r \perp \tilde{q}_s$ anisotropic. The form $\tilde{q}_r \perp \tilde{q}_s$ is uniquely determined up to isometry. In particular, i and j are uniquely determined.*

In view of this proposition, we call $\tilde{q}_r \perp \tilde{q}_s$ the *anisotropic part* of q and we write q_{an} for short. i resp. j will be called the *Witt index* resp. *defect* of q, denoted by $i_W(q) = i$ and $i_{\mathrm{ql}}(q) = j$. Furthermore, $i \times \mathbb{H} \perp q_{\mathrm{an}}$ will be called the *nondefective part* of q. For nondefective forms (the case $j = 0$), this result can be found in [9, page 160] or in [22, page 283]. Note that $i_W(q) + i_{\mathrm{ql}}(q)$ is exactly the dimension of any maximal totally isotropic subspace of q (a subspace W of the underlying vector space V of q is called totally isotropic if $q(w) = 0$ for all $w \in W$). Therefore, we call $i_{\mathrm{ti}}(q) = i_W(q) + i_{\mathrm{ql}}(q)$ the *total isotropy index* of q.

We call two forms φ and ψ *Witt-equivalent*, denoted by $\varphi \sim \psi$, if $\varphi_{an} \cong \psi_{an}$. If φ and ψ are nonsingular, then $\varphi \sim \psi$ iff $\varphi \perp -\psi$ is hyperbolic, the usual definition of Witt equivalence. Thus, the notion of Witt equivalence for nonsingular forms extends in a natural way to forms which are not necessarily nonsingular.

To prove the proposition, we need the following result on Witt cancellation in characteristic 2 (cf. [22, Prop. 1.2]).

Proposition 2.5. *Let q and q' be quadratic forms of the same dimension and let φ be a nonsingular quadratic form. If $q \perp \varphi \cong q' \perp \varphi$ then $q \cong q'$.*

We also need the following cancellation result.

Lemma 2.6. *Let q and q' be nondefective quadratic forms of the same dimension. If $q \perp j \times \langle 0 \rangle \cong q' \perp j \times \langle 0 \rangle$, then $q \cong q'$.*

Proof. Write $q \cong m \times \mathbb{H} \perp q_0$ and $q' \cong m' \times \mathbb{H} \perp q'_0$ with q_0 and q'_0 anisotropic and, say, $m' \geq m$. Put $n = m' - m$ and $q'_1 = n \times \mathbb{H} \perp q'_0$. By Witt cancellation for common nonsingular orthogonal summands (Proposition 2.5), it suffices to show that for $\varphi = q_0 \perp j \times \langle 0 \rangle$ and $\varphi' = q'_1 \perp j \times \langle 0 \rangle$, we have that $\varphi \cong \varphi'$ implies $q_0 \cong q'_1$.

Let $V = W \oplus U$ and $V' = W' \oplus U'$ be the underlying vector spaces of φ and φ', respectively, such that $\varphi|_W = q_0$, $\varphi'|_{W'} = q'_1$, and $\varphi|_U = j \times \langle 0 \rangle = \varphi'|_{U'}$.

Let now $\sigma : V \to V'$ be a vector space isomorphism which is an isometry of φ and φ', i.e. $\varphi'(\sigma(x)) = \varphi(x)$ for all $x \in V$. Let $\pi : V' = W' \oplus U' \to W'$ be the projection onto W', and define $\tau : W \to W'$ by $\tau = \pi \circ \sigma|_W$.

If $w \in W$ and $\sigma(w) = w' + u'$, $w' \in W'$, $u' \in U'$, then $\tau(w) = w'$ and thus

$$q'_1(\tau(w)) = \varphi'(\tau(w)) = \varphi'(w') = \varphi'(w' + u') = \varphi'(\sigma(w)) = \varphi(w) = q_0(w) \ ,$$

where the third equality follows from the fact that $u' \in U' \subset V'^{\perp}$ and $\varphi'(u') = 0$. To show that τ is an isometry, it suffices to show that τ is bijective. Now if $0 \neq w \in W$ then $q_0(w) \neq 0$ as q_0 is anisotropic. But then $0 \neq q'_1(\tau(w)) = q_0(w)$, hence $\tau(w) \neq 0$ and τ is therefore injective. The bijectivity follows as $\dim W = \dim W'$. $\square$

Proof of Proposition 2.4. We only have to prove uniqueness of the decomposition. Now $\tilde{q}_s \perp j \times \langle 0\rangle \cong \mathrm{ql}(q)$, and it follows immediately that j is uniquely determined and so is $\tilde{q}_s$ (up to isometry). So it suffices to show that if $i \times \mathbb{H} \perp \tilde{q}_r \perp \tilde{q}_s \perp j \times \langle 0\rangle \cong i' \times \mathbb{H} \perp \tilde{q}'_r \perp \tilde{q}_s \perp j \times \langle 0\rangle$ with $\tilde{q}_r$, $\tilde{q}'_r$ nonsingular and $\tilde{q}_r \perp \tilde{q}_s$, $\tilde{q}'_r \perp \tilde{q}_s$ anisotropic, then $\tilde{q}_r \perp \tilde{q}_s \cong \tilde{q}'_r \perp \tilde{q}_s$.

By Lemma 2.6, we have $i \times \mathbb{H} \perp \tilde{q}_r \perp \tilde{q}_s \cong i' \times \mathbb{H} \perp \tilde{q}'_r \perp \tilde{q}_s$. But then it follows immediately from Proposition 2.5 that $i = i'$ and $\tilde{q}_r \perp \tilde{q}_s \cong \tilde{q}'_r \perp \tilde{q}_s$. □

3. Subforms and dominated forms

Lemma 3.1. (i) *Let (q, V) and (φ, W) be quadratic spaces over F. Then the following are equivalent.*

(1) *There exists an injective isometry $t : (q, V) \to (\varphi, W)$ (i.e. t is an injective F-linear map $V \to W$ with $\varphi(t(x)) = q(x)$ for all $x \in V$).*

(2) *There exist nonsingular forms q_r and τ, nonnegative integers $s' \leq s \leq s''$, $c_i \in F$ $(1 \leq i \leq s'')$ and $d_j \in F$ $(1 \leq j \leq s')$ such that $q \cong q_r \perp \langle c_1, \cdots, c_s\rangle$ and*

$$\varphi \cong q_r \perp \tau \perp [c_1, d_1] \perp \cdots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1}, \cdots, c_{s''}\rangle \ .$$

(ii) *Suppose that the equivalent conditions in* (i) *hold. With the same notations, let $U = t(V)^\perp = \{x \in W \mid b_\varphi(x, t(V)) = 0\}$. Then $\varphi|_U \cong \tau \perp \langle c_1, \cdots, c_{s''}\rangle$.*

Proof. Part (ii) follows readily from part (i) and is left to the reader. As for part (i), the implication (2) ⇒ (1) is clear.

To prove the converse, we may assume that $V \subset W$ and that t is the canonical injection, so that $\varphi|_V = q$. Write $V = U \oplus \mathrm{rad}(q)$. We then have that $q_r := \varphi|_U = q|_U$ is nonsingular. Thus, with $U^\perp = \{x \in W \mid b_\varphi(x, U) = 0\}$, we get that $W = U \oplus U^\perp$. Clearly, $\mathrm{rad}(q) \subset U^\perp$. We also have $U^\perp = U' \oplus \mathrm{rad}\,\varphi$ with $\varphi|_{U'}$ nonsingular. We consider two cases.

First case : Suppose that $\mathrm{rad}(q) \subset \mathrm{rad}(\varphi)$. We choose any basis of $\mathrm{rad}(q)$ and extend it to a basis of $\mathrm{rad}(\varphi)$ such that, with respect to this basis, we get $q|_{\mathrm{rad}(q)} \cong \langle c_1, \cdots, c_s\rangle$ and $\varphi|_{\mathrm{rad}(\varphi)} \cong \langle c_1, \cdots, c_{s''}\rangle$, where $c_i \in F$ and $0 \leq s = \dim(\mathrm{rad}(q)) \leq s'' = \dim(\mathrm{rad}(\varphi))$. (2) follows readily by putting $\tau = \varphi|_{U'}$ and by noting that V (resp. W) decomposes orthogonally into $V = U \oplus \mathrm{rad}(q)$ (resp. $W = U \oplus U' \oplus \mathrm{rad}(\varphi)$).

Second case : Suppose that $\mathrm{rad}(q) \not\subset \mathrm{rad}(\varphi)$. Let $\{e_1, \cdots, e_s\}$ be a basis of $\mathrm{rad}(q)$ such that $e_1 \notin \mathrm{rad}(\varphi)$. With the same notations as before, we have that $\mathrm{rad}(q) \subset U^\perp = U' \oplus \mathrm{rad}(\varphi)$. Hence, there exists an $f_1 \in U'$ such that $b_\varphi(e_1, f_1) = \lambda_1 \neq 0$. In particular, f_1 is linearly independent of $\{e_1, \cdots, e_s\}$. Put $e'_1 = \frac{1}{\lambda_1} e_1$ and, for $2 \leq i \leq s$, put $e'_i = e_i - \frac{\lambda_i}{\lambda_1} e_1$, where $\lambda_i = b_\varphi(e_i, f_1)$. Then $\{e'_1, \cdots, e'_s\}$ is again a basis of $\mathrm{rad}(q)$, and furthermore $b_\varphi(e'_i, f_1) = \delta_{i1}$.

Let $\widetilde{U} = U \oplus Fe'_1 \oplus Ff_1$, let $c_1 = q(e'_1) = \varphi(e'_1)$ and $d_1 = \varphi(f_1)$. Then $\varphi|_{\widetilde{U}} \cong \varphi|_U \perp [c_1, d_1]$. Let $\widetilde{V} = V \oplus Ff_1$ and let $\widetilde{q} = \varphi|_{\widetilde{V}}$. It follows readily that $\mathrm{rad}(\widetilde{q})$ has basis $\{e'_2, \cdots, e'_s\}$ and that $\widetilde{V}$ has an orthogonal decomposition $\widetilde{V} = \widetilde{U} \oplus \mathrm{rad}(\widetilde{q})$, where $\widetilde{q}|_{\widetilde{U}} = \varphi|_{\widetilde{U}}$ is nonsingular. In particular, $\dim(\mathrm{rad}(\widetilde{q})) < \dim(\mathrm{rad}(q)) = s$. (2) follows now readily by induction on s together with the first case. □

Remark 3.2. The proof actually shows that q_r can be any nonsingular form with the property that $q \cong q_r \perp \mathrm{ql}(q)$.

Corollary 3.3. *Let (q,V) be a quadratic form of type (r,s) and let (φ,W) be a nonsingular quadratic form of dimension $2r+2s$. The following are equivalent.*

(1) *There exists an injective isometry $t:(q,V)\to(\varphi,W)$.*
(2) *There exist a nonsingular form q_r of dimension $2r$ and $c_j,d_j\in F$, $1\leq j\leq s$, such that for $\sigma=\langle c_1,\cdots,c_s\rangle$ and $\rho=[c_1,d_1]\perp\cdots\perp[c_s,d_s]$, we have $q\cong q_r\perp\sigma$ and $\varphi\cong q_r\perp\rho$.*

Definition 3.4. Let q and φ be forms over F.

(i) q is said to be a *subform* of φ if there exists a form τ such that $\varphi\cong q\perp\tau$. We then write $q\subset\varphi$.
(ii) q is said to be *dominated* by φ if the equivalent conditions in Lemma 3.1 hold. We then write $q\prec\varphi$ and call φ a dominant form of q. With the same notations as in Lemma 3.1, the form $\varphi|_U$ is called the *complement* of q in φ, and we write $\varphi|_U=q_\varphi^C$.
(iii) φ is said to be a *nonsingular completion* (*n.s.c.*) of q if the equivalent conditions in Corollary 3.3 hold.

Lemma 3.5. *Let $c_i,c_i',d_i\in F$, $1\leq i\leq n$, and suppose that $\langle c_1,\cdots,c_n\rangle\cong\langle c_1',\cdots,c_n'\rangle$. Then there exist $d_1',\cdots,d_n'\in F$ such that $[c_1,d_1]\perp\cdots\perp[c_n,d_n]\cong[c_1',d_1']\perp\cdots\perp[c_n',d_n']$.*

Proof. This follows readily from the standard relations and the fact that by assumption, $\{c_1,\cdots,c_n\}$ and $\{c_1',\cdots,c_n'\}$ span the same F^2-vector space. □

Remark 3.6. (i) The notion of domination was introduced in [27], [30] using condition (2) in Lemma 3.1 as definition, but under the somewhat more restrictive and less natural assumption that the c_i are already given. If the dominating form is nonsingular (the most important case), then this notion of domination coincides with the one given here as follows immediately from Lemma 3.5. In general, however, our notion is less restrictive. For example, let $F=K(X)$ be the rational function field in one variable over the field K. Then, according to our definition, the form $\langle 1,x\rangle$ is dominated by $\mathbb{H}\perp\langle 1+x\rangle$ as $\langle 1,x\rangle\cong\langle 1,1+x\rangle$, but not according to the definition in [27], [30] as $1+x\not\equiv 1,x \bmod F^{*2}$.

Condition (2) is useful for practical purposes, e.g. when doing explicit computations with coefficients in a normalized form. However, condition (1) is more natural and conceptual.

(ii) A subform of another form is obviously dominated by this other form. The notions of subform and dominated form coincide if either the dominated form is nonsingular, or if both forms are totally singular.

(iii) The notion of a complement has also been defined by Knebusch [25], but only in the case where the dominant form is nonsingular. In this situation and with above notations, Knebusch calls the complement $\varphi|_U$ the polar of φ with respect to the subspace U.

(iv) Assuming that $q\prec\varphi$ and with the same notations as in Lemma 3.1, we have that $q_\varphi^c\cong\tau\perp\langle c_1,\cdots,c_{s''}\rangle$.

The following easy lemma is quite useful and will be used frequently.

Lemma 3.7. *Let $q\prec\varphi$. Then $q\perp\varphi\sim q_\varphi^c$. In particular, if σ is a totally singular form over F and ρ is a nonsingular completion of σ, then $\rho\perp\sigma\sim\sigma$.*

Proof. This follows readily from the definitions of the complement and of a nonsingular completion, together with Remark 3.6(iv) and the fact that $[c,d] \perp \langle c\rangle \sim \langle c\rangle \sim \langle c,c\rangle$. □

Remark 3.8. If φ is anisotropic and $q \prec \varphi$, then for another form γ we have $\gamma \cong q_\varphi^c$ iff $q \perp \varphi \sim \gamma$ and γ is anisotropic.

If we assume neither anisotropy nor nondefectiveness for φ, we need to invoke other hypotheses to get a similar result. Suppose that $q \prec \varphi' \subset \varphi$ with φ' nonsingular. Let γ be another form. Then $\gamma \cong q_\varphi^c$ iff $q \perp \varphi \sim \gamma$, $\dim \varphi = \dim \gamma + \dim\ q$, $\dim \operatorname{rad}(\gamma) = \dim \operatorname{rad}(\varphi) + \dim \operatorname{rad}(q)$. Indeed, we have $q \perp \varphi \sim q_\varphi^c$ by the above Lemma, hence $\gamma \sim q_\varphi^c$, and the conditions on the dimensions plus the fact that q is dominated by a nonsingular form inside φ guarantee that q_φ^c and γ are of the same type.

Suppose that φ and ψ are nonsingular, σ is totally singular, and ρ is a nonsingular completion of σ. Then, in general, $\varphi \perp \sigma \cong \psi \perp \sigma$ does not imply $\varphi \perp \rho \cong \psi \perp \rho$. For example, suppose that $[a,b]$ is anisotropic. If we put $\varphi = \rho = [a,b]$, $\psi = [0,0]$ and $\sigma = \langle a\rangle$, then ρ is a nonsingular completion of σ and we clearly have $\varphi \perp \sigma \cong \psi \perp \sigma$. Now $\varphi \perp \rho \cong \mathbb{H} \perp \mathbb{H}$ is hyperbolic, but $\psi \perp \rho \cong \mathbb{H} \perp [a,b]$ isn't. However, we do have the following result which we will use in section 5.

Lemma 3.9. *Let φ, ψ be nonsingular, and let σ be totally singular. Let ρ be a fixed nonsingular completion of σ. If $\varphi \perp \sigma \cong \psi \perp \sigma$, then there exists a nonsingular completion ρ' of σ such that $\varphi \perp \rho \cong \psi \perp \rho'$.*

Proof. Let W (resp. V) be the underlying vector space of $\varphi \perp \sigma$ (resp. $\psi \perp \sigma$) with corresponding orthogonal decomposition $W = W_\varphi \oplus W_\sigma$ (resp. $V = V_\psi \oplus V_\sigma$). Let $W' \supset W$ be the underlying vector space of $\varphi \perp \rho$ such that $(\varphi \perp \rho)|_W = \varphi \perp \sigma$, and let $W' = W_\varphi \oplus W_\rho$ be the corresponding orthogonal decomposition with $W_\sigma \subset W_\rho$. The isometry $\psi \perp \sigma \cong \varphi \perp \sigma$ yields an injective isometry $t : (\psi \perp \sigma, V) \to (\varphi \perp \rho, W')$. The result now follows readily by noting that there is an orthogonal decomposition $W = t(V_\psi) \oplus t(V_\sigma)$ with $t(V_\sigma) = W_\sigma$ (an isometry maps radical onto radical) and $(\varphi \perp \sigma)|_{t(V_\psi)} \cong \psi$, and by applying Corollary 3.3 together with Remark 3.2. □

Corollary 3.10. *Let φ, ψ be nonsingular, and let σ, τ be totally singular. Let ρ be a fixed nonsingular completion of σ. If $\varphi \perp \sigma \perp \tau \cong \psi \perp \sigma \perp \tau$, then there exists a nonsingular completion ρ' of σ such that $\varphi \perp \rho \perp \tau \cong \psi \perp \rho' \perp \tau$.*

Proof. Let η be any n.s.c. of τ. By Lemma 3.9 and Lemma 3.5, there exist n.s.c.'s ρ' of σ and η' of τ such that $\varphi \perp \rho \perp \eta \cong \psi \perp \rho' \perp \eta'$. Adding τ on both sides and using $\eta \perp \tau \sim \eta' \perp \tau \sim \tau$ (Lemma 3.7), we get $\varphi \perp \rho \perp \tau \sim \psi \perp \rho' \perp \tau$. Since both forms are of the same type, we get $\varphi \perp \rho \perp \tau \cong \psi \perp \rho' \perp \tau$. □

It is known in characteristic $\neq 2$ that if the orthogonal difference of two nonsingular forms of dimension $\geq m$ has Witt index $\geq m$, then these two forms have isometric subforms of dimension m. This can be generalized to characteristic 2.

Proposition 3.11. *Let φ and ψ be forms over F and let $n(\varphi,\psi) = i_{\mathrm{ti}}(\varphi \perp \psi) - \max\{i_{\mathrm{ql}}(\varphi), i_{\mathrm{ql}}(\psi)\}$.*

(i) *Let $m \leq \min\{\dim \varphi, \dim \psi, n(\varphi,\psi)\}$. Then there exists a form q of dimension m with $q \prec \varphi$, $q \prec \psi$.*

(ii) *Let q be a form with $q \prec \varphi$, $q \prec \psi$. Then* $\dim q \leq n(\varphi, \psi)$.
(iii) $\min\{\dim \varphi, \dim \psi, n(\varphi, \psi)\}$ *is the maximal dimension of forms q with $q \prec \varphi$, $q \prec \psi$.*

Proof. (iii) follows readily from (i) and (ii).

Proof of (i). First, we reduce the statement to the case where one of the two forms is nondefective. Write $\varphi \cong \varphi_{\mathrm{nd}} \perp r \times \langle 0 \rangle$ and $\psi \cong \psi_{\mathrm{nd}} \perp s \times \langle 0 \rangle$ with φ_{nd}, ψ_{nd} nondefective, and assume $r \leq s$. Put $\varphi' = \varphi_{\mathrm{nd}}$ and $\psi' = \psi_{\mathrm{nd}} \perp (s-r) \times \langle 0 \rangle$. We have $\max\{i_{\mathrm{ql}}(\varphi), i_{\mathrm{ql}}(\psi)\} = s$, $\max\{i_{\mathrm{ql}}(\varphi'), i_{\mathrm{ql}}(\psi')\} = s - r$ and $i_{\mathrm{ti}}(\varphi \perp \psi) = i_{\mathrm{ti}}(\varphi' \perp \psi') + 2r$. Hence,

$$m - r \leq \min\{\dim \varphi', \dim \psi', n(\varphi', \psi')\} \ .$$

Assuming our result to be true in the case where one of the forms is nondefective, we conclude that there exists a form q' with $\dim q' = m - r$ such that $q' \prec \varphi'$ and $q' \prec \psi'$. The form $q = q' \perp r \times \langle 0 \rangle$ is the desired form.

So let us from now on assume that φ is nondefective and that $\psi \cong \psi_{\mathrm{nd}} \perp s \times \langle 0 \rangle$ with ψ_{nd} nondefective. We then have

$$i_{\mathrm{ti}}(\varphi \perp \psi) - \max\{i_{\mathrm{ql}}(\varphi), i_{\mathrm{ql}}(\psi)\} = i_{\mathrm{ti}}(\varphi \perp \psi) - s = i_{\mathrm{ti}}(\varphi \perp \psi_{\mathrm{nd}}) \ .$$

We use induction on m. If $m = 0$ there is nothing to show. So suppose $m > 0$.

Assume that φ and ψ have a common nonsingular subform α with $\dim \alpha > 0$. Then $\varphi \cong \varphi' \perp \alpha$, $\psi \cong \psi' \perp \alpha$ and $i_{\mathrm{ql}}(\psi') = i_{\mathrm{ql}}(\psi) = s$. We now use the induction hypothesis applied to $\varphi' \perp \psi'$. Indeed, Witt cancellation yields $i_{\mathrm{ti}}(\varphi' \perp \psi') - s \geq m - \dim \alpha$, and $\min\{\dim \varphi', \dim \psi', m - \dim \alpha\} = m - \dim \alpha$ then shows the existence of a form q' of dimension $m - \dim \alpha$ with $q' \prec \varphi'$, $q' \prec \psi'$. Thus, $q \cong q' \perp \alpha$ is the desired form.

Therefore, we may assume that φ and ψ do not contain a common nonsingular subform. Applied to the form $\mathbb{H}$, this implies in particular that φ or ψ_{nd} is anisotropic.

Suppose that φ is isotropic. Then, by the above, we may assume that ψ_{nd} is anisotropic. The nondefectiveness implies that $\varphi \cong \mathbb{H} \perp \varphi'$ and we get $i_{\mathrm{ti}}(\varphi' \perp \psi) - s \geq m - 1$

By induction, there exists a totally singular form q' such that $\dim q' = m' = \min\{\dim \varphi', \dim \psi, m-1\}$ and $q' \prec \varphi'$, $q' \prec \psi$. Since by assumption $m \leq \dim \varphi$, $\dim \psi$ and $\dim \varphi' = \dim \varphi - 2$, there are two cases : $m' = m - 1$ (if $\dim \varphi \geq m+1$) and $m' = m - 2$ (if $\dim \varphi = m$).

Suppose $m' = m - 1$. In this case, we can write $q' \cong \beta \perp \gamma$ with β, γ totally singular, $\psi \cong \beta_1 \perp \gamma \perp \delta$ with β_1 a nonsingular completion of β and some form δ. Now $\dim \psi > m'$ implies that $\dim \beta_1 + \dim \delta \geq 1$. If $\dim \delta \geq 1$, then for any $a \in D_F(\delta)$ we have $\langle a \rangle \prec \mathbb{H}$ as $D_F(\mathbb{H}) = F$, and hence $q' \perp \langle a \rangle \prec \varphi' \perp \mathbb{H} \cong \varphi$ and $q' \perp \langle a \rangle \prec \psi$. If $\dim \beta_1 \geq 1$, then there exist $b, d \in F$ such that $\langle b \rangle \subset \beta$, and $[b, d] \subset \beta_1$. Using the fact that $[b, d] \perp \langle b \rangle \cong \mathbb{H} \perp \langle b \rangle \prec \varphi$, we now conclude that $[b, d] \subset \varphi$ and $[b, d] \subset \psi$, a contradiction to our hypothesis of the nonexistence of common nonsingular subforms.

If $m' = \dim \varphi' = m - 2$, then $\dim \psi_{\mathrm{nd}} + m - 1 > \dim(\varphi' \perp \psi_{\mathrm{nd}})$, hence, for dimension reasons, the underlying subspace of ψ_{nd} inside the underlying vector space of $\varphi' \perp \psi_{\mathrm{nd}}$ intersects with the totally isotropic subspace of dimension $i_{\mathrm{ti}}(\varphi' \perp \psi_{\mathrm{nd}}) = i_{\mathrm{ti}}(\varphi' \perp \psi) - s \geq m - 1$, which in turn yields that ψ_{nd} is isotropic, which

by nondefectiveness implies $\mathbb{H} \subset \psi_{\mathrm{nd}}$, again a contradiction because by assumption $\mathbb{H} \subset \varphi$ as well.

The same reasoning can be applied when ψ_{nd} is isotropic. In this case we write $\psi \cong \mathbb{H} \perp \psi'$, we simply replace φ' by ψ', ψ and ψ_{nd} by φ in the above argument and we note that $i_{\mathrm{ti}}(\psi' \perp \varphi) \geq m - 1 + s \geq m - 1$.

Thus, we may finally assume that φ and ψ_{nd} are anisotropic, and that φ and ψ do not contain common nonsingular subforms.

Let V and W be the underlying vector spaces of φ and ψ, respectively. By induction hypothesis, there exist subspaces $V' \subset V$ and $W' \subset W$ with $\dim V' = \dim W' = m - 1$ such that $q' \cong \varphi|_{V'} \cong \psi|_{W'}$ with q' totally singular. Thus, there exist orthogonal bases $\{e_1, \cdots, e_{m-1}\}$ and $\{f_1, \cdots, f_{m-1}\}$ of V' and W', respectively, with $\varphi(e_i) = \psi(f_i) = c_i \in F^*$. In particular, $\{e_1 + f_1, \cdots, e_{m-1} + f_{m-1}\}$ is a basis of a totally isotropic subspace U of $(\varphi \perp \psi, V \oplus W)$.

Let $W = W_{\mathrm{nd}} \oplus W_0$ with $\psi|_{W_{\mathrm{nd}}} \cong \psi_{\mathrm{nd}}$ and $W_0 \subset \mathrm{rad}(\psi)$, $\psi|_{W_0} \cong s \times \langle 0 \rangle$. We fix a basis $\{g_1, \cdots, g_s\}$ of W_0. The anisotropy of φ implies that q' is anisotropic, hence $W' \cap W_0 = \{0\}$. In particular, we have a direct sum of sub-vector spaces $V' \oplus W' \oplus W_0$ of $V \oplus W$ and therefore also a direct sum $U \oplus W_0$. Clearly, $U \oplus W_0$ is a totally isotropic subspace of $(\varphi \perp \psi, V \oplus W)$ of dimension $m - 1 + s$. This totally isotropic subspace is contained in a maximal totally isotropic subspace of $(\varphi \perp \psi, V \oplus W)$ which, by assumption, is of dimension $\geq m + s$.

Hence, there exist $v \in V$ and $w \in W$ such that $v + w \notin U \oplus W_0$ and such that $\{e_1 + f_1, \cdots, e_{m-1} + f_{m-1}, g_1, \cdots, g_s, v + w\}$ is a basis for a totally isotropic subspace of dimension $m + s$. In particular, $\varphi(v) + \psi(w) = 0$. Suppose that $\varphi(v) = 0$. The anisotropy of φ implies that $v = 0$. In particular, $w \neq 0$ and $\psi(w) = 0$, and we have that $\{g_1, \cdots, g_s, w\}$ is in fact a totally isotropic subspace of (ψ, W) of dimension $s + 1$. But we assumed that $\psi \cong \psi_{\mathrm{nd}} \perp s \times \langle 0 \rangle$ with ψ_{nd} anisotropic, which in particular yields that $i_{\mathrm{ti}}(\psi) = s < s + 1$, a contradiction. Therefore, $v \neq 0$ and the anisotropy of φ, implies $a = \varphi(v) = \psi(w) \neq 0$. Note also that $0 = b_{\varphi \perp \psi}(e_i + f_i, v + w) = b_\varphi(e_i, v) + b_\psi(f_i, w)$ and therefore, $b_\varphi(e_i, v) = b_\psi(f_i, w)$.

Suppose $v \in V'$. Then there exists $w' \in W'$ with $a = \varphi(v) = \psi(w') = \psi(w)$. Also, $b_\varphi(e_i, v) = 0$ for all i and hence $b_\psi(f_i, w) = 0$ for all i and thus $b_\psi(w', w) = 0$. We also clearly have $b_\psi(g_i, w) = 0$ for all g_i. Suppose now that $w \notin W' \oplus W_0$. The above then shows that the quadratic subspace of (ψ, W) spanned by $\{w, w', g_1, \cdots, g_s\}$ is isometric to $\langle a, a \rangle \perp s \times \langle 0 \rangle \cong \langle a \rangle \perp (s + 1) \times \langle 0 \rangle$, again a contradiction to the fact that $i_{\mathrm{ti}}(\psi) = s$. Hence, $w \in W' \oplus W_0$. Let $v = \sum_{i=1}^{m-1} a_i e_i$ and $w = \sum_{i=1}^{m-1} b_i f_i + \sum_{j=1}^{s} c_j g_j$, $a_i, b_i, c_i \in F$. Then $\sum_{i=1}^{m-1} a_i^2 \varphi(e_i) = \varphi(v) = \psi(w) = \sum_{i=1}^{m-1} b_i^2 \psi(f_i)$. Now $\varphi(e_i) = \psi(f_i) = c_i$ and the fact that $\varphi|_{V'}$ and $\psi|_{W'}$ are anisotropic totally singular implies that the c_i are F^2-linearly independent, hence $a_i^2 = b_i^2$ for all $1 \leq i \leq m - 1$ and thus $a_i = b_i$. In particular, $v + w = \sum_{i=1}^{m-1} a_i(e_i + f_i) + \sum_{j=1}^{s} c_j g_j \in U \oplus W_0$, a contradiction.

We conclude that $v \notin V'$. A similar argument as above shows that $w \notin W'$. Let $\tilde{V} = V' \oplus Fv$ and $\tilde{W} = W' \oplus Fw$. One verifies readily that $\varphi|_{\tilde{V}} \cong \psi|_{\tilde{W}}$. $q \cong \varphi|_{\tilde{V}}$ is then the desired form of dimension m.

Proof of (ii). We may assume that $i_{\mathrm{ql}}(\psi) = s \geq r = i_{\mathrm{ql}}(\varphi)$. Suppose there exists a form q with $q \prec \varphi$, $q \prec \psi$ and $\dim q > n(\varphi, \psi) = i_{\mathrm{ti}}(\varphi \perp \psi) - s$.

First, we reduce to the case where q is totally singular. If q is not totally singular, then we can write $q \cong \rho \perp \sigma$ with ρ nonsingular of dimension ≥ 2 and

$\sigma = \mathrm{ql}(q)$. Now $q \prec \varphi$, $q \prec \psi$ and the nonsingularity of ρ readily imply that $\varphi \cong \rho \perp \varphi'$, $\psi \cong \rho \perp \psi'$ for certain forms φ', ψ' with $\sigma \prec \varphi'$, $\sigma \prec \psi'$. We have $\rho \perp \rho \cong (\dim \rho) \times \mathbb{H}$. Thus,

$$\dim \sigma = \dim q - \dim \rho > i_{\mathrm{ti}}(\varphi \perp \psi) - s - \dim \rho = i_{\mathrm{ti}}(\varphi' \perp \psi') - s \ .$$

We also have $i_{\mathrm{ql}}(\varphi') = i_{\mathrm{ql}}(\varphi) = r \leq s = i_{\mathrm{ql}}(\psi) = i_{\mathrm{ql}}(\psi')$ as the quasi-linear parts of φ (resp. ψ) and φ' (resp. ψ') are the same.

Hence, we get $\sigma \prec \varphi'$, $\sigma \prec \psi'$, and $\dim \sigma > n(\varphi', \psi')$ with σ totally singular. Thus, to get a contradiction, it suffices to consider the case where q is totally singular.

So let us assume that q is totally singular and, say, $\dim q = k$. Then $q \prec \varphi$ implies that there exist an integer ℓ, $0 \leq \ell \leq k$, $c_1, \cdots, c_k \in F$, $a_1, \cdots, a_\ell \in F$, and a form α such that $q \cong \langle c_1, \cdots, c_k \rangle$ and

$$\varphi \cong \alpha \perp [a_1, c_1] \perp \cdots \perp [a_\ell, c_\ell] \perp \langle c_{\ell+1}, \cdots, c_k \rangle \ .$$

Using the standard relations, we can rewrite $[a_1, c_1] \perp \cdots \perp [a_\ell, c_\ell] \perp \langle c_{\ell+1}, \cdots, c_k \rangle$ in such a way that we may assume that there exist $r_1, r_2 \geq 0$ with $\ell \geq r_1$, $k - \ell \geq r_2$ such that

$$c_1 = \cdots = c_{r_1} = c_{k-r_2+1} = \cdots = c_k = 0$$

and such that $q' = \langle c_{r_1+1}, \cdots, c_{k-r_2} \rangle$ is anisotropic. It follows that we can write $q \cong q' \perp (t \times \langle 0 \rangle)$ with q' anisotropic,

$$\varphi \cong \varphi' \perp (r_1 \times \mathbb{H}) \perp (r_2 \times \langle 0 \rangle)$$

with $q' \prec \varphi'$ and $t = r_1 + r_2$.

Using the fact that the anisotropic part q' of q is determined uniquely, we obtain in a similar way that there exist $s_1, s_2 \geq 0$ with $s_1 + s_2 = t$ such that

$$\psi \cong \psi' \perp (s_1 \times \mathbb{H}) \perp (s_2 \times \langle 0 \rangle)$$

and $q' \prec \psi'$. In fact, the anisotropy of q' implies that q' is dominated by the nondefective parts of φ and ψ, i.e. we have $q' \prec \varphi'_{\mathrm{nd}}, \psi'_{\mathrm{nd}}$.

By assumption, $\dim q' + t = \dim q > i_{\mathrm{ti}}(\varphi \perp \psi) - s$. We have

$$i_{\mathrm{ti}}(\varphi \perp \psi) = i_{\mathrm{ti}}(\varphi' \perp \psi') + r_1 + r_2 + s_1 + s_2 = i_{\mathrm{ti}}(\varphi' \perp \psi') + 2t \ .$$

Also, $i_{\mathrm{ql}}(\varphi') = r - r_2$, $i_{\mathrm{ql}}(\psi') = s - s_2$. Therefore,

$$i_{\mathrm{ti}}(\varphi' \perp \psi') = i_{\mathrm{ti}}(\varphi'_{\mathrm{nd}} \perp \psi'_{\mathrm{nd}}) + (r - r_2) + (s - s_2) \ .$$

Thus,

$$\begin{aligned} \dim q' \quad > \quad & i_{\mathrm{ti}}(\varphi'_{\mathrm{nd}} \perp \psi'_{\mathrm{nd}}) + t + (r - r_2) + (s - s_2) - s \\ & = i_{\mathrm{ti}}(\varphi'_{\mathrm{nd}} \perp \psi'_{\mathrm{nd}}) + s_1 + s_2 + (r - r_2) - s_2 \\ & = i_{\mathrm{ti}}(\varphi'_{\mathrm{nd}} \perp \psi'_{\mathrm{nd}}) + s_1 + (r - r_2) \\ & \geq i_{\mathrm{ti}}(\varphi'_{\mathrm{nd}} \perp \psi'_{\mathrm{nd}}) \ . \end{aligned}$$

Now $q' \perp q' \cong q' \perp (\dim q' \times \langle 0 \rangle) \prec \varphi'_{\mathrm{nd}} \perp \psi'_{\mathrm{nd}}$. Hence $i_{\mathrm{ti}}(\varphi'_{\mathrm{nd}} \perp \psi'_{\mathrm{nd}}) \geq \dim q'$, a contradiction. □

Corollary 3.12. *Let φ and ψ be nondefective forms and let q be a form of maximal dimension such that $q \prec \varphi$, $q \prec \psi$. Then $\dim q = \min\{\dim \varphi, \dim \psi, i_{\mathrm{ti}}(\varphi \perp \psi)\}$ and q is nondefective.*

Proof. The statement about $\dim q$ follows immediately from the previous proposition by noting that nondefectiveness of φ, ψ implies $n(\varphi,\psi) = i_{\mathrm{ti}}(\varphi \perp \psi)$.

Suppose that $\eta \prec \varphi$, $\eta \prec \psi$ and that η is defective, say, $\eta \cong \eta' \perp (t \times \langle 0\rangle)$ with η' nondefective and $t > 0$. Similar arguments as in the proof of part (ii) of the previous proposition together with the fact that φ, ψ are nondefective imply that we can write $\varphi \cong \varphi' \perp (t \times \mathbb{H})$, $\psi \cong \psi' \perp (t \times \mathbb{H})$ with $\eta' \prec \varphi'$, $\eta' \prec \psi'$. For $\tilde{\eta} = \eta' \perp (t \times \mathbb{H})$, we thus get $\tilde{\eta} \prec \varphi$, $\tilde{\eta} \prec \psi$ and $\dim \tilde{\eta} = \dim \eta + t > \dim \eta$. This implies the nondefectiveness of q. □

4. Further notions and results needed in proofs

4.1. **Pfister forms and quasi-Pfister forms.** Similar to the case of characteristic $\neq 2$, one can construct the Witt ring WF of nondegenerate symmetric bilinear forms over F and the Witt group W_qF of nonsingular quadratic forms over F. W_qF can be considered in a natural way as a WF-module by the operation $b \otimes q(x \otimes y) = b(x,x)q(y)$.

An n-fold bilinear Pfister form is a form of type $b = \langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b$ for some $a_i \in F^*$, we write $\langle\!\langle a_1, \cdots, a_n\rangle\!\rangle_b$ for short. If $q(X) = b(X,X)$ is the corresponding totally singular (hence quasi-linear) quadratic form, we write $q = \langle\!\langle a_1, \cdots, a_n\rangle\!\rangle$ and call it an *n-fold quasi-Pfister form*. An *$(n+1)$-fold Pfister form* is then a nonsingular quadratic form of type $\langle\!\langle a_1, \cdots, a_n\rangle\!\rangle_b \otimes [1, b]$ for some $a_i \in F^*$, $b \in F$. This Pfister form is then said to be of degree $n+1$ and we write $\langle\!\langle a_1, \cdots, a_n; b]\!]$ for short. $\langle\!\langle b]\!] = [1, b]$ is thus a 1-fold Pfister form. The set of forms isometric (resp. similar) to n-fold Pfister forms will be denoted by P_nF (resp. GP_nF). We denote the WF-submodule of W_qF generated by n-fold Pfister forms by I^nW_qF, and we obtain a filtration $W_qF = IW_qF \supset I^2W_qF \supset \cdots$.

Note that this definition of degree has also been given in [28, Sect. 6]. It differs from the one given in [6], [7], but ours is in closer analogy to the characteristic $\neq 2$ case, where n-fold Pfister forms are of degree n and dimension 2^n just as according to our definition in characteristic 2 (and not 2^{n+1} as in the definition in [11]).

As is the case in characteristic $\neq 2$, $x \in F^*$ is represented by a Pfister form π if and only if $x\pi \cong \pi$. This property is commonly referred to as π being round. The same holds for quasi-Pfister forms (cf. also [34, page 38]). In fact, anisotropic quasi-Pfister forms can be characterized in this way as was noted by Knebusch [25]. For completeness' sake we will sketch a proof of these facts in section 8 (see Proposition 8.5).

4.2. **Function fields of quadratic forms and splitting towers.** Suppose the quadratic form q (which is supposed to be not the zero form) is given by the polynomial $q(X,Y,Z) = \sum_{i=1}^{r}(a_iX_i^2 + X_iY_i + b_iY_i^2) + \sum_{j=1}^{s} c_jZ_j^2$. Then the corresponding polynomial $q(X,Y,Z)$ is reducible iff the nondefective part q_{nd} of q is either of type $(0,1)$ or of type $(1,0)$ and $q_{\mathrm{nd}} \cong \mathbb{H}$. It is absolutely irreducible iff q is of type (r,s) with $r \geq 1$ and $\dim q_{\mathrm{nd}} \geq 3$ (see, e.g. [1]). Suppose that $q(X,Y,Z)$ is irreducible. Then the function field $F(q)$ of q is defined to be the field of fractions of the quotient ring $F[X,Y,Z]/(q(X,Y,Z))$. To avoid case distinctions, we put $F(q) = F$ if $q(X,Y,Z)$ is reducible or $\dim q = 0$. We have that $F(q)/F$ is purely transcendental iff q_{nd} is isotropic. If $q(X,Y,Z)$ is absolutely irreducible and K/F is any field extension, then the free compositum $K \cdot F(q)$ can be identified with $K(q)$, the quotient field of the integral domain $K[X,Y,Z]/(q(X,Y,Z))$.

Suppose q is defective, say, $q \cong q_{\mathrm{nd}} \perp m \times \langle 0 \rangle$ with q_{nd} the nondefective part of q, then $F(q)$ is a purely transcendental extension of $F(q_{\mathrm{nd}})$ of transcendence degree m.

Of particular importance in this paper will be the isotropy behaviour of a quadratic form over the function field of another quadratic form. Since an anisotropic form stays anisotropic over purely transcendental extensions and by the previous remarks, this isotropy question is only of interest when both forms are anisotropic. The following lemma has been shown in [27, Cor. 3.1] and will be used frequently.

Lemma 4.1. *Let φ, ψ be anisotropic forms over F. Suppose that ψ is not totally singular. Then $\mathrm{ql}(\varphi)_{F(\psi)}$ is anisotropic and is the quasi-linear part of $(\varphi_{F(\psi)})_{\mathrm{an}}$. In particular, if φ is of type (r,s), then $(\varphi_{F(\psi)})_{\mathrm{an}}$ is of type (r',s) with $r' \leq r$.*

Of utmost importance are the following four results. The first part of it will be referred to later on as the *domination theorem*, the last part is an analogue in characteristic 2 of the Arason-Pfister Hauptsatz and will be referred to as such.

Theorem 4.2. (i) *Let φ, ψ be forms over F with φ anisotropic and ψ nondefective. If $\varphi_{F(\psi)}$ is hyperbolic, then $ab\psi \prec \varphi$ for each $a \in D_F(\varphi)$, $b \in D_F(\psi)$.*

(ii) *Let φ be an anisotropic nonsingular form. Then φ is similar (isometric) to a Pfister form if and only if $\varphi_{F(\varphi)}$ is hyperbolic (and $1 \in D_F(\varphi)$).*

(iii) *Let ψ be anisotropic nonsingular, and let φ be an anisotropic Pfister form. Then $\psi_{F(\varphi)}$ is hyperbolic if and only if there exist $a_1, \cdots, a_m \in F^*$ such that $\psi \cong a_1\varphi \perp \cdots \perp a_m\varphi$.*

(iv) *Let q be a nonzero anisotropic form in I^nW_qF. Then $\dim q \geq 2^n$. If $\dim q = 2^n$, then q is similar to an n-fold Pfister form.*

Remark 4.3. The corresponding version of the domination theorem in characteristic $\neq 2$ is an immediate consequence of the Cassels-Pfister subform theorem and has been first explicitly stated independently by Wadsworth [37], Arason [3] and Knebusch [23]. The characteristic 2 version of the Cassels-Pfister subform theorem was first proved by Amer (cf. [34, pp. 17–18]). This subform theorem plus the norm theorem in characteristic 2 by Baeza [12] immediately yield the above domination theorem. See also [27], [30], where the above domination theorem has been stated perhaps for the first time explicitly.

The characterization of forms similar (isometric) to Pfister forms is also an immediate consequence of the norm theorem [12] together with the characterization of Pfister forms as being the anisotropic (strictly) multiplicative forms, cf. [34, p.38].

Part (iii) is a direct consequence of (i) and (ii).

The Arason-Pfister Hauptsatz in characteristic $\neq 2$ was shown in [5] and later in characteristic 2 in [10]. It is a straightforward consequence of the first two results.

The next important theorem which we will use later on has been shown in characteristic $\neq 2$ in [16]. The characteristic 2 version below has been shown in [18]. Before that, a somewhat weaker version has been proved in [30].

Theorem 4.4. *Let φ and ψ be anisotropic forms over F such that, for some positive integer n, one has $\dim \varphi < 2^n \leq \dim \psi$. Then $\varphi_{F(\psi)}$ is anisotropic.*

In characteristic $\neq 2$, the foundations of a theory of generic splitting of quadratic forms, based on splitting towers as defined below, have been laid out by Knebusch

[22], [23]. Most of this generic theory goes through also in characteristic 2 for nonsingular forms, based on results in [10], [12]. The notion of *generic* splitting has recently been extended to forms which are possibly singular in a forthcoming book by Knebusch [25]. A theory of *standard* splitting towers for arbitrary forms in characteristic 2 has been developed in [28].

The standard splitting tower of q is defined to be the following sequence of field extensions of F. We put $F_0 = F$ and $q_0 = q_{\mathrm{an}}$, the anisotropic part of q. For $n \geq 1$, we put $F_n = F_{n-1}(q_{n-1})$ and $q_n = ((q_{n-1})_{F_n})_{\mathrm{an}}$. The *(standard) height* $h(q)$ of q is the smallest integer r with $\dim q_r \leq 1$. Let $h = h(q)$ be the height of q. The form q_i, $0 \leq i \leq h$ is called the i-th *kernel form*. If $h \geq 1$, then we call F_{h-1} the *leading field* of q, and q_{h-1} its *leading form*. Suppose that q is anisotropic of type (r, s) with $r \geq 1$. By the previous lemma, there exists an $1 \leq i \leq h(q)$ such that, for $1 \leq j \leq h(q)$ one has that q_j is of type (r_j, s) with $r_j < r_{j-1}$ if $j \leq i$, and $r_j = 0$ for $j \geq i$ (see also [28]). The value i will be called the *nondefective height* of q as it is the largest i such that the quasi-linear part $\mathrm{ql}(q)$ of q stays anisotropic over F_i, i.e. such that q_{F_i} stays nondefective, and we write $h_{\mathrm{nd}}(q) = i$. In Knebusch's generic theory for quadratic forms [25], it is shown that the standard splitting tower up to $F_{h_{\mathrm{nd}}}$ is a generic splitting tower according to his definition of genericity, and his definition of height corresponds to what we call nondefective height. Some results on the standard splitting tower and the height of singular forms will be shown in section 8. In particular, we will show

Theorem 4.5. *Let q be an anisotropic form of standard height h. Let $F = K_0 \subset K_1 \subset \cdots \subset K_m$ be any tower of fields such that $\dim(q_{K_{i-1}})_{\mathrm{an}} > \dim(q_{K_i})_{\mathrm{an}}$ for each $1 \leq i \leq m$, then $m \leq h$.*

It should be noted that the dimensions or types, respectively, of the forms $(q_{K_j})_{\mathrm{an}}$, $1 \leq j \leq m$ above might not necessarily be found among the dimensions or types, respectively, of the higher kernel forms q_i, $1 \leq i \leq h$ (see also Remark 8.15). However, we do have

Proposition 4.6. *Let q be an anisotropic form over F of type (r_0, s_0), and let (r_i, s_i) be the type of the i-th kernel form q_i in a standard splitting tower $F = F_0 \subset F_1 \subset \cdots \subset F_h$ of q. Let K/F be any field extension and let (r, s) be the type of $(q_K)_{\mathrm{an}}$. Then there exists $i \in \{0, \cdots, h\}$ such that $r = r_i$.*

Proof. If $s_0 = 0, 1$ then this has been shown in [26, Th. 1.3]. If $r_0 \leq 1$ or if $r = r_0$, the result is trivial.

So suppose that $r_0 \geq 2$, $s_0 \geq 1$ and let $i \in \{0, \cdots, h-1\}$ be such that $r < r_i$. To get the desired result, it suffices to show that $r \leq r_{i+1}$. Now $r_i \geq 1$ and we have $F_{j+1} = F_j(q_j)$ for $0 \leq j \leq i$. Define inductively $K_0 = K$ and $K_{j+1} = K_j(q_j)$ for $0 \leq j \leq i$. Clearly, $F_j \subset K_j$ and the nondefective part of $(q_j)_{K_j}$ is of dimension $\geq 2r_j + 1 \geq 3$ for all $0 \leq j \leq i$, so that indeed all K_j, $0 \leq j \leq i+1$ are well defined and can be identified with the free compositum $K \cdot F_j$.

Note that

$$\begin{aligned} i_W((q_j)_{K_j}) &= i_W(q_{K_j}) - i_W(q_{F_j}) \\ &\geq i_W(q_K) - i_W(q_{F_j}) \geq (r_0 - r) - (r_0 - r_j) = r_j - r > 0 \end{aligned}$$

for $0 \leq j \leq i$. Therefore, $K_j(q_j)/K_j$ is purely transcendental for $0 \leq j \leq i$, hence K_{i+1}/K is purely transcendental. Therefore, $r_0 - r = i_W(q_K) = i_W(q_{K_{i+1}}) \geq i_W(q_{F_{i+1}}) = r_0 - r_{i+1}$, and thus $r \leq r_{i+1}$ as desired. □

Suppose now that q is nonsingular. Then the degree of q is defined as follows. We put $\deg(q) = \infty$ if q is hyperbolic. Otherwise, $h(q) \geq 1$ and q_{h-1} becomes hyperbolic over its own function field, which implies that q_{h-1} is similar to an n-fold Pfister form over F_{h-1} for some $n \geq 0$. We then put $\deg(q) = n$. The set $J_nF = \{q \in W_qF \mid \deg(q) \geq n\}$ is a WF-submodule of W_qF with $I^nW_qF \subset J_nF$. Aravire and Baeza have shown in [6], [7] that in fact $J_nF = I^nW_qF$. It should be noted that the corresponding result $I^nF = J_nF$ in characteristic $\neq 2$ is known to be true for $n \leq 5$ due to the work on the Milnor Conjecture in small degrees by Merkurjev-Suslin and Rost (see, e.g. [21, Th. 2.8, Remarque]), and for all n due to Orlov-Vishik-Voevodsky [33] (see also [4, Theorem 1.5]).

Another important theorem by Aravire and Baeza which we will use is the following (cf. [7, Theorem 5.3]):

Theorem 4.7. *Let $\langle\langle a_1, \cdots, a_n\rangle\rangle$ be an anisotropic n-fold quasi-Pfister form. Let $\varphi \in I^{n+1}W_qF$ and $K = F(\langle\langle a_1, \cdots, a_n\rangle\rangle)$. If $\varphi_K \in I^{n+2}W_qK$ then there exists $b \in F$ such that $\varphi \equiv \langle\langle a_1, \cdots a_n, b]]$ mod $I^{n+2}W_qF$.*

4.3. **The Arf-invariant and the Clifford algebra.** For basic facts on Clifford algebras and quaternion algebras which we state without further comment, we refer the reader to [11], [31], [14].

Let q be a quadratic form defined on the vector space V and let $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes n}$ denote the tensor algebra of V. Let $J(q)$ be the ideal in $T(V)$ generated by elements of type $x \otimes x - q(x)$, $x \in V$. Then the Clifford algebra $C(q)$ of q is defined to be the quotient $T(V)/J(q)$. This algebra has a natural $\mathbb{Z}/2\mathbb{Z}$-gradation, and the subalgebra $C_0(q)$ of degree 0 elements is called the even Clifford algebra.

If q is nonsingular, then $C(q)$ is a central simple algebra over F which, up to isomorphism, only depends on the isometry class of q, and the center $Z(q)$ of $C_0(q)$ is a separable quadratic algebra over F, i.e. there exists $\delta \in F$ with $Z(q) = F[X]/(X^2 + X + \delta)$. If we put $\wp(a) = a^2 + a$, then $\wp(F) = \{\wp(a) \mid a \in F\}$ is an additive subgroup of F, and the class of δ modulo $\wp(F)$ is an invariant of the isometry class of q and will be called the Arf-invariant of q, denoted by $\Delta(q)$. By abuse of notation, we will often identify $\Delta(q)$ with a representative in F of its class in $F/\wp(F)$.

More explicitly, if $q \cong a_1[1, b_1] \perp \cdots \perp a_n[1, b_n]$, then $\Delta(q) = b_1 + \cdots + b_n \in F/\wp(F)$, and $C(q) = [b_1, a_1)_F \otimes \cdots \otimes [b_n, a_n)_F$, where $[b, a)_F$ ($a \in F^*$, $b \in F$) denotes the quaternion algebra generated by two elements u, v over F subject to the relations $u^2 = a$, $v^2 + v = b$, $uv = (v + 1)u$. This is a 4-dimensional central simple algebra which is trivial in the Brauer group iff a is a norm for the extension $F[X]/(X^2 + X + b)$ over F. For quaternion algebras, one has $[b, a)_F \otimes [b', a')_F \cong [b, aa')_F \otimes [b + b', a')_F$. In particular, considered as a symbol in the Brauer group, it is additive in the first and multiplicative in the second slot. Two central simple algebras A, B are said to be similar, $A \sim B$, if they represent the same element in the Brauer group $\mathrm{Br}(F)$.

We will need the following results.

Proposition 4.8. (Albert [2, Theorem 28, p. 108].) *Let A be a central simple algebra over F and let $K = F(\sqrt{b_1}, \cdots, \sqrt{b_n})$, $[K : F] = 2^n$. If $A_K \sim 0$ then there exist $a_1, \cdots, a_n \in F$ such that $A \sim [a_1, b_1)_F \otimes \cdots \otimes [a_n, b_n)_F$.*

Theorem 4.9. (Sah [35, Theorem 2]). *Let q be a nonsingular quadratic form over F.*

(i) *$q \in I^2W_qF$ iff $\Delta(q) = 0 \in F/\wp(F)$.*
(ii) *If $q \in I^2W_qF$, then $q \in I^3W_qF$ iff $C(q) \sim 0$.*

5. Fitzgerald's theorem

Let q be a nonsingular nonhyperbolic form over F. We define

$$N(q) = \dim q - 2^{\deg q} .$$

With this definition, we now give our characteristic 2 version of Fitzgerald's theorem [15, Th. 1.6].

Theorem 5.1. *Let q, η_r, φ_r be nonsingular forms over F, and let σ be a totally singular form over F. Let ρ be nonsingular with $\sigma \prec \rho$ and $\dim \rho = 2 \dim \sigma$, i.e. ρ is a nonsingular completion of σ. Suppose that*

(i) *q, $\eta = \eta_r \perp \sigma$, $\varphi = \varphi_r \perp \sigma$ are anisotropic,*
(ii) *$\dim q$, $\dim \varphi \geq 2$,*
(iii) *$q \sim \varphi_r \perp \eta_r \perp \rho$,*
(iv) *$\dim \eta < \dim \varphi + 2^{\deg q}$.*

Then q is hyperbolic over $F(\varphi)$ if and only if q is similar to a Pfister form and $q \cong \varphi_r \perp \eta_r \perp \rho$.

Proof. "if": Suppose that q is similar to a Pfister form (which is anisotropic by hypothesis), and that $q \cong \varphi_r \perp \eta_r \perp \delta$. Since $\varphi \prec q$, $\dim \varphi \geq 2$, we have that q becomes isotropic and hence hyperbolic over $F(\varphi)$.

"only if": We will prove this in a sequence of claims. So from now on, we suppose that q is hyperbolic over $F(\varphi)$.

Claim 5.1.1. *$q \cong \varphi_r \perp \eta_r \perp \rho$.*

Proof. Since $\varphi_r \perp \varphi_r \sim 0$ and $\rho \perp \sigma \sim \sigma$ (see Lemma 3.7), we have that $q \perp \varphi \sim \eta$. Now $\dim(q \perp \varphi) \geq \dim \varphi + 2^{\deg q} > \dim \eta$ by assumption (iv). Hence, $q \perp \varphi$ is isotropic and there exists an $a \in F^*$ represented both by q and φ because both these forms are anisotropic. Since $q_{F(\varphi)}$ is hyperbolic, it follows from the domination theorem that $\varphi \cong a^2\varphi \prec q$.

Thus, there exists a nonsingular form τ and a nonsingular completion ρ' of σ with $q \cong \varphi_r \perp \tau \perp \rho' \sim \varphi_r \perp \eta_r \perp \rho$. Witt cancellation for nonsingular forms implies $\tau \perp \rho' \sim \eta_r \perp \rho$. Now $\tau \perp \rho'$ is anisotropic and $\tau \perp \sigma \prec \tau \perp \rho'$. In particular, $\tau \perp \sigma$ is anisotropic. Since $\rho' \perp \sigma \sim \rho \perp \sigma \sim \sigma$, we get $\tau \perp \sigma \sim \eta_r \perp \sigma$. Since the forms on both sides are anisotropic, we therefore obtain $\tau \perp \sigma \cong \eta_r \perp \sigma$ and thus $\dim \tau = \dim \eta_r$. Thus, the anisotropic form $q \cong \varphi_r \perp \tau \perp \rho'$ and the form $\varphi_r \perp \eta_r \perp \rho$ have the same dimension, and since they are Witt-equivalent, it follows that $q \cong \varphi_r \perp \eta_r \perp \rho$. □

After scaling, we may therefore assume that $1 \in D_F(q) \cap D_F(\varphi)$.

Since an anisotropic nonsingular form becomes hyperbolic over its own function field iff the form is similar to a Pfister form (Theorem 4.2), we get that q is similar to a Pfister form (and thus isometric to a Pfister form as $1 \in D_F(q)$) iff $N(q) = 0$. Thus, it suffices to show that under the above assumptions we cannot have $N(q) > 0$.

So suppose that $N(q) = m > 0$. We will show by induction on m that this will yield a contradiction.

Claim 5.1.2. *Let $F_1 = F(q)$ and $q_1 = (q_{F_1})_{\mathrm{an}}$. Then φ_{F_1} is anisotropic, q_1 is a Pfister form of dimension $2^{\deg q}$, and $\varphi_{F_1} \prec q_1$.*

Proof. $N(q) > 0$ implies that $h(q) \geq 2$. Then q_1 is not hyperbolic. Furthermore, if $\dim \sigma > 0$, then σ_{F_1} stays anisotropic (cf. [27, Cor.3.1]). In particular, φ_{F_1} will stay nondefective, so that as a polynomial over F_1, φ_{F_1} will be irreducible. It follows that the compositum of F_1 and $F(\varphi)$ is just $F_1(\varphi)$. Therefore, $(q_1)_{F_1(\varphi)}$ is hyperbolic. Since the function field of a nondefective isotropic form is purely transcendental over the ground field, and since the non hyperbolic form q_1 becomes hyperbolic over $F_1(\varphi)$, we have that φ_{F_1} is anisotropic. Clearly, $\deg q = \deg q_1$ and thus $N(q_1) < N(q)$.

Put $\eta_1 = (\eta_{F_1})_{\mathrm{an}}$. As remarked above, the quasi-linear part σ of η will stay anisotropic (if $\dim \sigma > 0$). Hence, there exists a nonsingular form $\eta_{1,r}$ over F_1 such that $\eta_1 \cong \eta_{1,r} \perp \sigma$. Now over F_1 we have

$$\varphi_r \perp \eta_r \perp \sigma \sim_{F_1} \varphi_r \perp \eta_{1,r} \perp \sigma \ .$$

By Lemma 3.9, there exists a nonsingular completion ρ_1 of σ over F_1 such that

$$q_1 \sim_{F_1} \varphi_r \perp \eta_r \perp \rho \sim_{F_1} \varphi_r \perp \eta_{1,r} \perp \rho_1 \ .$$

Replacing q by q_1, η by η_1, η_r by $\eta_{1,r}$, φ by $\varphi_1 = \varphi_{F_1}$, φ_r by $\varphi_{1,r} = (\varphi_r)_{F_1}$, σ by $\sigma_1 = \sigma_{F_1}$, ρ by ρ_1, it is easily verified that these forms satisfy the hypotheses (i)–(iv). The same reasoning as above shows then that $q_1 \cong \varphi_{1,r} \perp \eta_{1,r} \perp \rho_1$. Furthermore, since $N(q_1) < N(q)$, it follows by induction that q_1 is similar to a Pfister form, and hence isometric to a Pfister form since $1 \in D_K(\varphi_1)$ and $\varphi_1 \prec q_1$. In particular, $\dim q_1 = 2^{\deg q_1} = 2^{\deg q}$. □

Claim 5.1.3. $D_F(q) = G_F(q)$.

Proof. Since $1 \in D_F(q)$ and q anisotropic, we have $G_F(q) \subset D_F(q) = D_F^*(q)$. Suppose there exists an $x \in D_F(q) \setminus G_F(q)$. Let $\beta = (\langle\langle x \rangle\rangle_b \otimes q)_{\mathrm{an}}$. By assumption, $\langle\langle x \rangle\rangle_b \otimes q$ is isotropic but not hyperbolic. It follows readily that $D_F(q) \cap D_F(\beta) \neq \emptyset$. Also, $\deg \beta \geq 1 + \deg q$ (see, e.g., [36, Ch.4, Theorem 7.5] in the case of characteristic $\neq 2$; the proof for the case of characteristic 2 is more or less identical).

Now let $\beta_1 = (\beta_{F_1})_{\mathrm{an}}$. Suppose that $\beta_1 \sim 0$. The domination theorem and the fact that $D_F(q) \cap D_F(\beta) \neq \emptyset$ imply that $q \subset \beta$. Then there exists a nonsingular form q' with $\beta \cong q \perp q' \sim q \perp xq$. Now $\dim \beta < 2 \dim q$, hence $\dim q' < \dim q$. But we have $q' \sim xq$, hence q is isotropic, a contradiction. It follows that $\beta_1 \not\sim 0$. In particular, $\dim \beta_1 \geq 2^{\deg \beta} \geq 2^{1+\deg q}$.

Since $x \in D_F(q)$, we have $1 \in D_F(xq)$. Now $1 \in D_F(\varphi)$, and xq becomes hyperbolic over $F_1 = F(\varphi)$. The domination theorem implies that there exists a nonsingular form η_r' and a nonsingular completion ρ' of σ such that $xq \cong \varphi_r \perp \eta_r' \perp \rho'$ and $\dim \eta_r = \dim \eta_r'$.

Over F_1 we have $\beta_1 \sim q_1 \perp xq_1$. Now $\varphi_{F_1} \prec q_1$ and, by the same reasoning as in the previous claim, $\varphi_{F_1} \prec xq_1$. In particular,

$$2^{1+\deg q} \leq \dim \beta_1 = \dim(q_1 \perp xq_1)_{\mathrm{an}} \leq 2 \dim q_1 - 2 \dim \varphi < 2^{1+\deg q} \ ,$$

which is absurd. Hence, $D_F(q) = G_F(q)$. □

Now let $K = F(X_1, \cdots, X_n)$ be the rational function field in $n = \dim q$ variables over F. Clearly, $\deg q_K = \deg q$, and working over K instead of F, it is obvious that the forms q_K, φ_K, $(\varphi_r)_K$, η_K, $(\eta_r)_K$, σ_K and ρ_K satisfy the hypotheses (i)–(v). By the previous claim, we therefore have $D_K(q) = G_K(q)$. Hence $q_K \cong q(X_1, \cdots, X_n)q_K$, i.e., the anisotropic form q is strictly multiplicative. It follows that q is a Pfister form, hence $N(q) = 0$, a contradiction to the assumption $N(q) > 0$ and the proof of the theorem is completed. □

Remark 5.2. To get Fitzgerald's original version in characteristic $\neq 2$ from our version above, one simply has to omit σ, i.e. φ and η are nonsingular, and one has to replace "+" by "−" in certain obvious places in the proof. Furthermore, in the original version, the condition on q being anisotropic was replaced by q nonhyperbolic and $q \cong \varphi \perp \eta$, and condition (iv) by the then equivalent condition $N(q) < 2 \dim \varphi$. As a consequence, the hyperbolicity of q over $F(\varphi)$ becomes equivalent to q being similar to an *anisotropic* Pfister form.

Our proof follows along the same lines as Fitzgerald's original proof. Certain modifications became necessary because of the possibility of having singular forms and dominated forms. On the other hand, some of the original arguments could be simplified by using the domination theorem more prominently in certain places.

Corollary 5.3. *Let φ and q be forms such that q is nonsingular and anisotropic, $\varphi \prec q$, and $2 \dim \varphi \geq \dim q$. If $q_{F(\varphi)}$ is hyperbolic, then q is similar to a Pfister form.*

6. Pfister neighbors and an analogue of a theorem by Knebusch in characterisic 2

A form q is said to be a *Pfister neighbor* if there exist $\pi \in P_n F$ for some n and an $a \in F^*$ such that $q \prec a\pi$ and $\dim q > 2^{n-1}$. We then say that q is a Pfister neighbor of π. In this case, the Pfister form π is uniquely determined by q up to isometry, and q is isotropic iff π is isotropic (and hence hyperbolic), cf. [27, Prop. 3.1]. Furthermore, it is rather obvious that if q is a Pfister neighbor of type (r, s), then there exists an integer $n \geq 1$ such that $2r + s > 2^{n-1}$ and $2r + 2s \leq 2^n$. In particular, q is cannot be totally singular. If q is a Pfister neighbor of $\pi \in P_n F$, and if $x, y \in F^*$ such that $q \prec x\pi$ and $q \prec y\pi$, then the roundness of π implies that $x\pi \cong y\pi$. This together with the uniqueness of π implies that $q^c_{x\pi}$ only depends on the Pfister neighbor q (see also [28, Prop. 5.2]). By abuse of language and notation, we will call it the *complement* of the Pfister neighbor q and write q^c for short.

Pfister forms can be characterized up to similarity by the property that they become hyperbolic over their own function field, see Theorem 4.2(ii). The behaviour of Pfister neighbors over their own function field is equally well understood.

Proposition 6.1. *Let q be an anisotropic Pfister neighbor. Then $(q_{F(q)})_{\mathrm{an}} \cong q^c_{F(q)}$. In particular, the anisotropic part of q over $F(q)$ is defined over F.*

Proof. Let n and $\pi \in GP_n F$ such that over F, $q \prec \pi$ and $\dim q > 2^{n-1}$. π is anisotropic as q is anisotropic, hence q^c is anisotropic as well. Furthermore, $\pi \perp q \sim q^c$. Now $\pi_{F(q)}$ is isotropic because $q_{F(q)}$ is isotropic, hence $\pi_{F(q)}$ is hyperbolic, and we get $(q_{F(q)})_{\mathrm{an}} \sim q^c_{F(q)}$. But $\dim q > 2^{n-1} > 2^n - \dim q = \dim q^c$, and by Theorem 4.4, the anisotropic form q^c stays anisotropic over $F(q)$. Thus, $(q_{F(q)})_{\mathrm{an}} \cong q^c_{F(q)}$. □

In characteristic $\neq 2$, Knebusch [24, Theorem 7.13] has shown that the converse of the last statement in the previous proposition also holds, more precisely, an anisotropic form q with the property that its anisotropic part over $F(q)$ is defined over F must be a Pfister neighbor, so that together with the above result, anisotropic Pfister neighbors can be characterized by this property. This also holds in characteristic 2 for nonsingular forms as follows readily from Theorem 5.1 (see also Theorem 6.6 below for more details). However, for singular forms the converse fails in general.

Example 6.2. Let σ be a totally singular anisotropic form. Then $\sigma_{F(\sigma)}$ is defined over F. This is clear by Lemma 2.2. However, σ is not a Pfister neighbor as we already mentioned above.

Example 6.3. Here, we construct counterexamples of forms of type $(1, s)$.

Let $q \cong q_{ns} \perp \sigma$ be an anisotropic form of type $(1, s)$ with q_{ns} nonsingular and σ totally singular. Lemma 4.1 immediately implies that $(q_{F(q)})_{\mathrm{an}} \cong \sigma_{F(q)}$. Now if q is a Pfister neighbor, then Proposition 6.1 implies that $q^c = \sigma$, hence there exists an n with $\dim q = 2 + s > 2^{n-1}$ and $2 + 2s = \dim q + \dim q^c = 2^n$. This implies $s = 2^{n-1} - 1$. Thus, any anisotropic form of type $(1, s)$ with $s + 1$ not a 2-power will be a non-Pfister neighbor with the property that its anisotropic part over $F(q)$ is defined over F. (See also [28].)

Now nondefective forms of type $(1, 1)$ are easily seen to be Pfister neighbors. However, there are fields such that for any $n \geq 3$ there exist anisotropic forms of type $(1, 2^{n-1} - 1)$ which are not Pfister neighbors. For one can construct fields with anisotropic nonsingular forms of arbitrarily large dimension, i.e. $u(F) = \infty$, but for which $I^3 W_q F = 0$. This can be done as in the characteristic $\neq 2$ case (where it was first done by Merkurjev [32]) using the methods in [31]. Over such a field, any anisotropic nonsingular form of dimension 2^n, $n \geq 3$, clearly contains a dominated form of type $(1, 2^{n-1} - 1)$. This anisotropic form of dimension $2^{n-1} + 1$ cannot be a Pfister neighbor as n-fold Pfister forms are hyperbolic for $n \geq 3$ because $I^3 W_q F = 0$.

Example 6.4. The construction we now describe can be used to produce counterexamples of type (r, s) subject to the following conditions : There exist integers $n, m, t \geq 0$ with $n < m$, $2r + t > 2^n$, $2r + 2t \leq 2^{n+1}$, and $s = t + 2^m$. In particular, we get counterexamples of type $(2^n, 2^{n+1})$.

Consider a Pfister form $\pi = \langle\langle a_1, \cdots, a_n; b]]$ and an anisotropic totally singular Pfister form $\tau = \langle\langle a_1, \cdots, a_m\rangle\rangle$ with $m > n$. Let $x \in F^*$ and put $\varphi = \pi \perp x\tau$. Let E/F be any field extension such that τ_E stays anisotropic but such that $\varphi_E = \pi_E \perp x\tau_E$ is isotropic. Then there exists $y \in D_E(\pi) \cap D_E(x\tau)$. The roundness of π and τ implies that $\varphi_E \cong y(\pi_E \perp \tau_E)$.

Now π is an orthogonal sum of the forms $(\prod_{i \in I} a_i)[1, b]$ with $I \subset \{1, \cdots, n\}$, and τ is an orthogonal sum of the forms $\langle \prod_{j \in J} a_j \rangle$ with $J \subset \{1, \cdots, m\}$. Since $[u, v] \perp \langle u \rangle \cong \mathbb{H} \perp \langle u \rangle$, it follows readily that $\pi \perp \tau \sim \tau$, hence $\varphi_E \sim y\tau_E \cong x\tau_E$.

Now let $\psi \prec \pi$ be a Pfister neighbor of π with complement ψ^c and let $\eta = \psi \perp x\tau$ for some $x \in F^*$. Thus, if E is a field extension with τ_E anisotropic, and if $\psi_E \perp x\tau_E$ is isotropic, then the above shows that $\eta_E \sim \psi^c_E \perp x\tau_E$ because $\pi \perp \psi^c \sim \psi$.

Suppose now that π is anisotropic and that $x \in F^*$ is such that $\pi \perp x\tau$ is anisotropic as well. In particular, this implies that η and $\psi^c \perp x\tau$ are anisotropic. Note that η is not totally singular as the Pfister neighbor ψ is not totally singular.

Also, η will not be a Pfister neighbor. Indeed, suppose η is a Pfister neighbor. Now $2^{m+1} \geq 2^m + 2^{n+1} \geq \dim \eta > 2^m + 2^n$, thus η is necessarily a Pfister neighbor of an $(m+1)$-fold Pfister form. If (r, s) is the type of η, this implies that $r + 2s \leq 2^{m+1}$, a contradiction as $r \geq 1$ and $s \geq 2^m$.

Let $E = F(\eta)$. Then, by the above, $\eta_E \sim (\psi^c \perp x\tau)_E$. Thus, if $(\psi^c \perp x\tau)_E$ is anisotropic, this gives an example of an anisotropic non-Pfister neighbor η whose anisotropic part over $F(\eta)$ is defined over the base field F.

This certainly holds if $\psi^c \perp x\tau$ is totally singular (cf. Lemma 4.1), i.e. ψ^c is totally singular. This is the case if and only if the Pfister neighbor ψ is of type (r', s') with $2r' + 2s' = 2^{n+1}$. (Knebusch [25] calls such Pfister neighbors *close* Pfister neighbors.) This incidentally yields examples of non-Pfister neighbors of nondefective height 1.

To construct explicit situations with $(\psi^c \perp x\tau)_{F(\eta)}$ anisotropic, let π and τ be as above, anisotropic, and with $a_i, b \in F_0$, and let $\psi \prec \pi$ be a Pfister neighbor over F_0. Let $F = F_0(x)$ with x an indeterminate. By the usual degree argument, we see that $\pi \perp x\tau$ is anisotropic. Let $K = F_0(\psi)$. Then ψ_K^c is anisotropic (cf. Proposition 6.1), and so is the totally singular form τ_K. Hence, over $F(\psi) = F_0(\psi)(x)$, we have that $(\psi^c \perp x\tau)_{F(\psi)}$ is anisotropic. Now $\eta_{F(\psi)}$ is nondefective and isotropic, hence $F(\psi)(\eta)/F(\psi)$ is purely transcendental, thus $(\psi^c \perp x\tau)_{F(\psi)(\eta)}$ is anisotropic and therefore, $(\psi^c \perp x\tau)_{F(\eta)}$ is anisotropic as desired.

A natural question is to ask under which conditions the converse of the last statement in Proposition 6.1 holds. In all the examples above, the counterexamples are of type (r, s) with $s \geq 2r$. In view of this, we venture the following

Conjecture 6.5. *Let q be an anisotropic form such that the anisotropic part of q over $F(q)$ is defined over F. If q is of type (r, s) with $s < 2r$, then q is a Pfister neighbor.*

Theorem 6.6. *Conjecture 6.5 is true if $s \leq 4$ or if $s = 5$ and there exist $a, b, c \in F$ such that the quasi-linear part of q is similar to $\langle\langle a, b\rangle\rangle \perp \langle c\rangle$.*

To prove the theorem, we need some lemmas.

Lemma 6.7. *Let $\varphi \cong \varphi_r \perp \sigma$ be anisotropic of type (r, s), $r \geq 1$, with $\sigma = \mathrm{ql}(\varphi)$. Suppose that the anisotropic part of $\varphi_{F(\varphi)}$ is defined over F. Then there exists a form $\eta \cong \eta_r \perp \sigma$ over F with η_r nonsingular such that*

- $(\varphi_{F(\varphi)})_{\mathrm{an}} \cong \eta_{F(\varphi)}$, *and*
- $(\varphi_r \perp \eta_r \perp \sigma)_{F(\varphi)} \sim \sigma_{F(\varphi)}$.

If, furthermore, there exists a nonsingular completion ρ of σ over F such that $q \cong \varphi_r \perp \eta_r \perp \rho \in I^n W_q F$ with $2^n > 2s$, then q is anisotropic and similar to a Pfister form. In particular, φ is a Pfister neighbor.

Proof. Let $K = F(\varphi)$. By Lemma 4.1, $\sigma_K \cong \mathrm{ql}(q_K)$. Hence, we may assume that the anisotropic part over K is defined over F by a form η whose quasi-linear part is σ, i.e., $\eta \cong \eta_r \perp \sigma$ with η_r nonsingular.

Now $(\varphi_r \perp \sigma)_K \sim (\eta_r \perp \sigma)_K$, hence $(\varphi_r \perp \eta_r \perp \sigma)_K \sim \sigma_K$

Let ρ be an n.s.c. of σ with $q \cong \varphi_r \perp \eta_r \perp \rho \in I^n W_q F$ and $2^n > 2s$. By Lemma 3.9, there exists over K an n.s.c. τ of σ_K such that $q_K \sim \tau_K \in I^n W_q F$. Now $\dim \tau = 2s < 2^n$, and the Arason-Pfister Hauptsatz implies $q_K \sim \tau \sim 0$. It follows now readily from Theorem 5.1 that q is anisotropic and similar to a Pfister form.

Since $\dim\varphi > \dim\eta$ and $\dim\varphi + \dim\eta = \dim q$, we have $\varphi \prec q$ and $2\dim\varphi > \dim q$. Hence, φ is a Pfister neighbor. □

Lemma 6.8. *Let $\varphi \cong \varphi_r \perp \langle 1, a_2, \cdots, a_s\rangle$ be nondefective of type (r,s) with $s \geq 2$ and φ_r nonsingular. Suppose that $2 \leq t \leq s$ is such that $F(\sqrt{a_2}, \cdots, \sqrt{a_s}) = F(\sqrt{a_2}, \cdots, \sqrt{a_t})$. Let q be nondefective of type (r', s') with $\dim q \geq 5$ and $r' \geq 2$. If $(\varphi_{F(q)})_{\mathrm{an}}$ is totally singular, then there exist $b_1, \cdots, b_t \in F^*$ such that $\psi = \varphi_r \perp [1, b_1] \perp a_2[1, b_2] \perp \cdots \perp a_t[1, b_t] \perp (t-s) \times \mathbb{H}$ is an n.s.c. of φ with $\psi \in I^3 W_q F$.*

Proof. Let $K = F(\sqrt{a_2}, \cdots, \sqrt{a_t})$. By assumption, $\langle 1, a_2, \cdots, a_s\rangle_K \sim \langle 1\rangle_K$.

Let q' be the nondefective part of q over K. Then q' is of type (r', s'') and $\dim q' \geq 5$. Indeed, if $r' \geq 3$, this is clear, and if $r' = 2$ then $s' \geq 1$ and hence $s' \geq s'' \geq 1$. Hence, $K(q)/K(q')$ is purely transcendental of transcendence degree $s' - s''$.

By assumption, $\varphi_{F(q)} \sim \langle 1, a_2, \cdots, a_s\rangle_{F(q)}$, thus $\varphi_{K(q)} \sim \langle 1, a_2, \cdots, a_s\rangle_{K(q)} \sim \langle 1\rangle_{K(q)}$, and since anisotropic forms stay anisotropic over purely transcendental extensions, we obtain $(\varphi_r \perp \langle 1\rangle)_{K(q')} \sim \varphi_{K(q')} \sim \langle 1\rangle_{K(q')}$. Let $\Delta = \Delta(\varphi_r)$. By Lemma 3.9 there exists $a \in K(q')$ such that $(\varphi_r \perp [1, \Delta])_{K(q')} \sim [1, a]$. Now $\varphi_r \perp [1, \Delta] \sim [1, a] \in I^2 W_q K(q')$ by Theorem 4.9 and therefore, by the Arason-Pfister Hauptsatz, $[1, a] \sim 0$. In particular, $C(\varphi_r \perp [1, \Delta])$ splits over $K(q')$.

Since $K(q')/K$ can be obtained as a purely transcendental extension followed by a separable quadratic extension, the index of $C(\varphi_r \perp [1, \Delta])_K$ can only be 1 or 2. If it is 2, then the class of $C(\varphi_r \perp [1, \Delta])$ in $\mathrm{Br}(K)$ is given by a quaternion division algebra which stays division over $K(q')$ as q'_K is nondefective and $\dim q' \geq 5$ (cf. [31, Theorem 3]), a contradiction. Hence, $C(\varphi_r \perp [1, \Delta])_K$ is split, and by Albert's result (Proposition 4.8), it follows that there exist $b_2, \cdots, b_t \in F$ with $C(\varphi_r \perp [1, \Delta]) = [b_2, a_2)_F \otimes \cdots \otimes [b_t, a_t)_F$. Let $b_1 = \Delta + b_2 + \cdots + b_t$ and put

$$\psi = \varphi_r \perp [1, b_1] \perp a_2[1, b_2] \perp \cdots \perp a_t[1, b_t] \perp (s-t) \times \mathbb{H} \, .$$

Then one verifies rapidly that $\psi \in I^3 W_q F$ and that ψ is a nonsingular completion of φ. □

Proof of Theorem 6.6. Let $\varphi \cong \varphi_r \perp \sigma$ be anisotropic of type (r,s) with quasi-linear part σ and $2r > s$. Suppose that $(\varphi_{F(\varphi)})_{\mathrm{an}} \cong \eta$ for some F-form η. Then, by Lemma 6.7, we may assume that $\eta \cong \eta_r \perp \sigma$ with η_r nonsingular, $\dim\eta_r < \dim\varphi_r$.

The case $s = 0$. In this case, $2 > 2s = 0$. Since $\varphi \perp \eta \cong \varphi_r \perp \eta_r \in I W_q F$, it follows from Lemma 6.7 that $\varphi \perp \eta$ is similar to an anisotropic Pfister form of which φ is a Pfister neighbor.

The case $s = 1$. After scaling, we may assume that $\sigma \cong \langle 1\rangle$. Let $d = \Delta(\varphi_r) + \Delta(\eta_r)$ Then $\rho \cong [1, d]$ is a regular completion of σ, and by putting $q := \varphi_r \perp \eta_r \perp \rho$, it follows from Theorem 4.9 that $q \in I^2 W_q F$. Now $2^2 > 2s = 2$, hence Lemma 6.7 implies that q is a Pfister form (note that q represents 1) and that φ is a Pfister neighbor of q.

The case $2 \leq s \leq 3$. $2r > s$ implies that $r \geq 2$ and $\dim\varphi \geq 6$. Let $K = F(\varphi)$. By assumption $(\varphi_r \perp \eta_r \perp \sigma)_K \sim \sigma_K$. By Lemma 6.8, there exists an n.s.c. ρ of σ such that for $q = \varphi_r \perp \psi_r \perp \rho$ we get $q \in I^3 W_q F$. Since $2^3 > 6 \geq 2s$, Lemma 6.7 yields that q is similar to an anisotropic Pfister form of which $\varphi \cong \varphi_r \perp \sigma$ is a Pfister neighbor.

The case $4 \leq s \leq 5$. By assumption in the case $s = 5$ and after scaling, we may assume that $\sigma \cong \langle 1, a, b, c\rangle$ or $\sigma \cong \langle 1, a, b, ab, c\rangle$, $a, b, c \in F^*$.

Let $L = F(\langle 1, a, b, ab\rangle)$. By Theorem 8.11(ii), $\langle 1, a, b, ab\rangle_L \sim \langle 1, a\rangle_L$. Let $\sigma' = (\sigma_L)_{\mathrm{an}}$. The above shows that $\sigma' \cong \langle 1, a\rangle$ or $\sigma' \cong \langle 1, a, c\rangle$.

By Lemma 6.8, there exists an n.s.c. ρ of σ such that $\psi := \varphi_r \perp \eta_r \perp \rho \in I^3W_qF$.

Now put $M = L(\varphi) = K(\langle 1, a, b, ab\rangle)$. Since $(\varphi_r \perp \eta_r \perp \sigma)_K \sim \sigma_K$, there exists an n.s.c. ρ' of σ over M such that $\psi_M \sim \rho'$. We have $i_{\mathrm{ti}}(\sigma_M) \geq s-3$, and $\sigma_M \prec \rho'$ with ρ' nonsingular. Hence, $i_W(\rho') \geq s-3$ and $\dim \rho'_{\mathrm{an}} \leq 2s - 2(s-3) = 6$. The Arason-Pfister Hauptsatz implies that $\rho' \sim 0$.

Clearly, $\deg \psi_L \geq 3$ as $\psi_L \in I^3W_qL = J_3L$. We claim that $\psi_L \in I^4W_qL = J_4L$. Suppose that $\deg \psi_L = 3$. Let N be the leading field of ψ_L and let $\pi \in GP_3N$ be the leading form. We have $\psi_N \sim \pi$ and $0 \sim \psi_{N(\varphi)} \sim \pi_{N(\varphi)}$. Let φ' be the nondefective part of φ_L which, by the above, is of type (r, s') with $s' = \dim \sigma' \in \{2, 3\}$ and $2r = \dim \varphi_r \geq 6$. Since $N(\varphi)/N(\varphi')$ is purely transcendental, we also have $0 \sim \psi_{N(\varphi')} \sim \pi_{N(\varphi')}$. Now N is obtained from L by taking function fields of nonsingular forms, hence the nondefective form φ' stays nondefective over N (Lemma 4.1). It follows from the domination theorem (Theorem 4.2) that there exists $x \in N^*$ with $x\varphi' \prec \pi$. In particular, $10 \leq 2r + 2s' \leq \dim \pi = 8$, a contradiction.

Now $\psi \in I^3W_qF$ and $\psi_L \in I^4W_qL$. Therefore, Theorem 4.7 yields that there exists $d \in F$ with $\psi \equiv \langle\!\langle a, b, d]] \bmod I^4W_qF$.

Consider first the case $s = 5$. Now $\langle 1, a, b, ab\rangle \prec \langle\!\langle a, b, d]]$ and $\langle 1, a, b, ab\rangle \prec \rho$. Hence, $i_{\mathrm{ti}}(\langle\!\langle a, b, d]] \perp \rho) = i_W(\langle\!\langle a, b, d]] \perp \rho) \geq \dim \langle 1, a, b, ab\rangle = 4$. Thus, $\langle\!\langle a, b, d]] \perp \rho \cong \rho' \perp (4 \times \mathbb{H})$ with ρ' nonsingular and $\dim \rho' = 10$. $\rho' \perp \sigma \sim \langle\!\langle a, b, d]] \perp \rho \perp \sigma \sim \langle\!\langle a, b, d]] \perp \sigma \sim \sigma$, which shows for dimension reasons that ρ' is an n.s.c. of σ. Furthermore, $q := \varphi_r \perp \eta_r \perp \rho' \in I^4W_qF$ and $2^4 > 2s = 10$. Lemma 6.7 implies that q is similar to a Pfister form of which $\varphi \cong \varphi_r \perp \sigma$ is a Pfister neighbor.

Suppose now that $s = 4$. Let $E = F(\sigma)$. Similarly as above, one shows that $\sigma'' = (\sigma_E)_{\mathrm{an}}$ is of dimension 2 or 3, that the nondefective part φ'' of φ_E is of type (r, s'') with $s'' = \dim \sigma''$, and that $\psi_{E(\varphi)} \sim 0$ and therefore $\psi_{E(\varphi'')} \sim 0$. Since $\psi \equiv \langle\!\langle a, b, d]] \bmod I^4W_qF$, this implies $\langle\!\langle a, b, d]]_{E(\varphi'')} \equiv 0 \bmod I^4W_qE(\varphi'')$ and thus, by the Arason Pfister Hauptsatz, $\langle\!\langle a, b, d]]_{E(\varphi'')} \sim 0$. If $\langle\!\langle a, b, d]]_E$ is anisotropic, then by the domination theorem, there exists $y \in E^*$ with $y\varphi'' \prec \langle\!\langle a, b, d]]$, hence $2r+2s'' \leq \dim \langle\!\langle a, b, d]] = 8$, a contradiction as $10 \leq 2r+2s''$. Hence, $\langle\!\langle a, b, d]]_E \sim 0$. If $\langle\!\langle a, b, d]]$ is anisotropic, then by the domination theorem and using the fact that both σ and $\langle\!\langle a, b, d]]$ represent 1, we get $\sigma \prec \langle\!\langle a, b, d]]$. This is clearly also true if $\langle\!\langle a, b, d]]$ is hyperbolic. In any case, similarly as above, there exists an n.s.c. ρ'' of σ with $\rho \perp \langle\!\langle a, b, d]] \sim \rho''$ and $\sigma \prec \rho''$. By putting this time $q := \varphi_r \perp \eta_r \perp \rho''$, we have $q \in I^4W_qF$ and we conclude as before. □

7. Forms of nondefective height 1

Let q be an anisotropic form of type (r, s) which is not totally singular, i.e. $r \geq 1$. Recall that q is of nondefective height 1 if $(q_{F(q)})_{\mathrm{an}}$ is totally singular, or in other words, by Lemmas 2.2 and 4.1, $(q_{F(q)})_{\mathrm{an}} \cong \mathrm{ql}(q)_{F(q)}$. Equivalently, $h_{\mathrm{nd}}(q) = 1$ if and only if $i_W(q_{F(q)}) = r$. (In Knebusch's terminology [25], such a form is said to be of height 1.) Note that anisotropic forms of nondefective height 1 have therefore the property that their anisotropic part over their own function field is defined over the base field, thus we can use our results from the previous section.

Example 7.1. Let q be anisotropic of type $(1,s)$. Then clearly, $h_{\mathrm{nd}}(q)=1$ (Lemma 4.1).

Example 7.2. Let q be an anisotropic Pfister neighbor of type (r,s). Let n be such that $2^{n-1}<\dim q=2r+s\leq 2^n$. Then q is a Pfister neighbor of some n-fold Pfister form and necessarily $2r+2s\leq 2^n$. The complement q^c has dimension $\dim q^c=2^n-2r-s$ and we have $\mathrm{ql}(q)\cong\mathrm{ql}(q^c)$ and $(q_{F(q)})_{\mathrm{an}}\cong q^c_{F(q)}$ by Proposition 6.1. It follows immediately that $h_{\mathrm{nd}}(q)=1$ if and only if $2r+2s=2^n$ if and only if $\mathrm{ql}(q)=q^c$, i.e. q is a close Pfister neighbor in Knebusch's terminology.

Example 7.3. We recall the construction in Example 6.4 of forms of nondefective height 1 which are not Pfister neighbors. Let ψ be a Pfister neighbor of type (r',s') of a Pfister form $\pi\cong\langle\langle a_1,\cdots,a_n;b]]$, let $\tau\cong\langle\langle a_1,\cdots,a_m\rangle\rangle$ with $m\geq n+1$, and suppose that $\pi\perp x\tau$, $x\in F^*$, is anisotropic. Suppose furthermore that $2r'+2s'=2^{n+1}$ so that ψ is a *close* Pfister neighbor of π. Let $q=\psi\perp x\tau$. Then $\mathrm{ql}(q)=\psi^c\perp x\tau$, q is not a Pfister neighbor, and $h_{\mathrm{nd}}(q)=1$ (see Example 6.4).

Lemma 7.4. *Suppose that q is anisotropic of type (r,s), $r\geq 2$. If there exists an integer $n\geq 1$ such that $r+s<2^n<2r+s$, then $h_{\mathrm{nd}}(q)\geq 2$.*

Proof. Suppose that $h_{\mathrm{nd}}(q)=1$. Let V be the underlying vector space of q, and let $W\subset V$ be a subspace with $\dim W=r+s+1$ and $q'=q|_W$. By assumption, $i_W(q_{F(q)})=r$. Hence, $V\otimes F(q)$ contains a totally isotropic subspace (with respect to $q_{F(q)}$) of dimension r which intersects with $W\otimes F(q)$ as $r+\dim W>\dim V$. In particular, $q'_{F(q)}$ is isotropic. But $q'\prec q$ and therefore, q' is anisotropic over F, and $\dim q'\leq 2^n<\dim q$, a contradiction to Theorem 4.4. □

We now come to the classification of forms of nondefective height 1 of type (r,s) with $r\geq 2$ and small values of s (recall that the case $r=1$ is trivial by Example 7.1).

Theorem 7.5. *Let q be anisotropic of type (r,s), $r\geq 2$. Suppose that $s\leq 4$ or $s=5$ and there exist $u,v,w\in F^*$ such that $\mathrm{ql}(q)$ is similar to $\langle\langle u,v\rangle\rangle\perp\langle w\rangle$. Then $h_{\mathrm{nd}}(q)=1$ if and only if q belongs to one of the following (mutually exclusive) classes of forms :*

(i) *There exists an integer $n\geq 1$ such that $r+s=2^n$ and q is a Pfister neighbor;*
(ii) *$(r,s)=(2,4)$ and there exist $a,b,c,d,e\in F^*$ such that $q\cong d\langle\langle a,b]]\perp e\langle\langle a,c\rangle\rangle$.*

Proof. If q is as in (i), then q is in fact a close Pfister neighbor of some $(n+1)$-fold Pfister form since $2r+2s=2^{n+1}$, and we have that $h_{\mathrm{nd}}(q)=1$ by Example 7.2.

If q is as in (ii), then $h_{\mathrm{nd}}(q)=1$ by Example 7.3. Note that in this case, q cannot be a Pfister neighbor as $\dim q=8$ and q is not similar to a 3-fold Pfister form.

Now assume that $h_{\mathrm{nd}}(q)=1$. Then $(q_{F(q)})_{\mathrm{an}}\cong\mathrm{ql}(q)_{F(q)}$. If $2r>s$, then by Theorem 6.6, q is a Pfister neighbor, say, of an $(n+1)$-fold Pfister form, and we have $(q_{F(q)})_{\mathrm{an}}\cong q^c_{F(q)}$ by Proposition 6.1. Hence, $\dim q^c=\dim\mathrm{ql}(q)$. Now $\mathrm{ql}(q)\subset q^c$ and thus $q^c\cong\mathrm{ql}(q)$. This readily yields that q is a Pfister neighbor of codimension s, i.e. $\dim q=2r+s=2^{n+1}-s$, and we get $r+s=2^n$.

Since $s\leq 5$ and $r\geq 2$, the only cases which are not yet covered are : $(r,s)\in\{(2,4),\ (2,5)\}$. The case $(r,s)=(2,5)$ cannot occur by Lemma 7.4.

Let us finally assume that $(r,s) = (2,4)$ and write (after possibly scaling) $q \cong \varphi \perp \langle 1,u,v,w\rangle$ with φ nonsingular of dimension 4 and $u,v,w \in F^*$. By Proposition 8.6, the anisotropy of $\langle 1,u,v,w\rangle$ yields two possible cases : Either $[F^2(u,v,w) : F^2] = 8$, or $[F^2(u,v,w) : F^2] = 4$, and in this latter case one has in fact $\langle 1,u,v,w\rangle \cong \langle 1,u,v,uv\rangle$.

Suppose first that $[F^2(u,v,w) : F^2] = 8$ and let $K = F(\langle\!\langle u,v\rangle\!\rangle)$. It follows readily from Corollary 8.14 that $\langle\!\langle u,w\rangle\!\rangle_K$ is anisotropic, hence also $\langle 1,u,w\rangle_K$. Now $\langle 1,u,v\rangle_K$ is isotropic (Proposition 8.9), hence $(\langle 1,u,v,w\rangle_K)_{\mathrm{an}} \cong \langle 1,u,w\rangle_K$. Let $q' \cong \varphi \perp \langle 1,u,w\rangle$. Then $q_K \sim q'_K$, q'_K is nondefective, and $K(q)$ is a purely transcendental extension of $K(q')$. Now $q_{F(q)} \sim \langle 1,u,v,w\rangle$. Hence,

$$\langle 1,u,v,w\rangle_{K(q)} \sim q_{K(q)} \sim q'_{K(q)} \sim \langle 1,u,w\rangle_{K(q)}$$

and thus $q'_{K(q')} \sim \langle 1,u,w\rangle_{K(q')}$.

If q'_K is anisotropic, then q'_K is of type $(r,s-1) = (2,3)$ and $h_{\mathrm{nd}}(q'_K) = 1$ which, by the above, implies that q'_K is a Pfister neighbor, necessarily of some 3-fold, which is impossible as $2r + 2(s-1) = 10 > 8$. If $(q'_K)_{\mathrm{an}}$ is of type $(1,3)$, then $i_W(q'_K) = 1$ and $K(q')/K$ is purely transcendental. This implies that the 5-dimensional form $(q'_K)_{\mathrm{an}}$ stays anisotropic over $K(q')$, impossible as $q'_{K(q')} \sim \langle 1,u,w\rangle_{K(q')}$. Hence, q'_K is of type $(0,3)$, i.e. $q'_K \sim \langle 1,u,w\rangle_K$.

Now by assumption, $q_{F(q)} \sim \langle 1,u,v,w\rangle_{F(q)}$, and by Lemma 6.8, there exists an n.s.c. ψ of q with $\dim\psi = 2r+2s = 12$ and $\psi \in I^3W_qF$. By Lemma 3.9, there exists an n.s.c. τ over K of $\langle 1,u,v,w\rangle_K \cong \langle 1,u,w,0\rangle_K$ such that $\psi_K \sim \tau \in I^3W_qK$. Since $\dim\tau = 8$ and since τ is isotropic, it follows from the Arason-Pfister Hauptsatz that $\psi_K \sim \tau \sim 0$. By Theorem 4.7, there exists $x \in F^*$ with $\psi \equiv \langle\!\langle u,v,x]] \bmod I^4W_qF$. Now $i_W(\psi \perp \langle\!\langle u,v,x]]) \geq 3$, hence $\dim(\psi \perp \langle\!\langle u,v,x]])_{\mathrm{an}} \leq 12 + 8 - 6 = 14 < 16$, and again by the Arason-Pfister Hauptsatz, $\psi \perp \langle\!\langle u,v,x]] \sim 0$. Hence,

$$\psi \perp \langle 1,u,v,w\rangle \sim q \sim \langle\!\langle u,v,x]] \perp \langle 1,u,v,w\rangle \sim uv[1,x] \perp \langle 1,u,v,w\rangle\ ,$$

and by comparing dimensions, we conclude that q is isotropic, a contradiction.

Thus, we have $[F^2(u,v,w) : F^2] = 4$ and $q \cong \varphi \perp \langle 1,u,v,uv\rangle$. Again by Lemma 6.8, there exist $x,y,z \in F$ such that $\varphi \perp [1,x] \perp [u,y] \perp [v,z] \perp [uv,0] \in I^3W_qF$. Now there are no anisotropic forms of dimension 10 in I^3W_qF ([11, pp. 129–130]), and thus, by the Arason-Pfister Hauptsatz, there exists $\pi \in GP_3F$ such that $\varphi \perp [1,x] \perp [u,y] \perp [v,z] \sim \pi$. Hence,

$$\varphi \perp [1,x] \perp [u,y] \perp [v,z] \perp \pi \perp \langle 1,u,v,uv\rangle \sim q \perp \pi \sim \langle 1,u,v,uv\rangle\ .$$

This implies that $(q \perp \pi)_{\mathrm{an}} \cong \langle 1,u,v,uv\rangle$. Now $\dim(q \perp \pi) = 16$, hence $i_W(q \perp \pi) = 6$, and by Proposition 3.11, there exists a form η such that $\dim\eta = 6$, $\eta \prec \pi$, $\eta \prec q$. In particular, η is a 6-dimensional Pfister neighbor, therefore of type $(3,0)$ or $(2,2)$. But q is of type $(2,4)$ and $\eta \prec q$, hence η is of type $(2,2)$. Then there exist $a,b,d,e \in F^*$ such that $\eta \cong d\langle\!\langle a,b]] \perp e\langle 1,a\rangle$ (see [27, Prop. 3.2]). Since $\eta \prec q$, it follows that there exist $c,f \in F^*$ such that $q \cong d\langle\!\langle a,b]] \perp e\langle 1,a,c,f\rangle$. Comparing quasi-linear parts, we obtain $e\langle 1,a,c,f\rangle \cong \langle 1,u,v,uv\rangle$. Propositions 8.5 and 8.6 readily imply that $\langle 1,u,v,uv\rangle \cong \langle 1,a,c,f\rangle \cong \langle 1,a,c,ac\rangle$. □

8. Totally singular forms and quasi-Pfister neighbors

In Theorem 6.6 in the case $s = 5$, we have considered only the case of totally singular forms similar to forms of type $\langle\!\langle a,b\rangle\!\rangle \perp \langle c\rangle$. In characteristic $\neq 2$, forms

similar to such diagonal forms are exactly the 5-dimensional Pfister neighbors, so it seems natural to ask whether there is a meaningful notion of totally singular Pfister neighbors in characteristic 2.

The purpose of this section is to develop some basic notions of a theory of totally singular forms in a way which is as much independent of the theory of nonsingular forms as possible and yet reflects some analogous properties, including a notion analogous to that of Pfister neighbors.

Some basic properties have been mentioned before as they have been used in the proofs of our results. Let us recall some of them to keep this section as self-contained as possible. (See also Lemma 2.2 and the remarks preceding it.)

Recall that the anisotropic part of a totally singular form $\sigma = \langle a_1, \cdots, a_n \rangle$ is uniquely determined by the F^2-vector space spanned by $\{a_1, \cdots, a_n\}$ inside F, and this F^2-vector space is nothing but $D_F(\sigma) \cup \{0\}$. This immediately implies

Proposition 8.1. (i) *There is a one-to-one correspondence*

$$\begin{array}{rcl} \left\{ \begin{array}{l} \textit{isometry classes of} \\ \textit{totally singular forms} \end{array} \right\} & \rightleftharpoons & \left\{ \begin{array}{l} \textit{finite-dimensional} \\ F^2\textit{-vector spaces inside } F \end{array} \right\} \times \mathbb{N}_0 \\ \sigma & \to & (V, \dim \sigma - \dim_{F^2} V) \quad \textit{for } V = D_F(\sigma) \cup \{0\} \\ \langle a_1, \cdots, a_n \rangle \perp m \times \langle 0 \rangle & \leftarrow & (V, m) \quad \textit{for an } F^2\textit{-basis } \{a_1, \cdots, a_n\} \textit{ of } V \end{array}$$

(ii) $\varphi = \langle a_1, \cdots, a_n \rangle \subset \psi = \langle b_1, \cdots, b_m \rangle$ *if and only if* $m \geq n$ *and the sub-*F^2*-vector space inside* F *spanned by* $\{a_1, \cdots, a_n\}$ *is contained in the sub-*F^2*-vector space inside* F *spanned by* $\{b_1, \cdots, b_n\}$ *if and only if* $m \geq n$ *and* $D_F(\varphi) \cup \{0\} \subset D_F(\psi) \cup \{0\}$.

(iii) (Cf. Lemma 2.2.) *Let* $\varphi = \langle a_1, \cdots a_n \rangle$. *If* E/F *is any field extension, then there exist (after possibly reindexing)* $0 \leq m \leq n$ *such that* $(\varphi_E)_{\mathrm{an}} \cong \langle a_1, \cdots, a_m \rangle$.

Recall that over a field K of characteristic $\neq 2$, the norm group of a quadric Q associated to a (nonsingular) quadratic form φ is given by the subgroup of K^* generated by elements ab with $a, b \in K^*$ represented by φ (see [13, Lemme 2.2]). In analogy to this we make the following definition :

Definition 8.2. Let σ be a totally singular form over F. The *norm field* of σ over F, denoted by $N_F(\sigma)$, is defined to be the field $F^2(ab \,|\, a, b \in D_F(\sigma))$. The *norm degree* of σ over F is defined to be $\mathrm{ndeg}_F(\sigma) = [N_F(\sigma) : F^2]$.

The proof of the following lemma is easy and left to the reader.

Lemma 8.3. *Let* $\sigma \cong \langle a_1, \cdots, a_n \rangle$ *be a nonzero totally singular form with* $a_1 \neq 0$. *Then, for each* $a \in F^*$, *we have* $N_F(a\sigma) = N_F(\sigma) = F^2(a_1a_2, \cdots, a_1a_n)$.

This implies in particular that the norm field and the norm degree are invariants of the similarity class of a totally singular form.

This lemma also readily implies

Corollary 8.4. *Let* σ *be a nonzero totally singular form and let* K/F *be a field extension. Let* $b_1, \cdots b_m \in F$ *be such that* $N_F(\sigma) = F^2(b_1, \cdots, b_m)$. *Then* $N_K(\sigma_K) = K^2(b_1, \cdots, b_m)$.

We now come to the characterization of quasi-Pfister forms.

Proposition 8.5. (i) *Let $q = \langle\langle a_1, \cdots, a_n\rangle\rangle$ be a quasi-Pfister form over F. Then $N_F(q) = F^2(a_1, \cdots, a_n) = D_F^*(q) \cup \{0\}$. In particular, if $x \in D_F^*(q)$, then $q \cong xq$.*

(ii) *Conversely, let q be a nonzero totally singular form such that $q \cong xq$ for each $x \in D_F^*(q)$. Then q_{an} is a quasi-Pfister form.*

(iii) *An anisotropic totally singular form q is similar to a quasi-Pfister form if and only if $\dim q = \mathrm{ndeg}_F(q)$.*

(iv) *There exists a natural bijection between anisotropic n-fold quasi-Pfister forms and purely inseparable extensions of F^2 of degree 2^n inside F which is given by $\langle\langle a_1, \cdots, a_n\rangle\rangle \leftrightarrow F^2(a_1, \cdots, a_n)$.*

Proof. (i) Let $q = \langle\langle a_1, \cdots, a_n\rangle\rangle$ with $a_i \in F^*$. Clearly, $N_F(q) = F^2(a_1, \cdots, a_n)$. Also, we obviously have $F^2[a_1, \cdots, a_n] = D_F(q) \cup \{0\} \subset N_F(q)$. Since the a_i are algebraic over F^2, we have $D_F(q) \cup \{0\} = N_F(q)$. It follows readily that the coefficients of the diagonal forms q and xq, $x \in N_F(q)^* = D_F^*(q)$ span the same sub-vector space of the F^2-vector space F, namely $N_F(q)$. Hence, $q \cong xq$.

(ii) Since $\langle 0\rangle \cong x\langle 0\rangle$ for each $x \in F$, we clearly may assume by Lemma 2.6 that q is anisotropic. $R = D_F(q) \cup \{0\}$ is an additive subgroup of F, and the roundness of q implies that $D_F(q)$ is closed under multiplication. Hence, R is a finite-dimensional F^2-algebra inside the field F and one obviously has $\dim_{F^2}(R) = \dim q$. In particular, R is a field. Let $q = \langle a_1, \cdots, a_m\rangle$ and let $n \leq m$ be minimal such that, after possibly reindexing, $F^2(a_1, \cdots, a_m) = F^2(a_1, \cdots, a_n)$. Similarly as in part (i), we get $R = F^2(a_1, \cdots, a_n)$ which, by the minimality of n, yields $m = \dim_{F^2} R = 2^n = \dim q$. Now $\{a_1, \cdots, a_m\}$ is an F^2-basis of R, as is $\{\prod_{i\in I} a_i \,|\, I \subset \{1, \cdots, n\}\}$. It follows readily that $q \cong \langle\langle a_1, \cdots, a_n\rangle\rangle$.

(iii) Let $\mathrm{ndeg}_F(q) = 2^n$ and $a_1, \cdots, a_n$ with $N_F(q) = F^2(a_1, \cdots, a_n)$. Suppose that $\dim q = \mathrm{ndeg}(q)$. We may assume that $q \cong \langle 1, \cdots\rangle$ and thus $V = D_F(q) \cup \{0\} \subset N_F(q) = D_F(\langle\langle a_1, \cdots, a_n\rangle\rangle) \cup \{0\}$. By Proposition 8.1(ii), we get $q \subset \langle\langle a_1, \cdots, a_n\rangle\rangle$ and hence $q \cong \langle\langle a_1, \cdots, a_n\rangle\rangle$ for dimension reasons.

The converse is clear.

(iv) is obvious and left to the reader. □

Proposition 8.6. *Let σ be a nonzero totally singular form.*

(i) *There exists an $m \geq 0$ such that $\mathrm{ndeg}_F(\sigma) = 2^m$.*

(ii) *With m as in (i), we have $m + 1 \leq \dim \sigma_{\mathrm{an}} \leq 2^m$.*

(iii) *With m as in (i) and if $a_1, \cdots, a_m \in F^*$ are chosen such that $N_F(\sigma) = F^2(a_1, \cdots, a_m)$, then for each $a \in D_F^*(\sigma)$ we have $a\sigma_{\mathrm{an}} \subset \langle\langle a_1, \cdots, a_m\rangle\rangle$. Furthermore, if $b_1, \cdots, b_m \in F^*$ are such that $a\sigma_{\mathrm{an}} \subset \langle\langle b_1, \cdots, b_m\rangle\rangle$, then $\langle\langle a_1, \cdots, a_m\rangle\rangle \cong \langle\langle b_1, \cdots, b_m\rangle\rangle$, and this m-fold quasi-Pfister form is anisotropic.*

Proof. We may assume that σ is anisotropic and $\sigma \cong \langle 1, a_1, \cdots, a_n\rangle$. Hence, $N_F(\sigma) = F^2(a_1, \cdots, a_n)$. Let $m \leq n$ be minimal such that (after reindexation) $N_F(\sigma) = F^2(a_1, \cdots, a_m)$. It is clear that $2^m = \mathrm{ndeg}(\sigma)$, which shows (i). To prove (ii), we note that the anisotropy of σ implies that the $\{1, a_1, \cdots, a_n\} \subset N_F(\sigma)$ are F^2-linearly independent and can therefore be extended to an F^2-basis of the extension $N_F(\sigma)/F^2$, which immediately yields that $\sigma \subset \langle\langle a_1, \cdots, a_m\rangle\rangle$. Hence, $m + 1 \leq \dim \sigma \leq 2^m$. Part (iii) follows readily from Proposition 8.5(iv). □

Proposition 8.7. *Let σ be anisotropic totally singular, $\mathrm{ndeg}_F(\sigma) = 2^m$ and let $a_1, \cdots, a_m \in F^*$ be such that $N_F(\sigma) = F^2(a_1, \cdots, a_m)$. Let K/F be any field*

extension such that σ_K is isotropic. Then $\operatorname{ndeg}_K(\sigma_K) = 2^\ell < 2^m$ *and there exist* $\{a_{i_1}, \cdots, a_{i_\ell}\} \subset \{a_1, \cdots, a_m\}$ *such that* $N_K(\sigma_K) = K^2(a_{i_1}, \cdots, a_{i_\ell})$. *In particular, for each* $a \in D_K^*(\sigma)$, *we have* $a(\sigma_K)_{\mathrm{an}} \subset \langle\langle a_{i_1}, \cdots, a_{i_\ell}\rangle\rangle$

Proof. After scaling, we may assume that $1 \in D_F(\sigma)$. By Proposition 8.6(iii), $\sigma \subset \langle\langle a_1, \cdots, a_m\rangle\rangle$. Now σ_K is isotropic, hence also $\langle\langle a_1, \cdots, a_m\rangle\rangle_K$. By Proposition 8.5(iv), this implies that $[K^2(a_1, \cdots, a_m) : K^2] = 2^\ell < 2^m$. Clearly, we may find $\{a_{i_1}, \cdots, a_{i_\ell}\} \subset \{a_1, \cdots, a_m\}$ with $K^2(a_1, \cdots, a_m) = K^2(a_{i_1}, \cdots, a_{i_\ell})$. Now $N_K(\sigma_K) = K^2(a_1, \cdots, a_m)$. All this, together with Proposition 8.6(iii) implies the desired result. □

Definition 8.8. A totally singular form σ is called a *quasi-Pfister neighbor* if there exists $a \in F^*$ and an anisotropic m-fold quasi-Pfister form π ($m \geq 0$) such that $a\sigma \subset \pi$ and $2 \dim \sigma > \dim \pi = 2^m$. In this situation, σ is said to be a quasi-Pfister neighbor of π.

Proposition 8.9.
(i) *If σ is a quasi-Pfister neighbor of the anisotropic m-fold quasi-Pfister form π, then $N_F(\sigma) = N_F(\pi)$. In particular, if σ is also a quasi-Pfister neighbor of another anisotropic m-fold quasi-Pfister form τ, then $\pi \cong \tau$.*
(ii) *A totally singular form σ is a quasi-Pfister neighbor if and only if σ is anisotropic and $2 \dim \sigma > \operatorname{ndeg}_F(\sigma)$.*
(iii) *Let σ be a quasi-Pfister neighbor of the anisotropic m-fold quasi-Pfister form π, $m \geq 1$. Let E/F be a field extension. Then σ_E is isotropic iff π_E is isotropic.*

Proof. (i) Clearly, $N_F(\sigma) \subset N_F(\pi)$. Hence $\operatorname{ndeg}_F(\sigma) \leq \operatorname{ndeg}_F \pi = 2^m$. Since σ is anisotropic of dimension $> 2^{m-1}$, it follows from Proposition 8.6 that $\operatorname{ndeg}_F(\sigma) = 2^m$ and thus $N_F(\sigma) = N_F(\pi)$. Now if σ is also a quasi-Pfister neighbor of τ, then $N_F(\pi) = N_F(\tau)$ and $\pi \cong \tau$ by Proposition 8.5.

(ii) If σ is a quasi-Pfister neighbor, say, of the anisotropic m-fold quasi-Pfister form π, then by (i) we have $2 \dim \sigma > \dim \pi = 2^m = \operatorname{ndeg}_F(\pi) = \operatorname{ndeg}_F(\sigma)$.

Conversely, let σ be anisotropic and $\operatorname{ndeg}_F(\sigma) = 2^m$. Then there exist $a_1, \cdots, a_m \in F^*$ with $N_F(\sigma) = F^2(a_1, \cdots, a_m)$ and $\pi = \langle\langle a_1, \cdots, a_m\rangle\rangle$ is anisotropic (Proposition 8.5) and for each $a \in D_F^*(\sigma)$, we have $a\sigma \subset \pi$ (Proposition 8.6). Now $2 \dim \sigma > \operatorname{ndeg}_F(\sigma) = 2^m = \dim \pi$ implies that σ is a Pfister neighbor of π.

(iii) Clearly, the isotropy of σ_E implies that of π_E. Now the definition of a quasi-Pfister neighbor and its associated quasi-Pfister form imply by Proposition 8.6 that $N_E(\sigma_E) = N_E(\pi_E)$. Hence $\operatorname{ndeg}_E(\sigma_E) = \operatorname{ndeg}_E(\pi_E)$. Now π_E being isotropic implies that $\operatorname{ndeg}_E(\pi_E) \leq 2^{m-1}$, hence $\dim(\sigma_E)_{\mathrm{an}} \leq 2^{m-1} < \dim \sigma$. □

Recall that in characteristic $\neq 2$, the splitting pattern of a quadratic form is the increasing sequence of Witt indices of this form over the fields in a generic splitting tower (see, e.g., [19], [20], [17]). In characteristic 2, it it more reasonable to define a standard splitting pattern of a (possibly singular) form to be the sequence of pairs (r, s) denoting the types of the anisotropic parts of this form over the fields in the standard splitting tower (see, e.g. [28]). For totally singular forms, this just means the sequence of the dimensions of the anisotropic parts over the fields in the standard splitting tower (cf. subsection 4.2). Hence, if $F = F_0 \subset F_1 \subset \cdots \subset F_{h-1} \subset F_h$ denotes the standard splitting tower of the nonzero totally singular form

σ of height h, then the standard splitting pattern is given by $(s_0, \cdots, s_{h-1}, s_h)$ with $s_i = \dim(\sigma_{F_i})_{\mathrm{an}}$.

The following result is a more precise version of Proposition 8.7 in the case where $K = F(\sigma)$.

Lemma 8.10. *Let $\sigma \cong \langle 1\rangle \perp \sigma'$ be an anisotropic totally singular form of dimension ≥ 2 and let $K = F(\sigma)$. Suppose that $\mathrm{ndeg}_F(\sigma) = 2^m$ and let $b_1, \cdots, b_m \in F^*$ be such that $N_F(\sigma) = F^2(b_1, \cdots, b_m)$. Then $\mathrm{ndeg}_K(\sigma_K) = 2^{m-1}$, $N_K(\sigma_K) = K^2(b_2, \cdots, b_m)$. In particular, $(\sigma_K)_{\mathrm{an}} \subset \langle\!\langle b_2, \cdots, b_m\rangle\!\rangle_K$.*

Proof. Since $1 \in D_F(\sigma)$, we have $N_F(\sigma) = F^2(a \,|\, a \in D_F(\sigma'))$. Hence, there exists $c_1 \in D_F(\sigma')$ with $c_1 \notin F^2(b_2, \cdots, b_m)$. Thus, $F^2(b_1, \cdots, b_m) = F^2(c_1, b_2, \cdots, b_m)$. Let $n = \dim \sigma'$ and let $c_2, \cdots c_n \in F^*$ such that $\sigma' \cong \langle c_1, \cdots, c_n\rangle$. For $i \geq 2$, write $c_i = u_i + c_1 v_i$ with $u_i, v_i \in F^2(b_2, \cdots, b_m)$.

Let $X_1, \cdots, X_n$ be variables and put $L = F(X_1, \cdots, X_n)$, $d = \sum_{i=1}^n c_i X_i^2$. Then $K = L(\sqrt{d})$ and $K^2 = L^2(d)$. If we put $r = X_1^2 + \sum_{i=2}^n v_i X_i^2$ and $s = \sum_{i=2}^m u_i X_i^2$, then clearly $r, s \in L^2(b_2, \cdots, b_m)$, $r \neq 0$ and $d = s + c_1 r$. Hence, $L^2(b_1, \cdots, b_m) = L^2(c_1, b_2, \cdots, b_m) = L^2(d, b_2, \cdots, b_m) = K^2(b_2, \cdots, b_m)$.

By Corollary 8.4, $N_K(\sigma_K) = K^2(b_1, \cdots b_m) = L^2(d, b_1, \cdots, b_m)$, and thus, by the above, $N_K(\sigma_K) = K^2(b_2, \cdots, b_m)$.

Since L^2/F^2 is purely transcendental, we clearly have $2^m = [F^2(b_1, \cdots, b_m) : F^2] = [L^2(b_1, \cdots, b_m) : L^2] = [K^2(b_2, \cdots, b_m) : L^2]$. Now $L^2(d) = K^2$ and $[K^2 : L^2] = 2$. Thus, $\mathrm{ndeg}_K(\sigma_K) = [N_K(\sigma_K) : K^2] = [K^2(b_2, \cdots, b_m) : K^2] = 2^{m-1}$.

Finally, $(\sigma_K)_{\mathrm{an}} \subset \langle\!\langle b_2, \cdots, b_m\rangle\!\rangle_K$ follows from Proposition 8.6 and the fact that $1 \in D_K(\sigma_K)$. □

It is now easy, to compute the standard splitting pattern for quasi-Pfister neighbors.

Theorem 8.11. *Let σ be an anisotropic totally singular form. Let $F = F_0 \subset F_1 \subset \cdots \subset F_h$ be the standard splitting tower of σ and let $\sigma_i = (\sigma_{F_i})_{\mathrm{an}}$, $s_i = \dim \sigma_i$. Let n_i and m be such that $2^{n_i} < s_i \leq 2^{n_i+1}$ and $\mathrm{ndeg}_F(\sigma) = 2^m$. Let $b_1, \cdots b_m \in F^*$ be such that $N_F(\sigma) = F^2(b_1, \cdots b_m)$.*

(i) *For $1 \leq i \leq h$, we have $\mathrm{ndeg}_{F_i}(\sigma_i) = 2^{m-i}$, $N_{F_i}(\sigma_i) = F_i^2(b_1, \cdots, b_{m-i})$, and $\max\{m - i + 1, 2^{n_{i-1}}\} \leq \dim \sigma_i \leq 2^{m-i}$. In particular, $h(\sigma) = m$. Furthermore, for every $a \in D_F(\sigma)$, we have $a\sigma_i \subset \langle\!\langle b_1, \cdots, b_{m-i}\rangle\!\rangle_{F_i}$.*

(ii) *σ is a Pfister neighbor iff $(s_0, s_1, \cdots, s_h) = (\dim \sigma, 2^{m-1}, 2^{m-2}, \cdots, 1)$. In this situation, $a\sigma_i \cong \langle\!\langle b_1, \cdots, b_{m-i}\rangle\!\rangle_{F_i}$ for every $a \in D_F(\sigma)$.*

Proof. (i) $\mathrm{ndeg}_{F_i}(\sigma_i) = 2^{m-i}$, $N_{F_i}(\sigma_i) = F_i^2(b_1, \cdots, b_{m-i})$, $m - i + 1 \leq \dim \sigma_i \leq 2^{m-i}$ and $a\sigma_i \subset \langle\!\langle b_1, \cdots, b_{m-i}\rangle\!\rangle_{F_i}$ follow by induction from Proposition 8.6 and Lemma 8.10.

Now $2^{n_{i-1}} < \dim \sigma_{i-1} \leq 2^{n_{i-1}+1}$. Theorem 4.4 yields that any subform of σ_{i-1} of dimension $2^{n_{i-1}}$ stays anisotropic over F_i and will therefore be also a subform of $((\sigma_{i-1})_{F_i})_{\mathrm{an}} = \sigma_i$. Hence $\dim \sigma_i \geq 2^{n_{i-1}}$.

$h(\sigma) = m$ now follows readily by induction on m and the definition of height.

(ii) If σ is a Pfister neighbor, then $2^{m-1} < \dim \sigma \leq 2^m = \mathrm{ndeg}_F(\sigma)$ by Proposition 8.9, and it follows immediately from (i) that σ has standard splitting pattern $(\dim \sigma, 2^{m-1}, 2^{m-2}, \cdots, 1)$.

Conversely, if the standard splitting pattern is of that type, then $h(\sigma) = m$, hence $\mathrm{ndeg}_F(\sigma) = 2^m$ and $\dim \sigma > 2^{m-1}$. Hence, σ is a quasi-Pfister neighbor by Proposition 8.9.

The remaining statement follows immediately from (i) by comparing dimensions. □

We now give characterizations of quasi-Pfister neighbors of small dimension.

Proposition 8.12. *Let σ be an anisotropic totally singular form.*

(i) *If $\dim \sigma \leq 3$, then σ is a quasi-Pfister neighbor.*
(ii) *If $\dim \sigma = 2^n$, then σ is a quasi-Pfister neighbor iff σ is similar to a quasi-Pfister form.*
(iii) *If $\dim \sigma = 5$, then σ is a quasi-Pfister neighbor iff there exist $a, b, c \in F^*$ such that σ is similar to $\langle 1, a, b, ab, c\rangle$.*
(iv) *If $\dim \sigma = 6$, then σ is a quasi-Pfister neighbor iff there exist $a, b, c \in F^*$ such that σ is similar to $\langle 1, a, b, ab, c, ac\rangle$.*
(v) *If $\dim \sigma = 7$, then σ is a quasi-Pfister neighbor iff there exist $a, b, c \in F^*$ such that σ is similar to $\langle a, b, c, ab, ac, bc, abc\rangle$.*

Proof. (i) and (ii) are easy and left to the reader. It is also obvious that anisotropic forms similar to forms of type $\langle 1, a, b, ab, c\rangle$, $\langle 1, a, b, ab, c, ac\rangle$, $\langle a, b, c, ab, ac, bc, abc\rangle$ are quasi-Pfister neighbors. For the converse, let σ be a quasi-Pfister neighbor with $5 \leq \dim \sigma \leq 7$

The case $\dim \sigma = 5$. After scaling, we may assume that $\sigma \cong \langle 1, p, q, r, s\rangle$, and by the definition of a quasi-Pfister neighbor and Proposition 8.6, we may furthermore assume that $\sigma \subset \langle\langle p, q, r\rangle\rangle$ with $[F^2(p, q, r) : F^2] = 8$. Then $0 \neq s = u + v$ with $v \in D_F(\langle 1, p, q, r\rangle) \cup \{0\}$ and $u = pqx^2 + pry^2 + qrz^2 + pqrt^2$ for suitable $x, y, z, t \in F$. If $u = 0$ then $s \in D_F(\langle 1, p, q, r\rangle)$ and σ is isotropic, a contradiction. Hence, $u \neq 0$ and we have $\langle 1, p, q, r, s\rangle \cong \langle 1, p, q, r, u\rangle$.

Suppose first that $t \neq 0$. Then, after multiplying u by a square, we may assume $t = 1$. Now $\langle 1, p, q, r\rangle \cong \langle 1, z^2 + p, y^2 + q, x^2 + r\rangle$ and $(z^2+p)(y^2+q)(x^2+r) = u+w$ with $w \in D_F(\langle 1, p, q, r\rangle)$. Put $f = z^2 + p$, $g = y^2 + q$, $h = x^2 + r$. Thus, we get $\sigma \cong \langle 1, p, q, r, u\rangle \cong \langle 1, p, q, r, u + w\rangle \cong \langle 1, f, g, h, fgh\rangle$. By putting $a = fg$, $b = fh$, $c = fgh$, we get $c\sigma \cong \langle 1, a, b, ab, c\rangle$.

Suppose now that $t = 0$. Since $u \neq 0$, it suffices by symmetry to consider only the case $x \neq 0$. After scaling by a square, we may assume $x = 1$. If $y = z = 0$ then $\sigma \cong \langle 1, p, q, r, u\rangle \cong \langle 1, p, q, pq, r\rangle$ and we are done. Otherwise, again by symmetry, it suffices to consider the case $y \neq 0$. This time, we put $a = p + q(z/y)^2$, $b = q + ry^2$. Then $ab = u + w$ with $w \in F^2$ and we get $\sigma \cong \langle 1, p, q, r, u\rangle \cong \langle 1, a, b, r, u\rangle \cong \langle 1, a, b, r, u + w\rangle \cong \langle 1, a, b, ab, r\rangle$.

The case $\dim \sigma = 6$. By the 5-dimensional case, we may assume after scaling that $\sigma \cong \langle 1, p, q, pq, r, s\rangle \subset \langle\langle p, q, r\rangle\rangle$. Then we can write $s = u + rv$ with $u, v \in D_F(\langle\langle p, q\rangle\rangle) \cup \{0\}$, and the anisotropy of σ implies $v \neq 0$. Hence, $\sigma \cong \langle 1, p, q, pq, r, rv\rangle$. Also, we can write $v = x^2 + w$ with $w \in D_F(\langle p, q, pq\rangle)$. Since $\langle r, rx^2\rangle$ is isotropic, we must have $w \neq 0$, and thus $w \in F^2(p, q) \setminus F^2$. But then there exists $t \in \{p, q\}$ with $F^2(p, q) = F^2(w, t)$ and we have $\langle\langle p, q\rangle\rangle \cong \langle\langle w, t\rangle\rangle$. Also, $\langle r, rv\rangle \cong \langle r, rw\rangle$ and thus $\sigma \cong \langle 1, p, q, pq, r, rw\rangle \cong \langle 1, w, t, tw, r, rw\rangle$.

The case $\dim \sigma = 7$. By the 6-dimensional case, we may assume after scaling that $\sigma \cong \langle 1, p, q, pq, r, pr, s\rangle \subset \langle\langle p, q, r\rangle\rangle$. We can write $s = u + v$ with $u \in D_F(\langle qr, pqr\rangle)$

and $v \in D_F(\langle 1, p, q, pq, r, pr\rangle)$. The anisotropy implies $u \neq 0$ and we get $\sigma \cong \langle 1, p, q, pq, r, pr, u\rangle$. Now $qru \in D_F(\langle 1, p\rangle)$ and thus, by Proposition 8.5 (or Proposition 8.1), $up\langle 1, p\rangle \cong u\langle 1, p\rangle \cong qr\langle 1, p\rangle$. It follows that $up\sigma \cong \langle p, q, r, pq, pr, qr, pqr\rangle$. □

It should be noted that the description of quasi-Pfister neighbors in dimensions $5, 6, 7$ bears a striking resemblance to the well known characterization of Pfister neighbors of the same dimensions in characteristic $\neq 2$ (see, e.g. [24, p.10]). However, our methods to obtain the characterization are naturally very different from those usually employed in characteristic $\neq 2$.

Let us now give a necessary condition for the isotropy of an anisotropic totally singular form over the function field of another totally singular form, which then yields criteria for the isotropy of anisotropic quasi-Pfister forms over function fields of totally singular forms. Recall that anisotropic totally singular forms stay anisotropic over function fields of forms which are not totally singular, so there is no need to consider this case.

Proposition 8.13. *Let φ, ψ be nonzero totally singular forms with φ anisotropic. Let $K = F(\psi)$. If φ_K is isotropic, then $N_F(\psi) \subset N_F(\varphi)$.*

Proof. Without loss of generality, we may assume that ψ is anisotropic, $\dim \psi \geq 2$, and, after scaling, $\psi \cong \langle 1, b_1, \cdots, b_n\rangle$.

Let $\operatorname{ndeg}_F(\varphi) = 2^m$ and let $a_1, \cdots a_m \in F^*$ such that $N_F(\varphi) = F^2(a_1, \cdots a_m)$. Let $\pi = \langle\langle a_1, \cdots a_m\rangle\rangle$. Then π is anisotropic by Proposition 8.5, and by Proposition 8.6, φ is similar to a subform of π. Hence, φ_K isotropic implies that π_K is isotropic.

Let $i \in \{1, \cdots, n\}$ and put $L' = F(\langle 1, b_i\rangle) = F(X)(\sqrt{b_i X^2})$ (where X denotes a variable), $L = F(\sqrt{b_i})$. By a standard specialization argument ([27, Lemma 4.5]), the isotropy of π_K implies that of $\pi_{L'}$, and therefore also that of π_L. Hence, necessarily $\operatorname{ndeg}_L(\pi_L) < \operatorname{ndeg}_F(\pi) = 2^m$. But $N_F(\pi) = N_F(\varphi) = F^2(a_1, \cdots a_m)$, and $N_L(\pi_L) = F^2(a_1, \cdots a_m, b_i)$. If $b_i \notin F^2(a_1, \cdots a_m)$, then $[F^2(a_1, \cdots a_m, b_i) : F^2] = 2^{m+1}$ and thus $[F^2(a_1, \cdots a_m, b_i) : F^2(b_i)] = [L^2(a_1, \cdots a_m) : L^2] = \operatorname{ndeg}_L(\pi_L) = 2^m$, a contradiction. Thus, $b_i \in F^2(a_1, \cdots a_m)$ for every i, hence we obtain $N_F(\psi) = F^2(b_1, \cdots, b_n) \subset N_F(\varphi)$. □

Corollary 8.14. *Let π be an anisotropic totally singular m-fold Pfister form, $m \geq 1$, and let σ be a totally singular form. Let $K = F(\sigma)$. The following are equivalent :*

(i) *π_K is isotropic.*
(ii) *$\dim \sigma_{\mathrm{an}} \geq 2$ and for any $a \in D_F^*(\sigma)$ one has $a\sigma_{\mathrm{an}} \subset \pi$.*
(iii) *$\dim \sigma_{\mathrm{an}} \geq 2$ and $N_F(\sigma) \subset N_F(\pi)$.*

Proof. Since $F(\sigma)/F(\sigma_{\mathrm{an}})$ is purely transcendental, we have that π_K is isotropic iff $\pi_{F(\sigma_{\mathrm{an}})}$ is isotropic. Also, $N_F(\sigma) = N_F(\sigma_{\mathrm{an}})$, so we may assume without loss of generality that σ is anisotropic.

By the previous proposition, (i) implies (iii). (iii) implies (ii) by Propositions 8.5 and 8.6, and trivially (ii) implies (i). □

We conclude this section with some remarks on splitting towers and heights of forms (singular or not). One of the important properties of generic splitting towers in characteristic $\neq 2$ is the fact that if φ is an anisotropic form over the field E, and if K is any field extension of E, then there exists a field E_i in the generic splitting tower $E = E_0 \subset E_1 \subset \cdots \subset E_h$ such that $\dim(\varphi_K)_{\mathrm{an}} = \dim(\varphi_{E_i})_{\mathrm{an}}$.

This property no longer holds in general for standard splitting towers of totally singular forms as the following example shows.

Example 8.15. Consider a totally singular anisotropic form $q = \langle\langle a_1, \cdots, a_n\rangle\rangle \perp \langle b\rangle$, $n \geq 2$. Then q is a quasi-Pfister neighbor of $\langle\langle a_1, \cdots, a_n, b\rangle\rangle$, and thus the standard splitting pattern will be $(2^n+1, 2^n, 2^{n-1}, \cdots, 2, 1)$ by the preceding theorem. Now for $K_m = F(\sqrt{a_1}, \cdots, \sqrt{a_m})$, $1 \leq m \leq n-1$, it is not difficult to show that $(q_{K_m})_{\mathrm{an}} \cong \langle\langle a_{m+1}, \cdots, a_n\rangle\rangle \perp \langle b\rangle$, and it follows that $2^{n-m}+1 = \dim(q_{K_m})_{\mathrm{an}}$ does not show up in the standard splitting pattern.

However, we do know by Proposition 4.6 that the dimension of the nonsingular part of the anisotropic part of a form over a field extension will appear as dimension of the nonsingular part of one of its higher kernel forms in the standard splitting tower. With our results on totally singular forms over field extensions, we are now able to prove Theorem 4.5 which shows that the maximal height of a tower of fields over which the anisotropic parts of a form are of strictly decreasing dimension is given by the standard height. We restate the theorem for the readers convenience.

Theorem 8.16. *Let q be an anisotropic form of standard height $h(q) = h$. Let $F = K_0 \subset K_1 \subset \cdots \subset K_m$ be any tower of fields such that $\dim(q_{K_{i-1}})_{\mathrm{an}} > \dim(q_{K_i})_{\mathrm{an}}$ for each $i \in \{1, \cdots, m\}$. Then $m \leq h$.*

Proof. Let $q_{(i)} = (q_{K_i})_{\mathrm{an}}$ and denote by (r_i, s_i) its type. By assumption, $\dim q_{(i)} < \dim q_{(i-1)}$ and thus $r_i < r_{i-1}$ or $s_i < s_{i-1}$ for $1 \leq i \leq m$. Let $u = \mathrm{card}\{i \,|\, 1 \leq i \leq m,\ r_i < r_{i-1}\}$ and $v = \mathrm{card}\{i \,|\, 1 \leq i \leq m,\ s_i < s_{i-1}\}$. Then $u + v \geq m$. On the other hand, by Proposition 4.6, we have $u \leq h_{\mathrm{nd}}(q)$.

Let $\sigma_i = \mathrm{ql}(q_{(i)})$ and $\sigma = \sigma_0 = \mathrm{ql}(q)$. Now $\sigma_i = (\sigma_{K_i})_{\mathrm{an}}$ and $\dim \sigma_i = s_i$. If $\mathrm{ndeg}_{K_i}(\sigma_i) = 2^{m_i}$ and $s_i < s_{i-1}$, then by Proposition 8.7, $m_i < m_{i-1}$ because the anisotropic form σ_{i-1} becomes isotropic over K_i. Consequently, $v \leq m_0$. But $h(\sigma) = m_0$ by Theorem 8.11(i), and therefore $v \leq h(\sigma)$. We have $h(q) = h_{\mathrm{nd}}(q) + h(\sigma)$ (cf. [28, Th. 4.6]), hence $h(q) \geq u + v \geq m$. □

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