Beilinson's formula

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Abstract

The goal of this thesis is to present, following the papers by F. Brunault [Bru07] and the recent presentation of Bertolini-Darmon [BD], an explicit version of Beilinson's formula that relates the product of special values of the L-function of an elliptic curve (at $s = 0$ and $s = 2$) and the complex regulator of an anti-holomorphic differential for certain modular units.

Contents

1 | Introduction

The main goal of this thesis is to present an explicit version of Beilinson's formula that relates the product of special values of the L-function of an elliptic curve (at $s = 0$ and $s = 2$) and the complex regulator of an anti-holomorphic differential for certain modular units. This result sits in the wide framework of Beilinson's conjecture, of which we will only consider a very special and explicit case. The general theme in which these results and conjecture sit is the problem of describing special values of L-series of motives by means of some of its relevant arithmetic invariants, generally referred to as regulators.

As already remarked, in this thesis we will be only concerned with an explicit version of Beilinson's theorem, following the basic references of F. Brunault [Bru07] and the presentation (and generalization) of this result offered recently by M. Bertolini and H. Darmon in [BD]. Therefore, we will not discuss generalities of motives, their L-functions and regulators. On the contrary, we will present explicit (non holomorphic) Eisenstein series and define regulators in terms of an explicit integral involving logarithmic derivatives of modular units attached to these Eisenstein series.

In 1984 a Russian mathematician Alexander Beilinson proposed a vast generalization of Birch and Swinnerton-Dyer like conjectures. As a result supporting his conjectures, he proved that there is relation between complex regulator of modular units and value of L-function. More precisely, the formula we are interested in is (cf. [BD, Prop. 2.3]):

$$L^*(f, \chi_1, 2) \cdot L^*(f, \chi_2, 1) = C_{f,\chi_1,\chi_2} \cdot \mathrm{reg}_{\mathbb{C}}\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ah}})$$

where the notation is as follows:

- f is the weight 2 modular form of level $\Gamma_0(N)$ attached by modularity to an elliptic curve E, of conductor N which for simplicity we assume torsion free, defined over the field of rational numbers.

- χ_1 and χ_2 are two Dirichlet characters of conductors N_1 and N_2, respectively, with $N = N_1 N_2$;

- $\chi = \bar{\chi}_1 \bar{\chi}_2$ is primitive, and we suppose $\chi(-1) = 1$;

- $\eta_f^{\mathrm{a}h}$ is the anti-holomorphic differential attached to f;

- C_{f,χ_1,χ_2} is a non-zero explicit algebraic number.

Here it should be remarked that the constant C_{f,χ_1,χ_2} arise from the computation, it can be given explicitly, for instance for $s = \frac{k}{2} + l - 1$ we have $C_{f,\chi_1,\chi_2} = \frac{i2^{k-1}}{N^{l-1}}$. We will now describe more clearly the objects introduced above, and present the main steps in the proof of this result.

Congruence groups and modular forms. For a positive integer N, we use the usual symbols $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ to denote congruence subgroups of $SL_2(\mathbb{Z})$ (see §5.2). Let Γ be one of the above groups. A modular form of weight k with respect to Γ is, roughly speaking, a holomorphic function defined on an upper complex half-plane: $f : H \to \mathbb{C},\ H := \{z \in \mathbb{C} :\ \mathrm{Im}(z) > 0\}$ such that

- $f = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right),\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$

- $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ is holomorphic at infinity for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

Modular forms of weight k with respect to Γ form a vector space over $\mathbb{C}$. We will denote this space by $\mathcal{M}_k(\Gamma)$. We will use the notation $\mathcal{M}_k(N,\chi)$ for the subspace consisting of $f \in \mathcal{M}_k(\Gamma_1(N))$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \chi(d) f,\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

However, for our purposes we have to introduce another space $\mathcal{M}_k^{\mathrm{an}}(N,\chi)$ that is closely related to $\mathcal{M}_k(N,\chi)$; this is defined as the vector space of real analytic functions with the same transformation properties under $\Gamma_0(N)$ and having bounded growth at the cusps of $X_0(N)$, the compact modular curve of level $\Gamma_0(N)$ (this is defined as the compactification of the open Riemann surface

$\Gamma_0(N)\backslash H$, and the action of $SL_2(\mathbb{R})$ on H is via fractional linear transformations; see also below).

An important subspace of $\mathcal{M}_k(\Gamma)$ is the space of modular forms vanishing at infinity, called cusp forms. We will denote it by $\mathcal{S}_k(\Gamma)$ and $\mathcal{S}_k(N,\chi)$. We will let $\mathcal{S}_k^{\mathrm{an}}(N,\chi)$ be the subspace of $\mathcal{M}_k^{\mathrm{an}}(N,\chi)$ consisting of forms which have rapid decay at cusps.

Eisentstein series - an important example of modular forms. One of the most significant example of modular forms are Eisenstein series. For $\Gamma = SL_2(\mathbb{Z})$ and weight $k \geq 3,\ k$ integer, Eisenstein series is defined:

$$G_k(z) = \sum_{(c,d)}{}' \frac{1}{(cz+d)^k},\ z \in H.$$

By the prime sign it is meant that the summation is over $\mathbb{Z}^2 \setminus \{(0,0)\}$. It can be showed that such a series has q-expansion:

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where

$$\sigma_{k-1}(n) = \sum_{\substack{m>0 \\ m|n}} m^{k-1},\ q = e^{2\pi i z}.$$

Note that we assumed $k \geq 3$. With this assumption Eisenstein series defined as above has uniform and absolute convergence and is a modular form. In the thesis we shall also briefly describe the case $k=2$. When $k=2$ convergence of Eisenstein series is only conditional and it fails to be a modular form. In general, one can introduce much more complicated Eisenstein series. For instance they can be defined on more variables or be attached to many Dirichlet characters. For our purposes we will need in particular non-holomorphic Eisenstein series and Eisenstein series attached to a pair of Dirichlet characters. Non holomorphic Eisenstein series are defined to be:

$$\tilde{E}_{k,\chi}(z,s) = \sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi^{-1}(n)}{(mz+n)^k}\cdot\frac{y^s}{|mz+n|^{2s}},$$

where $z \in H,\ s \in \mathbb{C},\ \chi$ a primitive Dirichlet character mod N. As a function of s it is convergent for $\mathrm{Re}(s) > 1 - \frac{k}{2}$ and admits a meromorphic continuation to all $s \in \mathbb{C}$. As a function of z it transforms like a modular form on $\Gamma_0(N)$. Choosing

$s = 0$ we get, $\tilde{E}_{k,\chi} := \tilde{E}_{k,\chi}(z, 0)$. For $k > 2$, we also have $\tilde{E}_{k,\chi} \in \mathcal{M}_k(N, \chi)$.

Now we shall define an Eisenstein series attached to a pair of Dirichlet characters χ_1 and χ_2. We will not provide an explicit definition of this series but present its q-expansion. First define:

$$\delta_{\chi_1} = \begin{cases} \frac{1}{2} \text{ if } N_1 = 1 \\ 0 \text{ otherwise} \end{cases}$$

and

$$\sigma_{k-1}(\chi_1, \chi_2)(n) = \sum_{d|n} \chi_1(n/d)\chi_2(d)d^{k-1}.$$

For $k \geq 1$ and $(\chi_1, \chi_2) \neq (1, 1)$, the q-expansion of normalized Eisentstein series attached to Dirichlet characters $\chi_1,\ \chi_2$ is:

$$E_k(\chi_1, \chi_2)(z) = \delta_{\chi_1} L(1 - k, \chi_1^{-1}\chi_2) + \sum_{n=1}^{\infty} \sigma_{k-1}(\chi_1, \chi_2)(n)q^n.$$

Here $L(s, \psi)$ is the Dirichlet series attached to the character ψ, described in Section. 4. In the thesis we shall only consider case $k = 2$.

Petersson scalar product. Let Γ be any congruence subgroup of $SL_2(\mathbb{Z})$ as above. The corresponding modular curve is defined as the quotient space $\Gamma\backslash H$:

$$Y(\Gamma) = \{\Gamma z : z \in H\}.$$

Here, for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ and $z \in H$, we set

$$\gamma(z) = \frac{az + b}{cz + d}.$$

This is equipped with a structure of open Riemann surface. If $\Gamma = \Gamma_1(N)$ we denote $Y(\Gamma_1(N)) = Y_1(N)$. Let $P_N = \Gamma\backslash(\mathbb{Q} \cup \{\infty\})$. Define

$$X_1(N)(\mathbb{C}) = Y_1(N) \sqcup P_N.$$

This set can be equipped with a canonical structure of compact Riemann surface.

Petersson scalar product is defined on $\mathcal{S}_k^{\mathrm{an}}(N, \chi) \times \mathcal{M}_k^{\mathrm{an}}(N, \chi)$ in the following way:

$$\langle f_1, f_2 \rangle_{k,N} := \int_{X_1(N)(\mathbb{C})} y^k \bar{f}_1(z) f_2(z) \frac{dxdy}{y^2}.$$

The definition of $\mathcal{S}_k^{\mathrm{an}}(N,\chi)$ and $\mathcal{M}_k^{\mathrm{an}}(N,\chi)$ guarantees that this integral is well defined and convergent. It induces a scalar product on $\mathcal{S}_k^{\mathrm{an}}(N,\chi)$. Having introduced the notion of the Petersson scalar product we can define an anti-holomorphic differential attached to f as

$$\eta_f^{\mathrm{ah}} := \frac{\bar{f}(z)d\bar{z}}{\langle f,f\rangle_{2,N}}.$$

Modular units. With a slight abuse of notation, we still use the symbol $Y_1(N)$ for Shimura's model, defined over $\mathbb{Q}$, of the open Riemann surface $Y_1(N)$, and we adopt the same convention for $X_1(N)$. Let also $\bar{Y}_1(N)$ be the scalar extension of $Y_1(N)$ to $\bar{\mathbb{Q}}$. By $\mathbb{C}(\bar{Y}_1(N))$ we will mean the field of meromorphic functions on $\bar{Y}_1(N)$. Further, for arbitrary field F we will define the algebraic K_2-group as

$$K_2(F) = (F^* \otimes_{\mathbb{Z}} F^*)/\langle a\otimes(1-a)|a\in F\setminus\{0,1\}\rangle.$$

We will denote elements of $K_2(F)$ as $\{x,y\}$. A modular unit is a meromorphic function

$$u \in \mathbb{C}(X_1(N))^*$$

such that

$$\mathrm{Supp}(u) \subset P_N.$$

The group of modular units is denoted $\mathcal{O}^*(Y_1(\mathbb{C}))$. There is a strong relation joining modular units and Eisenstein series. We have a surjective homomorphism:

$$\mathcal{O}(\bar{Y}_1(N))^* \otimes F \xrightarrow{\mathrm{dlog}} \mathrm{Eis}_2(\Gamma_1(N),F),$$

where

$$\mathrm{dlog}(u) := \frac{1}{2\pi i}\frac{u'(z)}{u(z)},$$

F is an arbitrary field and

$$\mathrm{Eis}_l(\Gamma_1(N),F) \subset \mathcal{M}_l(\Gamma_1(N),F)$$

is a subspace of $\mathcal{M}_l(\Gamma_1(N),F)$ spanned by weight l Eisenstein series with Fourier coefficients in F. The following proposition, [Bru07, Prop. 5.3], gives us an explicit construction of the above elements and represents the first key step in the proof of Beilinson's formula. Before stating this result, we introduce a couple of more notation: for a modular unit u, write

$$u(z) = \sum_{n=n_0}^{\infty} a_n q^n$$

for its Fourier expansion, and set

$$\hat{u}(\infty) := a_{n_0};$$

define

$$E_{u,v}(z,s) = {\sum}'_{m\equiv_N u,\, n\equiv_N v} \frac{\operatorname{Im}(z)^s}{|mz+n|^{2s}}$$

where the sum is over all non-zero pairs of integers m and n congruent to u and v, respectively, mod N, and set

$$E_l^* = \sum_{v\in\frac{\mathbb{Z}}{N\mathbb{Z}}} l(v) E_{0,v}^*,$$

$$E_{u,v}^*(z) = \lim_{s\to 1}\left(E_{u,v}(z,s) - \frac{\pi}{N^2(s-1)}\right),$$

where $l : \frac{\mathbb{Z}}{N\mathbb{Z}} \to \mathbb{C}$ is a function of sum zero.

Theorem 1. *For a function of sum zero: $l : \frac{\mathbb{Z}}{N\mathbb{Z}} \to \mathbb{C}$ there exists a unique modular unit*

$$u_l \in \mathcal{O}^*(Y_1(N)(\mathbb{C})) \otimes \mathbb{C}$$

satisfying

$$log|u_l| = \frac{1}{\pi}\cdot E_l^* \text{ and } \hat{u}_l(\infty) = 1 \in \mathbb{C}^* \otimes \mathbb{C}.$$

For our purposes we shall take modular units

$$\{u_\chi, u(\chi_1,\chi_2)\} \in K_2(\mathbb{C}(\bar{Y}_1(N)))$$

satisfying:

$$\operatorname{dlog}(u_\chi) = E_{2,\chi},\ \operatorname{dlog}(u(\chi_1,\chi_2)) = E_2(\chi_1,\chi_2).$$

Complex regulator. Let $u, v \in F^*$ be rational functions (here, as above, $F = \mathbb{C}(\bar{Y}_1(N))$). Let

$$\eta(u,v) = \log|u| \cdot d \arg v - \log|v| \cdot d \arg u.$$

Then $\mathrm{reg}_{\mathbb{C}}$ is defined as the map

$$\mathrm{reg}_{\mathbb{C}} : K_2(\mathbb{C}(X_1(N))) \to \ \mathrm{Hom}_{\mathbb{Q}}(\Omega^1(X_1(N)), \mathbb{R})$$

which takes

$$\{u, v\} \to \left(\omega \mapsto \int_{X_1(N)(\mathbb{C})} \eta(u, v) \wedge \omega \right).$$

A computation, applying Stokes Theorem, yields:

$$\mathrm{reg}_{\mathbb{C}}\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ah}}) = \frac{\int_{X_1(N)(\mathbb{C})} \bar{f} \cdot \log|u_\chi| \cdot \mathrm{dlog}(u(\chi_1, \chi_2)(z))dxdy}{\langle f, f\rangle_{2,N}}.$$

Rankin method. The second key ingredient to prove Beilinson's formula is result of Shimura, an application of Rankin method. For a cusp form f in $\mathcal{S}_k(N, \chi_f)$, with Fourier expansion $\sum_{n\geq 1} a_n q^n$, let $f^* := \sum_{n\geq 1} \bar{a}_n q^n$ be the modular form in $\mathcal{S}_k(N, \bar{\chi}_f)$ obtained by applying complex conjugation to the Fourier coefficients; here, $q = \exp(2\pi i s)$.

Theorem 2. *For a weight k cusp form f we have*

$$\langle f^*(z), \tilde{E}_{k-l,\chi}(z, s) \cdot g(z)\rangle_{k,N} = 2\frac{\Gamma(s + k - 1)}{(4\pi)^{s+k-1}} L(\chi^{-1}, 2s+k-l)D(f, g, s+k-1).$$

Here, as above,

$$L(\psi, s) := \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s},$$

is the Dirichlet L-function attached to a Dirichlet character ψ, and $D(f, g, s)$ is Rankin-Selberg L-function attached to the pair of modular forms f and g; more precisely, if f and g have Fourier expansions $\sum_{n\geq 1} a_n(f)q^n$ and $\sum_{n\geq 0} a_n(g)q^n$, respectively,

$$D(f, g, s) := \sum_{n\geq 1} a_n(f)a_n(g)n^{-s}.$$

Beilinson's formula. The proof of Beilinson's formula comes then from a combination of the two above results after a direct computation, by considering separately the values $s = \frac{k}{2} + l - 1$ and $s = 2$.

Organization of the material. After introducing some basic preliminary in the first chapter, in the second chapter, we shall briefly discuss the Riemann Zeta function, which turns out to be one of the most simple and important example of Dirichlet L-functions. Dirichlet L-functions are infinite series related to Dirichlet characters. Both of them will be described in the separate chapter.

Further we will define modular forms. Set of modular forms turns out to be a vector space with operators, called Hecke operators, acting on them. It shall be shown that Hecke operators have properties which allow to treat modular forms with tools of linear algebra. In the successive sections we shall discuss most significant examples of modular forms - Eisenstein series. Not all Eisenstein series satisfy the definition of modular forms as some of them fail to be holomorphic, and therefore we will describe this case separately. We shall also pay a lot of attention to cusp forms, special case of modular forms.

Finally we will define Petersson scalar product. It will be given as an integral. As it turns out, it is essential to compute the complex regulator on class of anti-holomorphic differential. Complex regulator provides information about the density of modular units in algebraic number field. In the last chapter we shall use this value in Beilinson's formula.

2 | Preliminaries

For further reference, we collect in this chapter some definitions and basic notation.

Bernoulli numbers. Bernoulli numbers are used in the definition of normalized Eisenstein series. They are defined as coefficients of a formal power series expansion:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Character. A character is a group homomorphism between a finite abelian multiplicative group A and a multiplicative group of invertible complex numbers $\mathbb{C}^*$:

$$\chi : A \to \mathbb{C}^*.$$

We will say that a character is trivial if it maps all non-zero elements to 1. Interesting property of characters is that it maps elements of abelian group of order n to n-th roots of unity in $\mathbb{C}^*$. This is because $a^n = 1 \forall a \in A$ so by the properties of group homomorphism $(\chi(a))^n = 1$. It follows that $\chi\bar{\chi} = |\chi| = 1$, where $\bar{\chi}$ is a complex conjugate of χ. In other words, $\bar{\chi} = \chi^{-1}$.

Induced character. Let M, N be two integers such that $M|N$ and suppose that we have two characters:

$$\varphi : (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}^*, \ \chi : (\mathbb{Z}/M\mathbb{Z}) \to \mathbb{C}^*.$$

We will say that the character φ is induced by the character χ if it factors like this:

$$(\mathbb{Z}/N\mathbb{Z}) \twoheadrightarrow (\mathbb{Z}/M\mathbb{Z}) \to \mathbb{C}^*.$$

Primitive character. We say that a character χ modulo N is primitive if it is not induced by any character φ modulo M such that $M|N$.

Conductor of a character. Suppose that χ is a Dirichlet character modulo N. Then conductor of the character χ is the smallest divisor M of N such that χ is induced by a character modulo M.

Analytic function. Let $D \underset{\text{open}}{\subset} \mathbb{C},\ z_0 \in D,\ f : D \to \mathbb{C}$. We call f analytic, or equivalently holomorphic, in z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In case the following holds for all points in D we call a function analytic on D, and if f is holomorphic on $\mathbb{C}$ we call it entire.

Pole of a function. Let $U \underset{\text{open}}{\subset} \mathbb{C},\ z_0 \in U$ and $f : U \setminus \{z_0\} \to \mathbb{C}$ be analytic on its domain. Further let f have a Laurent series expansion (for suitable z):

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

We say f has a pole of order k, if k is a positive integer such that $c_{-k} \neq 0,\ c_{-n} = 0$ for $n > k$. If $k = 1$ pole is called simple.

Meromorphic function. We call a complex function meromorphic on an open set U if it is analytic on that set apart from discrete subset D. What is more, all points in D are poles of that function.

Function holomorphic at infinity. We call a function $f : H \to \mathbb{C}$ holomorphic at infinity if it has a Fourier expansion:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n \left(e^{2n\pi i z}\right)$$

Analytic continuation. Suppose f is analytic function on an open set $U \subset \mathbb{C}$ with values in $\mathbb{C}$. Suppose further $U \subset V,\ V \underset{\text{open}}{\subset} \mathbb{C}$ and g is another analytic function from V to $\mathbb{C}$. We call g analytic continuation of f if $g|_U = f$.

Logarithmic derivative. By logarithmic derivative of a function f we will mean $\frac{f'}{f}$, where f' is a classical derivative.

Operator. Let V, W be two vector spaces. An operator is any mapping from V to W.

Group action. We define an action of a group G on a non-empty set Ω to be a function

$$\begin{aligned} \cdot : \Omega \times G &\to \Omega \\ (a, g) &\mapsto a \cdot g \end{aligned}$$

such that it satisfies conditions:

1. $a \cdot 1 = a \ \forall a \in \Omega$
2. $(a \cdot g) \cdot h = a \cdot (gh) \ \forall a \in \Omega,\ g, h \in G$.

Orbit of a group action. For a group action of G on Ω we define an orbit to be:

$$\mathcal{O}_\alpha = \{\alpha \cdot g : g \in G\}.$$

Orbits are disjoint, i.e if $\beta \in \mathcal{O}_\alpha$ it means that $\mathcal{O}_\alpha = \mathcal{O}_\beta$.

Fundamental domain. If a group G is acting on a topological space S then a fundamental domain F is a closed subset of G consisting of exactly one representative of each orbit of the group action.

Weakly multiplicative function. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function. We say that f is weakly multiplicative if:

- $f(1) = 1$
- $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$.

3 | The Riemann Zeta function

In this chapter we want to define the Riemann Zeta function and talk about its properties. The function is defined for complex numbers with real part greater than one, but by means of Mellin transform we will manage to show, that it has analytical continuation for the whole complex space, apart from $z = 1$. Let's take a closer look at Mellin transform and other objects that will be necessary to prove analytic continuation of the Riemann Zeta function.

Mellin transform. Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a function on positive real axis with complex values. We define Mellin transform of this function as:

$$g(z) := \int_{t=0}^{\infty} f(t) t^z \frac{\mathrm{dt}}{t}.$$

It is defined for complex z such that the integral converges absolutely.

The Gamma function. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ we define the Gamma function as follows:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^z \frac{\mathrm{dt}}{t}.$$

Notice that the Gamma function is a Mellin transform of e^{-t}. Other feature of Mellin transform and the Gamma function is the following:

$$\int_0^{\infty} e^{-ct} t^z \frac{\mathrm{dt}}{t} = c^{-z} \Gamma(z), \; c \text{ const.}$$

It can be obtained the following way:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^z \frac{\mathrm{dt}}{t} = \int_0^{\infty} e^{-ct} (ct)^z \frac{c\mathrm{dt}}{ct} = c^z \int_0^{\infty} e^{-ct} t^z \frac{\mathrm{dt}}{t}.$$

The Gamma function has some interesting properties. Let's recall them:

- Γ is analytic on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$

- $\Gamma(z+1) = z\Gamma(z),\ z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$
- $\Gamma(n) = (n-1)!,\ n \in \mathbb{N}$
- $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},\ z \in \mathbb{C} \setminus \mathbb{Z}$
- $\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = \sqrt{\pi}2^{1-z}\Gamma(z)$

The Theta function. Let's define the Theta function.

$$\theta(t) := \sum_{n=-\infty}^{n=\infty} e^{-\pi t n^2}.$$

It satisfies the following equation:

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).$$

Soon we shall also need the following inequality related to the theta function:

$$|\theta(t) - t^{\frac{-1}{2}}| < e^{\frac{-C}{t}},\ C > 0,\ C \text{ const.}$$

All the statements above along with the proves can be found in [Kob93, II.4].

3.1 The Riemann Zeta function

For complex values s the Riemann Zeta function is defined:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ \mathrm{Re}(s) > 1.$$

Due to Euler we know also that the following holds:

$$\zeta(s) = \prod_{\text{p prime}} \frac{1}{1-p^{-s}},\ \mathrm{Re}(s) > 1.$$

Theorem 3. *The Riemann zeta function $\zeta(z)$ extends analytically onto the whole z-plane, except for a simple pole at $z=1$ with residue 1. Let*

$$\Lambda(z) := \pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z).$$

Then $\Lambda(z) = \Lambda(1-z)$. In other words, $\zeta(z)$ satisfies the functional equation:

$$\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{\frac{z-1}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z).$$

Proof: We will follow [Kob93]. First let us define:

$$\phi(z) := \int_1^\infty t^{\frac{z}{2}}(\theta(t)-1)\frac{\mathrm{dt}}{t} + \int_0^1 t^{\frac{z}{2}}\left(\theta(t) - \frac{1}{\sqrt{t}}\frac{\mathrm{dt}}{t}\right).$$

Now notice that by the definition of the theta function:

$$\theta(t) - 1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

Therefore the first integral converges. Further, we have seen that $|\theta(t) - \frac{1}{\sqrt{t}}|$ is bounded by $e^{\frac{-C}{t}}$, so the other integral converges as well. Let's calculate the integrals:

$$\int_1^\infty t^{\frac{z}{2}}(\theta(t)-1)\frac{\mathrm{dt}}{t} + \int_0^1 t^{\frac{z}{2}}\left(\theta(t) - \frac{1}{\sqrt{t}}\frac{\mathrm{dt}}{t}\right)$$

$$= 2\sum_{n=1}^\infty \int_1^\infty e^{-\pi n^2 t} t^{\frac{z}{2}} \frac{\mathrm{dt}}{t} + \int_0^1 t^{\frac{z}{2}}\theta(t)\frac{\mathrm{dt}}{t} - \int_0^1 t^{\frac{z-1}{2}}\frac{\mathrm{dt}}{t}$$

$$= 2\sum_{n=1}^\infty \int_1^\infty e^{-\pi n^2 t} t^{\frac{z}{2}} \frac{\mathrm{dt}}{t} + \int_0^1 t^{\frac{z}{2}}(\theta(t)-1)\frac{\mathrm{dt}}{t} + \int_0^1 t^{\frac{z}{2}}\frac{\mathrm{dt}}{t} - \frac{2}{z-1}$$

$$= 2\sum_{n=1}^\infty \int_1^\infty e^{-\pi n^2 t} t^{\frac{z}{2}} \frac{\mathrm{dt}}{t} + 2\sum_{n=1}^\infty \int_0^1 e^{-\pi n^2 t} t^{\frac{z}{2}} \frac{\mathrm{dt}}{t} + \frac{2}{z} - \frac{2}{1-z}$$

$$= 2\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{\frac{z}{2}} \frac{\mathrm{dt}}{t} + \frac{2}{z} - \frac{2}{1-z}.$$

Now recall the following property of the Mellin transform:

$$\int_0^\infty e^{-ct} t^z \frac{\mathrm{dt}}{t} = c^{-z}\Gamma(z),\ c > 0,\ c \text{ const.}$$

Replacing c with πn^2 and z with $\frac{z}{2}$ we get the following:

$$2\phi(z) = 2\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{\frac{z}{2}} \frac{\mathrm{dt}}{t} + \frac{2}{z} - \frac{2}{1-z}$$

$$= 2\sum_{n=1}^\infty (\pi n^2)^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right) + \frac{2}{z} + \frac{2}{1-z}$$

$$= 2\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) + \frac{2}{z} + \frac{2}{1-z}.$$

So we obtain the equation for the Riemann Zeta function:

$$\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma(z/2)}\left(\frac{1}{2}\phi(z) - \frac{1}{z} - \frac{1}{1-z}\right) := S,\ \mathrm{Re}\ z > 1.$$

We know that $\pi^{\frac{z}{2}}$, $\frac{1}{\Gamma(z/2)}$, $\phi(z)$ are entire functions, so we only have to consider the behaviour of $\frac{1}{z}$, $\frac{1}{1-z}$. Those functions have poles at $z = 0$ and $z = 1$.

Recalling important property of the the Gamma function: $\Gamma(z+1) = z\Gamma(z)$, we can rewrite:

$$\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma(z/2)} \left(\frac{1}{2}\phi(z) - \frac{1}{z} - \frac{1}{1-z} \right) = \frac{\pi^{\frac{z}{2}}}{\Gamma(z/2)} \left(\frac{1}{2}\phi(z) - \frac{1}{1-z} \right) - \frac{\pi^{\frac{z}{2}}}{2\Gamma(z/2+1)}.$$

Therefore S does not have a pole at $z = 0$. Now recall

$$\Lambda(z) := \pi^{\frac{-z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z),$$

so inserting what we have already figured out we obtain:

$$\Lambda(z) = \frac{1}{2}\phi(z) - \frac{1}{z} - \frac{1}{1-z}.$$

We have to show $\Lambda(z) = \Lambda(1-z)$. First let's try to show $\phi(z) = \phi(1-z)$.

$$\phi(z) = \left[\int_1^\infty t^{\frac{z}{2}} (\theta(t) - 1) \tfrac{\mathrm{dt}}{t} + \int_0^1 t^{\frac{z}{2}} \left(\theta(t) - \tfrac{1}{\sqrt{t}} \tfrac{\mathrm{dt}}{t} \right) \right] |_{t=\frac{1}{s}}$$

$$= \left[- \int_0^1 \left(\tfrac{1}{s}\right)^{\frac{z}{2}} \left(\theta\left(\tfrac{1}{s}\right) - 1 \right) s \cdot \tfrac{-\mathrm{ds}}{s^2} - \int_1^\infty \left(\tfrac{1}{s}\right)^{\frac{z}{2}} \left(\theta\left(\tfrac{1}{s}\right) - \sqrt{s} \right) s \cdot \tfrac{-\mathrm{ds}}{s^2} \right] |_{\theta\left(\frac{1}{s}\right) = \sqrt{s}\theta(s)}$$

$$= \int_0^1 \left(\tfrac{1}{s}\right)^{\frac{z}{2}} \left(\sqrt{s}\theta(s) - 1 \right) \tfrac{\mathrm{ds}}{s} + \int_1^\infty \left(\tfrac{1}{s}\right)^{\frac{z}{2}} \left(\sqrt{s}\theta(s) - \sqrt{s} \right) \tfrac{\mathrm{ds}}{s}$$

$$= \int_0^1 (s)^{\frac{1-z}{2}} \left(\theta(s) - \tfrac{1}{\sqrt{s}} \right) \tfrac{\mathrm{ds}}{s} + \int_1^\infty (s)^{\frac{1-z}{2}} (\theta(s) - 1) \tfrac{\mathrm{ds}}{s} = \phi(1-z).$$

Now it is clear that $\Lambda(z) = \Lambda(1-z)$ and therefore the Riemann Zeta function satisfies the equation stated in the theorem.$\square$

Later on we shall use the following identity, which shall be given without the proof:

$$\zeta(k) = \frac{-(2\pi i)^k B_k}{2k!},$$

where B_k stands for Bernoulli numbers.

4 | L-function

In this chapter we shall describe Dirichlet L-functions, generalization of the Riemann Zeta function.

4.1 Dirichlet character

In order to define Dirichlet L-functions we first need to define Dirichlet character. We already know that character is a group homomorphism between a finite abelian multiplicative group and $\mathbb{C}^*$. Now given a character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}^*$ we are going to lift it to a character $\chi' : \mathbb{Z} \to \mathbb{C}^*$, so it satisfies certain conditions:

- $\chi'(ab) = \chi'(a)\chi'(b)$
- $\chi'(a) = \chi'(b),\ a \equiv b \bmod N$
- $\chi'(a) = 0,\ \gcd(a, N) > 1$

Such a character is called Dirichlet character. Dirichlet character that has been obtained by lifting trivial character is called principal Dirichlet character. It is defined as follows:

$$\chi : \mathbb{Z} \to \mathbb{C}^*,\ \chi(a) = \begin{cases} 1 \text{ if } \gcd(a, N) = 1 \\ 0 \text{ if } \gcd(a, N) > 1 \end{cases}$$

Trivial Dirichlet character is the unique Dirichlet character mod 1. Thus it sends all non-zero elements to 1.

In the thesis we shall use the notion of the Gauss sum. The Gauss sum depends on a character χ and is defined:

$$\tau(\chi) = \sum_{a=1}^{N_\chi} \chi(a) e^{2\pi i a/N_\chi},\ N_\chi = \text{ cond } (\chi).$$

It has the following properties:

- $\tau(\bar{\chi}) = \chi(-1)\overline{\tau(\chi)}$
- $\tau(\chi_1\chi_2) = \chi_1(N_2)\chi_2(N_1)\tau(\chi_1)\tau(\chi_2),\ N_1 = \text{ cond}(\chi_1),\ N_2 = \text{ cond}(\chi_2)$

4.2 Dirichlet L-function

Suppose χ is a Dirichlet character modulo N. Dirichlet L-function associated to this character is defined as follows:

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

It can be proved that

$$L(s,\chi) = \prod_{p\in \mathrm{P}} (1-\chi(p)p^{-s})^{-1}.$$

Example 1 One of the examples of Dirichlet L-function is Riemann zeta function, that has been already introduced. Given by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

it is Dirichlet L-function of trivial Dirichlet character.

Example 2 An interesting example of a Dirichlet character modulo a prime number p is a Legendre symbol. It is defined as follows:

$$\left(\frac{n}{p}\right) = \begin{cases} 1 \text{ if } x^2 \equiv n \bmod p \text{ has solution} \\ 0 \text{ if } p|n \\ -1 \text{ otherwise} \end{cases}$$

An attached Dirichlet L-function is:

$$L\left(s, \left(\frac{\cdot}{p}\right)\right) = \sum_{n=1}^{\infty} \frac{\left(\frac{n}{p}\right)}{n^s}.$$

Example 3 Suppose we have a character $\chi : \mathbb{Z}/10\mathbb{Z} \to \mathbb{C}^*$ given by:

n	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\bar{10}$
$\chi(n)$	1	-i	i	-1	0	1	-i	i	-1	0

Then the induced Dirichlet character $\chi' : \mathbb{Z} \to \mathbb{C}^*$ is given by:

n mod 10	1	2	3	4	5	6	7	8	9	10
$\chi(n)$	1	0	i	0	0	0	-i	0	-1	0

Thus the attached Dirichlet L-function is given as follows:

$$L(s, \chi') = \sum_{n=1}^{\infty} \frac{\chi'(n)}{n^s} = \frac{1}{1^s} + \frac{i}{3^s} - \frac{i}{7^s} - \frac{1}{9^s} + \dots$$

5 | Modular forms

5.1 Modular group

Let us start this section with defining a modular group: $SL_2(\mathbb{Z})$.

$$SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

As a matrix of this form induces a linear automorphism of a vector space, elements of $SL_2(\mathbb{Z})$ can be also given as

$$T: \qquad \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

$$z \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} (z) = \tfrac{az+b}{cz+d}.$$

It needs to be specified that here we mean:

$$T(\infty) = \frac{a}{c},\ T\left(\frac{-d}{c}\right) = \infty.$$

In what follows, we are going to denote the upper complex half-plane by H, in other words:

$$H := \{z \in \mathbb{C} :\ \mathrm{Im}(z) > 0\}.$$

5.2 Congruence subgroup of level N

An important subgroup of $SL_2(\mathbb{Z})$, which is also a normal subgroup, is a congruence subgroup of level N defined as follows:

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod N \right\}.$$

In what follows $*$ indicates an arbitrary entry. We shall define $\Gamma_0(N)$ and $\Gamma_1(N)$, subgroups of $SL_2(\mathbb{Z})$ satisfying:

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z}).$$

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod N \right\}.$$

5.3 Modular forms for congruence subgroups

In what follows let $\Gamma \subset SL_2(\mathbb{Z})$ be such that there exists $N \in \mathbb{Z}^+$ such that $\Gamma(N) \subset \Gamma$. For f a function $f : H \to \mathbb{C}$ and a matrix

$$\gamma \in SL_2(\mathbb{Z}),\ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we define an operator $[\gamma]_k$:

$$(f[\gamma]_k)(z) = (cz+d)^{-k} f(\gamma(z)),\ z \in H.$$

We call a function f weakly modular of weight k with respect to Γ if the following is satisfied:

$$f[\gamma]_k = f,\ \forall \gamma \in \Gamma.$$

Definition 1. *We call a function $f : H \to \mathbb{C}$ a modular form of weight k with respect to Γ if it satisfies the following conditions:*

- *f is holomorphic*
- *f is weakly modular of weight k with respect to Γ*
- *$f[\gamma]_k$ is holomorphic at infinity for all $\gamma \in SL_2(\mathbb{Z})$.*

We will denote the set of modular forms of weight k with respect to Γ by $\mathcal{M}_k(\Gamma)$.

5.4 Cusp forms

Definition 2. *A cusp form of weight k with respect to Γ is a modular form of weight k with respect to Γ such that for all $\gamma \in SL_2(\mathbb{Z})$ in the Fourier expansion of $f[\gamma]_k,\ a_0 = 0$ i.e.*

$$f[\gamma]_k = \sum_{n=1}^{\infty} a_n q^n.$$

We will denote the set of cusp forms with respect to Γ by $\mathcal{S}_k(\Gamma)$. This set with operations defined like in the previous section is again a vector space over a field $\mathbb{C}$ and again if $\Gamma_1 \subset \Gamma_2$ then $\mathcal{S}_k(\Gamma_2) \subset \mathcal{S}_k(\Gamma_1)$. Analogously as for modular forms we will denote with $\mathcal{S}_k(N, \chi_f)$ the subspace of space $\mathcal{S}_k(\Gamma_1(N))$ with the property that $f[\gamma]_k = \chi(d) f,\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

For $f \in \mathcal{S}_k(N, \chi_f)$ by f^* we will denote the modular form $f^* \in \mathcal{S}_k(N, \chi_f^{-1})$ such that the following condition is satisfied: $a_n(f^*) = \bar{a}_n$.

Let's make some comments regarding sets $\mathcal{M}_k(\Gamma),\ \mathcal{S}_k(\Gamma)$.

1. Let $f, g \in \mathcal{M}_k(\Gamma),\ \alpha \in \mathbb{C},\ z \in H$ and let

   $$(f+g)(z) = f(z) + g(z),\ (\alpha f)(z) = \alpha(f(z)).$$

   It can be checked that $\mathcal{M}_k(\Gamma)$ with those operations forms a vector space over a field $\mathbb{C}$.

2. Suppose that $f \in \mathcal{M}_{k_1}(\Gamma),\ g \in \mathcal{M}_{k_2}(\Gamma)$. Then it follows from the definition that $fg \in \mathcal{M}_{k_1+k_2}(\Gamma)$.

3. Suppose that $\Gamma_1 \subset \Gamma_2$. Notice that if a modular form is weight-k invariant under Γ_2 then it is also weight-k invariant under Γ_1. Therefore it holds $\mathcal{M}_k(\Gamma_2) \subset \mathcal{M}_k(\Gamma_1)$.

4. Let now $-I \in \Gamma$ and k be an odd integer. Suppose further that $f \in \mathcal{M}_k(\Gamma)$. As the condition $(f[\gamma]_k)(z) = (cz+d)^{-k} f(\gamma(z))$ holds for all matrices in Γ, in particular it holds for $-I$. Therefore $f = -f = 0$ and so we conclude that if $-I \subset \Gamma$ then there are no, different than zero, modular forms with respect to Γ of odd weight. However if $-I \notin \Gamma$, such modular forms might occur.

5. We shall briefly discuss so called twist of a modular form by a Dirichlet character χ. Let $f \in \mathcal{M}_k(\Gamma(N))$ and let χ be a Dirichlet character modulo N. Let f have a q-expansion: $f = \sum_{n=0}^{\infty} a_n q^n$. Then the twist by χ is a modular form: $f_\chi = \sum_{n=0}^{\infty} a_n \chi(n) q^n$. We have to be careful, because although the twist is still a modular form, it doesn't hold that $f_\chi \in \mathcal{M}_k(\Gamma(N))$. We will denote by $\mathcal{M}_k(N,\chi)$ the subspace consisting of $f \in \mathcal{M}_k(\Gamma_1(N)) : f[\gamma]_k = \chi(d) f,\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

6. [BD, section 2] One can introduce spaces $\mathcal{S}_k^{\mathrm{an}}(N,\chi) \subset \mathcal{M}_k^{\mathrm{an}}(N,\chi)$ which are closely related to $\mathcal{S}_k(N,\chi) \subset \mathcal{M}_k(N,\chi)$. Let $\mathcal{S}_k(N,\chi) \subset \mathcal{M}_k(N,\chi)$ be as above and denote by $\mathcal{S}_k^{\mathrm{an}}(N,\chi) \subset \mathcal{M}_k^{\mathrm{an}}(N,\chi)$ their real analytic counterparts consisting of real analytic functions on H with the same transformation properties under $\Gamma_0(N)$ and having bounded growth at the cusps of $\mathcal{M}_k^{\mathrm{an}}(N,\chi)$ (rapid decay at the cusps of $\mathcal{S}_k^{\mathrm{an}}(N,\chi)$).

See also [Kob93, III.3]

6 | Hecke operators

6.1 Hecke operators: $\langle d\rangle,\ T_p, \langle n\rangle, T_n$

In this section we shall define $\langle d\rangle,\ T_p, \langle n\rangle, T_n$ Hecke operators. Hecke operators are described with details in [DS05, Chapter 5]. Other approach, using lattices in definition of Hecke operators, is presented in [Kob93, III.5]. Let

$$GL_2^+(\mathbb{Q}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Q},\ ad - bc > 0 \right\}.$$

Let $\alpha \in GL_2^+$ and $\Gamma_1,\ \Gamma_2$ be the two congruence subgroups of $SL_2(\mathbb{Z})$. By double coset in $GL_2^+(\mathbb{Q})$ we mean the set

$$\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 :\ \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2\}.$$

It can be shown that $\Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\beta_j$, where the union is finite and disjoint and β_j are representatives of orbits of an action by left multiplication of the group Γ_1 on the double coset $\Gamma_1\alpha\Gamma_2$.

Definition 3. *For $\beta,\ \alpha$ as above, $k \in \mathbb{Z}$ the weight-k β operator on $f : H \to \mathbb{C}$ is given by*

$$(f[\beta]_k)(z) = (det\ \beta)^{k-1}(cz + d)^{-k} f(\beta(z)),\ z \in H,\ \beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Definition 4. *For congruence subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{Z})$ and $\alpha \in GL_2^+(\mathbb{Q})$, the weight-$k$ operator takes functions $f \in \mathcal{M}_k(\Gamma_1)$ to*

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k.$$

Definition 5. *The Hecke diamond operator is the following weight k double-coset operator :*

$$\langle d \rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$$

$$\langle d \rangle f = f[\alpha]_k,\ \Gamma_0(N) \ni \alpha = \begin{bmatrix} a & b \\ c & \delta \end{bmatrix},\ \delta \equiv d \ mod\ N.$$

Definition 6. *The Hecke T_p operator is the following weight k double-coset operator :*

$$T_p : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$$

$$T_p f = f[\Gamma_1(N) \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \Gamma_1(N)],\ p\ prime.$$

The two operators commute.

Examining the representatives of orbits (see [DS05]) we are able to give explicit form of T_p operator. Namely for operator T_p on $\mathcal{M}_k(\Gamma_1(N))$ we have([DS05, Proposition 5.2.1]):

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} f[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}]_k & \text{if } p|N \\ \sum_{j=0}^{p-1} f[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}]_k + f[\begin{bmatrix} m & n \\ N & p \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}]_k & \text{if } p \nmid N,\ mp - nN = 1 \end{cases}$$

Using this result we shall prove the following theorem.

Theorem 4. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ have a Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n,\ q = e^{2\pi i z}.$$

Then:

1. *Let 1_N be a trivial character modulo N. Then $T_p f$ has Fourier expansion*

$$(T_p f)(z) = \sum_{n=0}^{\infty} (a_{np}(f) + 1_N(p) p^{k-1} a_{n/p}(\langle p \rangle f)) q^n,$$

   in other words

$$a_n(T_p f) = a_{np}(f) + 1_N(p) p^{k-1} a_{n/p}(\langle p \rangle f).$$

2. *Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a character. If $f \in \mathcal{M}_k(N,\chi)$ then $T_pf \in \mathcal{M}_k(N,\chi)$ and has a Fourier expansion:*

$$(T_pf)(z) = \sum_{n=0}^{\infty}(a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f))q^n,$$

in other words

$$a_n(T_pf) = a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f).$$

Proof: We will follow [DS05, Proposition 5.2.2]. For the first part suppose first that $p|N$. Then the definition of β weight-k operator computation yields:

$$f[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}]_k(z) = p^{k-1}(0z+p)^{-k} f\left(\frac{z+j}{0z+p}\right) = \frac{1}{p}\sum_{n=0}^{\infty} a_n(f)e^{2\pi i n}(z+j)/p.$$

From the representation of T_pf we found we deduce:

$$T_pf(z) = \sum_{j=0}^{p-1} f[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}]_k = \sum_{j=0}^{p-1}\frac{1}{p}\sum_{n=0}^{\infty} a_n(f)e^{2\pi i n}(z+j)/p = \sum_{n=0}^{\infty} a_{np}(f)q^n.$$

For $p \nmid N$ we need to add the term:

$$f[\begin{bmatrix} m & n \\ N & p \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}]_k(z) = (\langle p\rangle f)[\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}]_k(z) =$$

$$p^{k-1}(0z+1)^{-k}(\langle p\rangle f)(pz) = p^{k-1}\sum_{0}^{\infty} a_n(\langle p\rangle f)q^{np}.$$

As $1_N(p)$ takes value 0 if $p|N$ and 1 otherwise, we can combine results to one formula:

$$(T_pf)(z) = \sum_{n=0}^{\infty}(a_{np}(f) + 1_N(p)p^{k-1}a_{n/p}(\langle p\rangle f))q^n.$$

The second part follows from the first part. $\square$

Note that operator T_p is defined only for prime numbers and $\langle d\rangle$ for d such that $(d,N) = 1$. We can extend those operators to all $n \in \mathbb{Z}^+$.

Definition 7. *For $n \in \mathbb{Z}^+, n = \Pi p_i^{e_i}$, p_i primes we define an operator T_n to be $T_n = \Pi T_{p_i^{e_i}}$ where T_p is like in the previous definition and*

$$T_{p^r} = T_pT_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}},\ r \geq 2.$$

To extend $\langle d\rangle$ to all $n \in \mathbb{Z}^+$, let $\langle n\rangle = 0$ when $(n, N) > 1$ and use total multiplicativity:

$$\forall_{m,n\in\mathbb{Z}^+} \langle mn\rangle = \langle m\rangle\langle n\rangle.$$

For $T_n f$ we have theorem analogous to the one for $T_p f$ describing Fourier expansion of $T_n f$.

Theorem 5. *Let $f \in \mathcal{M}_k(\Gamma_1(N))$ have Fourier expansion*

$$f(z) = \sum_{m=0}^{\infty} a_m(f) q^m.$$

Then for all $n \in \mathbb{Z}^+$, $T_n f$ has Fourier expansion

$$(T_n f)(z) = \sum_{m=0}^{\infty} a_m(T_n f) q^m$$

where

$$a_m(T_n f) = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2}(\langle d\rangle f).$$

In particular, if $f \in \mathcal{M}_k(N, \chi)$ then

$$a_m(T_n f) = \sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f).$$

The proof is mostly computation and inserting results for $T_p f$. It will not be given, but it can be found in [DS05, Proposition 5.3.1]. Further we shall need the definition of Hecke eigenforms:

Definition 8. *A non-zero modular form $f \in \mathcal{M}_k(\Gamma_1(N))$ that is an eigenform for the Hecke operators T_n and $\langle n\rangle$ for all $n \in \mathbb{Z}^+$ is a Hecke eigenform or simply an eigenform. The eigenform*

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n$$

is normalized when $a_1(f) = 1$.

We have the following theorem joining cusp forms and Hecke operators:

Theorem 6. *The space $\mathcal{S}_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $\{\langle n\rangle, T_n : (n, N) = 1\}$.*

Proof:[DS05, Proposition 5.5.4].

We shall close this section with the following theorem:

Theorem 7. *Let $f \in \mathcal{M}_k(N, \chi)$. Then f is normalized eigenform if and only if its Fourier coefficients satisfy the conditions*

1. $a_1(f) = 1$
2. $a_{p^r}(f) = a_p(f)a_{p^{r-1}}(f) - \chi(p)p^{k-1}a_{p^{r-2}}(f)$ *for all p prime and* $r \geq 2$
3. $a_{mn}(f) = a_m(f)a_n(f)$ *when* $(m, n) = 1$.

Proof:[DS05, Proposition 5.8.5].

7 | Eisenstein series

7.1 Eisenstein series of weight higher than 2

Eisenstein series are very important examples of modular forms. In this section we shall define Eisenstein series of weight higher than 2 and find their q-expansion. First we shall discuss the case $\Gamma = SL_2(\mathbb{Z})$ and then move to arbitrary $\Gamma(N)$.

Eisenstein series for $\Gamma = SL_2(\mathbb{Z})$. Fix an even integer $k > 2$ (recall from section about modular forms that it does not make sense to chose even integers for $\Gamma = SL_2(\mathbb{Z})$, as all such forms are equal to zero). The Eisenstein series of weight k is defined to be:

$$G_k(z) = \sum_{(c,d)}{}' \frac{1}{(cz+d)^k},\ z \in H.$$

By the prime sign it is meant that the summation is over $\mathbb{Z}^2 \setminus \{(0,0)\}$. In what follows we will use Bernoulli numbers B_k. It can be proved that Eisenstein series have expansion:

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where

$$\sigma_{k-1}(n) = \sum_{\substack{m>0 \\ m|n}} m^{k-1},\ q = e^{2\pi i z}.$$

Sketch of the proof is the following. Using the formula for $\zeta(k)$:

$$\zeta(k) = \frac{-(2\pi i)^k B_k}{2k!}$$

and identity
$$\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n m z}$$
we obtain
$$\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} = -\frac{2k}{B_k}\zeta(k) \sum_{d=1}^{\infty} d^{k-1} q^{dm},$$
where $q = e^{2\pi i z}$. So
$$G_k(z) = 2\zeta(k) + 2\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \tfrac{1}{(mz+n)^k} =$$
$$2\zeta(k)\left(1 - \tfrac{2k}{B_k}\sum_{m,d=1}^{\infty} d^{k-1} q^{dm}\right) =$$
$$2\zeta(k)\left(1 - \tfrac{2k}{B_k}\sum_{n=1}^{\infty} \sigma(n) q^n\right).$$
More shall be said about the identity
$$\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n m z}.$$
It is obtained from the product formula for sine:
$$\sin(\pi z) = \pi z \Pi_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$
Taking logarithmic derivative of this equation we obtain:
$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$
Using series representation of cot we moreover have:
$$\pi \cot(\pi z) = \pi i - 2\pi i \sum_{m=0}^{\infty} (e^{2\pi i m z}),$$
so
$$\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = \pi i - 2\pi i \sum_{m=0}^{\infty} (e^{2\pi i m z})$$
and the identity is obtained by differentiating both sides of the equation $k-1$ times with respect to z.
Using this expansion for Eisenstein series we are able to define normalized Eisenstein series of weight k:
$$E_k(z) = \frac{G_k(z)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$
For more details see [Kob93, III.2].

Eisenstein series for arbitrary modular group. Choose a row vector $\bar{v} \in (\mathbb{Z}/N\mathbb{Z})$ and an integer $k \geq 3$. Define level N Eisenstein series:

$$G_k^{\bar{v}}(z) := \sum_{\substack{\bar{m} \in \mathbb{Z}^2 \\ \bar{m} \equiv \bar{v} \bmod N}} \frac{1}{(m_1 z + m_2)^k}.$$

Theorem 8. $G_k^{\bar{v}} \in \mathcal{M}_k(\Gamma(N))$.

Proof: We will follow [Kob93, Proposition 21]. $G_k^{\bar{v}}(z)$ is absolutely and uniformly convergent, hence holomorphic. To show invariance under $\Gamma(N)$ let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\gamma \in \Gamma(N)$ and notice:

$$G_k^{\bar{v}}(z)[\gamma]_k = (cz+d)^{-k} \textstyle\sum_{\bar{m} \equiv \bar{v} \bmod N} \frac{1}{(m_1 \frac{az+b}{cz+d} + m_2)^k} =$$

$$\textstyle\sum_{\bar{m} \equiv \bar{v} \bmod N} \frac{1}{((m_1 a + m_2 c)z + (m_1 b + m_2 d))^k} =$$

$$G_k^{\bar{v}\gamma}(z) = G_k^{\bar{v}}(z)$$

where two last equalities hold because

$$(m_1 a + m_2 c, m_1 b + m_2 d) = m\gamma, \ m \equiv v\gamma \bmod N$$

and we have by definition $\gamma \equiv I \bmod N$. The last condition we need to check is holomorphy at infinity. Observe that

$$\lim_{z \to i\infty} G_k^{\bar{v}} = \sum_{\bar{m} \equiv \bar{v} \bmod N, m_1 = 0} m_2^{-k} = \begin{cases} 0 & \text{if } v_1 \neq 0 \\ \sum_{n \equiv v_2 \bmod N} n^{-k} & \text{if } v_1 = 0 \end{cases}$$

As we have chosen $k \geq 3$ the sum converges and therefore $G_k^{\bar{v}}$ is holomorphic at infinity and that concludes the proof. □

Recall that for $\Gamma = SL_2(\mathbb{Z})$ we found q-expansion for Eisenstein series. Similarly we can give q-extension for $G_k^{\bar{v}}$. It is given by the following theorem:

Theorem 9. *Let*

$$c_k := \frac{(-1)^{k-1} 2k\zeta(k)}{N^k B_k},$$

$$b_{0,k}^{\bar{v}} := \begin{cases} 0 & \text{if } v_1 \neq 0 \\ \zeta^{v_2}(k) + (-1)^k \zeta^{-v_2}(k) & \text{if } v_1 = 0 \end{cases}$$

$$\xi := e^{2\pi i/N}, \ q_N = e^{2\pi i z/N},$$

where

$$\zeta^v(k) := \sum_{n\geq 1, n\equiv a \ mod \ N} n^{-k}.$$

For $k \geq 3$, $G_k^{\bar{v}}$ *has q-expansion*

$$G_k^{\bar{v}} = b_0^{\bar{v}} + c_k \left(\sum_{m_1 \equiv v_1 \ mod \ N, m_1 > 0} \sum_{j=1}^{\infty} j^{k-1} \xi^{jv_2} q_N^{jm_1} \right.$$
$$\left. + (-1)^k \sum_{m_1 \equiv -v_1 \ mod \ N, m_1 > 0} \sum_{j=1}^{\infty} j^{k-1} \xi^{-jv_2} q_N^{jm_1} \right).$$

Proof: [Kob93, Proposition 22].

7.2 Eisenstein series of weight 2

One shall pay attention to the case $k = 2$. So far we always assumed $k \geq 3$ and that guaranteed uniform and absolute convergence of Eisenstein series. With $k = 2$ Eisenstein series does not longer satisfy the definition of modular form, and therefore it shall be described in separate section. This time we will only focus on case $\Gamma = SL_2(\mathbb{Z})$.

Eisenstein series of weight 2 for $SL_2(\mathbb{Z})$. Recall, that Eisenstein series of weight 2 is given by:

$$G_2(z) = \sum_{(c,d)}{}' \frac{1}{(cz+d)^2}.$$

Convergence of such a series is only conditional. However, it still holds:

$$G_2(z) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n.$$

Eisenstein series in such form is not a modular form as it fails to be a weakly modular form. It holds, however, that

$$(G_2[\gamma]_2)(z) = G_2(z) - \frac{2\pi i c}{cz+d}.$$

Unfortunately,the corrected series $G_2(z) - \frac{\pi}{\mathrm{Im}(z)}$ which is weight-2 invariant under $SL_2(\mathbb{Z})$ fails to be holomorphic. More details can be found in [DS05, Chapter 1.2] and in [Kob93, p.112-114].

7.3 Non-holomorphic Eisenstein series

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a primitive character. Then we can attach to it a non-holomorphic Eisenstein series of weight k and level N:

$$\tilde{E}_{k,\chi}(z,s) = \sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi^{-1}(n)}{(mz+n)^k} \cdot \frac{y^s}{|mz+n|^{2s}}.$$

As a function of s it is convergent for $\mathrm{Re}(s) > 1 - \frac{k}{2}$ and admits a meromorphic continuation to all $s \in \mathbb{C}$. As a function of z it transforms like a modular form on $\Gamma_0(N)$, i.e.

$$\tilde{E}_{k,\chi}(\gamma z,s) = \chi(d)(cz+d)^k \tilde{E}_{k,\chi}(z,s),\ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$$

and so $\tilde{E}_{k,\chi} \in \mathcal{M}_k^{\mathrm{an}}(N,\chi)$. An easier case to analyze is: $\tilde{E}_{k,\chi}(z,0)$. We will denote is as $\tilde{E}_{k,\chi}(z)$. Such a series is actually a modular form: $\tilde{E}_{k,\chi}(z) \in \mathcal{M}_k(N,\chi)$ and we can give a normalized Eisenstein series in a form of equation:

$$\tilde{E}_{k,\chi}(z,0) = 2N^{-k}\tau(\chi^{-1})\frac{(-2\pi i)^k}{(k-1)!}E_{k,\chi}(z,0),$$

where $\tau(\chi^{-1})$ is the Gauss sum, and $E_{k,\chi}(z,0)$ is a normalized Eisenstein series. To find out more about non-holomorphic Eisenstein series see [DS05, Chapter 4.10] or [BD, section 2.1].

Further in the thesis we shall need equation joining non-holomorphic Eisenstein series of different weight. First let's define Shimura-Maass operator (see also [Hid93, 10.1] and [BD, section 2.1]) sending $\mathcal{M}_k^{\mathrm{an}}(N,\chi)$ to $\mathcal{M}_{k+2}^{\mathrm{an}}(N,\chi)$:

$$\delta_k := \frac{1}{2\pi i}\left(\frac{d}{dz} + \frac{ik}{2y}\right).$$

Then the following holds:

$$\delta_k \tilde{E}_{k,\chi}(z,s) = -\frac{(s+k)}{4\pi}\tilde{E}_{k+2,\chi}(z,s-1).$$

Now let:

$$\delta_k^t := \delta_{k+2t-2}\ldots\delta_{k+2}\delta_k.$$

Thus:

$$\delta_k^t \tilde{E}_{k,\chi}(z,s) = \frac{(-1)^t}{(4\pi)^t}(s+k)\ldots(s+k+t-1)\tilde{E}_{k+2t,\chi}(z,s-t),$$

so

$$\delta_k^t \tilde{E}_{k,\chi}(z,s)\Big|_{\substack{k=k-2t \\ s=0}} = \frac{(k-2t-1)!}{(k-t-1)!}(-4\pi)^t \delta_{k-2t}^t \tilde{E}_{k-2t,\chi}(z).$$

7.4 Eisenstein series attached to pair of characters

Let N_1 and N_2 be integers satisfying $N_1 N_2 = N$ and χ_1, χ_2 Dirichlet characters modulo N_1, N_2. Further suppose $\chi_1\chi_2$ satisfies the following parity condition: $\chi_1\chi_2(-1) = (-1)^k$. Let

$$\delta_{\chi_1} = \begin{cases} \frac{1}{2} \text{ if } N_1 = 1 \\ 0 \text{ otherwise} \end{cases}$$

and

$$\sigma_{k-1}(\chi_1,\chi_2)(n) = \sum_{d|n} \chi_1(n/d)\chi_2(d)d^{k-1}.$$

For $k \geq 1$ and $(\chi_1, \chi_2) \neq (1,1)$ q-expansion of normalized Eisentstein series attached to Dirichlet characters $\chi_1,\ \chi_2$ is:

$$E_k(\chi_1,\chi_2)(z) = \delta_{\chi_1} L(1-k, \chi_1^{-1}\chi_2) + \sum_{n=1}^{\infty} \sigma_{k-1}(\chi_1,\chi_2)(n) q^n.$$

For more information see [DS05, 4.5].

8 | L-functions of modular forms

Let $f \in \mathcal{M}_k(\Gamma)$, $f = \sum_{n=0}^{\infty} a_n q^n$. We can attach to it an L-function. For $s \in \mathbb{C}$ we define L-function of f to be:

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

When it comes to convergence of such a series we have the following theorem:

Theorem 10. *If $f \in \mathcal{M}_k(\Gamma_1(N))$ is a cusp form then $L(s, f)$ converges absolutely for all s with $Re(s) > \frac{k}{2} + 1$. If f is not a cusp form then $L(s, f)$ converges absolutely for all s with $Re(s) > k$.*

Proof: [DS05, Proposition 5.9.1].
What is more, the following theorem gives us Euler's factorization for modular forms that happen to be a normalized eigenform:

Theorem 11. *Let $f \in \mathcal{M}_k(N, \chi)$, $f(z) = \sum_{n=0}^{\infty} a_n q^n$. Then the following are equivalent:*

- *f is a normalized eigenform*
- *$L(s, f)$ has an Euler product expansion*

$$L(s, f) = \Pi_p (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1}.$$

Proof: [DS05, Theorem 5.9.2].
As a consequence of this theorem we have:

$$L(E_k(\chi_1, \chi_2), s) = L(\chi_1, s) L(\chi_2, s - k + 1)$$

.

For further results we will need something more, an L-series that would be attached to two modular forms. So let $l < k$ and

$$f := \sum_{n=1}^{\infty} a_n(f)q^n \in \mathcal{S}_k(N, \chi_f),\ g := \sum_{n=1}^{\infty} a_n(g)q^n \in \mathcal{M}_k(N, \chi_g).$$

To those forms we shall attach Rankin L-series:

$$D(f, g, s) := \sum a_n(f)a_n(g)n^{-s}.$$

From now on let's make another, stronger, assumption about f and g, namely let's assume that both of them are normalized eigenforms of level N. According to the last point of the last theorem in chapter about Hecke operators coefficients of f and g are weakly multiplicative. It is a well known fact from analytic number theory that for L-function attached to a weakly multiplicative function f we have the following factorization:

$$L(f, s) = \Pi_{p \text{ prime}} \sum_{n=0}^{\infty} f(p^n)p^{-ns}.$$

Applying it to our Rankin L-function we get the following factorization:

$$D(f, g, s) = \Pi_{p \text{ prime}} D_{(p)}(f, g, s),\ D_{(p)}(f, g, s) = \sum_{n=0}^{\infty} a_{p^n}(f)a_{p^n}(g)p^{-ns}.$$

For f, g normalized eigenforms of level N, simultaneous eigenvector for the Hecke operators $T_n : (n, N) = 1$ and a prime p we will denote by $(\alpha_p(f), \beta_p(f))$ the roots of the following Hecke polynomial:

$$x^2 - a_p(f)x + \chi_f(p)p^{k-1}.$$

By $(\alpha_p(g), \beta_p(g))$ we mean roots of polynomial

$$x^2 - a_p(g)x + \chi_g(p)p^{l-1}.$$

We shall use them to construct the following Rankin convolution L-series:

$$L(f \otimes g, s) := \prod_p L_p(f \otimes g, s)$$

where:

$$L_{(p)}(f \otimes g, s) := (1 - \alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \alpha_p(f)\beta_p(g)p^{-s})^{-1}$$
$$(1 - \beta_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \beta_p(f)\beta_p(g)p^{-s})^{-1}.$$

It can be computed that:

$$D_{(p)}(f,g,s) = (1-\chi^{-1}(p)p^{k+l-2-2s})L_{(p)}(f\otimes g,s)$$

and therefore:

$$L(f\otimes g,s) = L(\chi^{-1},2s-k-l+2)D(f,g,s).$$

We shall now say few words about expression $L^*(j,f,\varphi)$ (it shall be called algebraic part of special value $L(j,f,\varphi)$). Let φ be any Dirichlet character, $\eta = \varphi(-1)(-1)^{j-1}$. By $\mathbb{Q}_{f,\varphi}$ we will denote the field generated by the Fourier coefficients of f and the values of φ. Further chose complex periods to satisfy:

$$\Omega_f^+\Omega_f^- = (2\pi)^2\langle f,f\rangle_{k,N}.$$

Define:

$$L^*(j,f,\varphi) := \frac{(j-1)!\tau(\bar{\varphi})}{(-2\pi i)^{j-1}\Omega_f^{\epsilon}}L(j,f,\varphi).$$

Here $\tau(\bar{\varphi})$ is the value of Gauss sum for the character $\bar{\varphi}$. It holds that $L^*(f,\varphi,j)\in\mathbb{Q}_{f,\varphi}$. Proof and more information about chosen complex periods and $L^*(j,f,\varphi)$ can be found in [Shi77].

9 | Petersson scalar product

Let $z \in H,\ z = x + yi$.

Definition 9. *Let Γ be any congruence subgroup of $SL_2(\mathbb{Z})$. The corresponding modular curve is defined as the quotient space $\Gamma\backslash H$:*

$$Y(\Gamma) = \{\Gamma z : z \in H\}.$$

If $\Gamma = \Gamma_1(N)$ we denote $Y(\Gamma_1(N)) = Y_1(N)$. Let $P_N = \Gamma\backslash(\mathbb{Q} \cup \{\infty\})$. Denote $X_1(N)(\mathbb{C}) = Y_1(N)(\mathbb{C}) \sqcup P_N$.

Definition 10. *Petersson scalar product is defined on $\mathcal{S}_k^{an}(N,\chi) \times \mathcal{M}_k^{an}(N,\chi)$ in the following way:*

$$\langle f_1, f_2 \rangle_{k,N} := \int_{X_1(N)(\mathbb{C})} y^k \bar{f}_1(z) f_2(z) \frac{dxdy}{y^2}.$$

We shall briefly explain why the integral is well defined. First notice that the hyperbolic measure

$$dz = \frac{\mathrm{dxdy}}{y^2}$$

is $SL_2(\mathbb{Z})$-invariant. Let further F be a fundamental domain for the action of $\Gamma_0(N)$ on the upper half-plane H. Then the integral from definition of scalar product becomes in fact:

$$\int_{X_1(N)(\mathbb{C})} y^k \bar{f}_1(z) f_2(z) \frac{\mathrm{dxdy}}{y^2} = \sum_j \int_F \varphi(\alpha_j(z)) dz$$

where

$$\varphi(z) := y^k \bar{f}_1(z) f_2(z)$$

and $\alpha_j(z)$ are chosen orbit representatives in the fundamental domain. Now to make sure that the definition of integral is correct we need to check that it does not depend on chosen representatives. But:

$$\varphi(\gamma(z)) = y_{\gamma_z}^k \bar{f}_1(\gamma(z)) f_2(\gamma(z)) =$$

$$\overline{(f_1[\gamma]_k)(z)(cz+d)^k}(f_2([\gamma]_k)(z)(cz+d)^k y_z^k |(cz+d)|^{-2k} =$$

$$\overline{(f_1[\gamma]_k)(z)} f_2([\gamma]_k)(z)(cz+d)^k y_z^k =$$

$$y^k \bar{f}_1(z) f_2(z) = \varphi(z)$$

where we mean:

$$y_{\gamma(z)} = \text{ Im } (\gamma(z)),\ y_z = \text{ Im } (z).$$

Further we shall say something about convergence of the integral. Let's start with the following two definitions:

Definition 11. *A function f is called slowly increasing if for any $\alpha \in SL_2(\mathbb{Z})$ there exist positive numbers A, B such that $f[\alpha]_k \leq A(1 + y^{-B})$ as $y \to \infty$.*

Definition 12. *A function f is called rapidly decreasing if for any $B \in \mathbb{R}$ and $\alpha \in SL_2(\mathbb{Z})$, there exist a positive constant A such that $f[\alpha]_k \leq A(1 + y^B)$ as $y \to \infty$.*

As it is stated in [Hid93, p.297]: if f_1 and f_2 are modular forms of weight k with respect to $\Gamma_0(N)$ and if f_1 is rapidly decreasing and f_2 is slowly increasing, then the integral defining Petersson scalar product $\langle f_1, f_2 \rangle_{k,N}$ is absolutely convergent. Therefore the domain chosen to define Petersson scalar product guarantees that the integral is convergent.
We are ready to define:

Definition 13. *The anti-holomorphic differential attached to f is:*

$$\eta_f^{ah} := \frac{\bar{f}(z) d\bar{z}}{\langle f, f \rangle_{2,N}}.$$

Apart from [Hid93] more detailed description of Petersson scalar product can be found for example in [DS05, 5.4] or [Kob93, p.168 - 172].

10 | Complex regulator

Let $Y_1(N)$ be a modular curve over $\mathbb{Q}$ and $\bar{Y}_1(N)$ its extension to $\bar{\mathbb{Q}}$. By $\mathbb{C}(\bar{Y}_1(N))$ we will mean the field of meromorphic functions on $\bar{Y}_1(N)$. Further, for arbitrary field F we will define algebraic K-group coming from K-theory. This group will be defined as a quotient:

$$K_2(F) = (F^* \otimes_{\mathbb{Z}} F^*)/\langle a \otimes (1-a) | a \in F \setminus \{0,1\}\rangle.$$

We will denote elements of $K_2(F)$ as $\{x, y\}$. We shall give the definition of a complex regulator.

Definition 14. *Let $u, v \in F^*$ be rational functions. Let*

$$\eta(u,v) = log|u| \cdot d\ arg\ v - \ log\ |v| \cdot d\ arg\ u.$$

Then $reg_{\mathbb{C}}$ is:

$$reg_{\mathbb{C}} : K_2(\mathbb{C}(X_1(N))) \to \ Hom_{\mathbb{Q}}(\Omega^1(X_1(N)), \mathbb{R})$$

$$\{u,v\} \to \left(\omega \mapsto \int_{X_1(N)(\mathbb{C})} \eta(u,v) \wedge \omega\right).$$

We shall point out few facts:

$$\log z := \log |z| + i \arg (z)$$

$$\mathrm{dlog}(u) := \frac{1}{2\pi i}\frac{u'(z)}{u(z)}$$

$$d \arg u = d\ \mathrm{Im}(\log u) = d\left(\frac{\log u - \overline{\log u}}{2i}\right) = \frac{-1}{4\pi}\left(\frac{du}{u} - \frac{d\bar{u}}{\bar{u}}\right)$$

$$d \log |u| = d\ \mathrm{Re}(\log u) = d\left(\frac{\log u - \overline{\log u}}{2}\right) = \frac{1}{4\pi i}\left(\frac{du}{u} - \frac{d\bar{u}}{\bar{u}}\right)$$

$$d \arg u \wedge \omega_f = -\frac{1}{2i}\frac{d\bar{u}}{\bar{u}} \wedge \omega_f = id(\log|u| \cdot \omega_f).$$

Further the Stokes theorem gives us:

$$\int_{X_1(N)(\mathbb{C})} \log |u| \cdot d(\log |v| \cdot \omega_f) = -\int_{X_1(N)(\mathbb{C})} \log |v| \cdot d(\log |u| \cdot \omega_f).$$

Let's perform some computations:

$$\begin{aligned}
&\textstyle\int_{X_1(N)(\mathbb{C})} \eta(u,v) \wedge \omega_f = \\
&\textstyle\int_{X_1(N)(\mathbb{C})} (\log|u| \cdot d \arg v - \log |v| \cdot d \arg u) \wedge \omega_f = \\
&\textstyle\int_{X_1(N)(\mathbb{C})} \log|u| \cdot d \arg v \wedge \omega_f - \int_{X_1(N)(\mathbb{C})} \log|v| \cdot d \arg u \wedge \omega_f = \\
&\textstyle 2\int_{X_1(N)(\mathbb{C})} \log|u| \cdot d \arg v \wedge \omega_f = \\
&\textstyle 2i\int_{X_1(N)(\mathbb{C})} \log|u| \cdot d(\log v \cdot \omega_f).
\end{aligned}$$

Definition 15. *A modular unit is a meromorphic function u in $\mathbb{C}(X_1(N))^*$ such that Supp$(u) \subset P_N$. The group of modular units is denoted: $\mathcal{O}^*(Y_1(\mathbb{C}))$.*

There is a strong relation joining modular units and Eisenstein series. We have a surjective homomorphism:

$$\mathcal{O}(\bar{Y}_1(N))^* \otimes F \xrightarrow{\text{dlog}} \text{Eis}_2(\Gamma_1(N), F),$$

where

$$\text{dlog}(u) := \frac{1}{2\pi i}\frac{u'(z)}{u(z)},$$

F is an arbitrary field and

$$\text{Eis}_l(\Gamma_1(N), F) \subset \mathcal{M}_l(\Gamma_1(N), F)$$

is a subspace of $\mathcal{M}_l(\Gamma_1(N), F)$ spanned by weight l Eisenstein series with coefficients in F. See also [Ste85, p.521] and [BD, section 2.4]. The following proposition, [Bru07, Prop. 5.3], gives us an explicit construction of the above elements and represents the first key step in the proof of Beilinson's formula. Before stating this result, we introduce a couple of more notation: for a modular unit u, write

$$u(z) = \sum_{n=n_0}^{\infty} a_n q^n$$

for its Fourier expansion, and set

$$\hat{u}(\infty) := a_{n_0};$$

define

$$E_{u,v}(z,s) = \sum\nolimits'_{m \equiv_N u,\, n \equiv_N v} \frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}}$$

where the sum is over all non-zero pairs of integers m and n congruent to u and v, respectively, mod N, and set

$$E_l^* = \sum_{v \in \frac{\mathbb{Z}}{N\mathbb{Z}}} l(v) E_{0,v}^*,$$

$$E_{u,v}^*(z) = \lim\nolimits_{s \to 1} \left(E_{u,v}(z,s) - \frac{\pi}{N^2(s-1)} \right).$$

Theorem 12. *For a function of sum zero:* $l : \frac{\mathbb{Z}}{N\mathbb{Z}} \to \mathbb{C}$ *there exists a unique modular unit*

$$u_l \in \mathcal{O}^*(Y_1(N)(\mathbb{C})) \otimes \mathbb{C}$$

satisfying

$$log|u_l| = \frac{1}{\pi} \cdot E_l^* \text{ and } \hat{u}_l(\infty) = 1 \in \mathbb{C}^* \otimes \mathbb{C}.$$

Let

$$u_\chi, u(\chi_1, \chi_2) \in K_2(\mathbb{C}(\bar{Y}_1(N)))$$

be modular units such that

$$\mathrm{dlog}(u_\chi) = E_{2,\chi},\ \mathrm{dlog}(u(\chi_1,\chi_2)) = E_2(\chi_1,\chi_2).$$

Recall formula for anti-holomorphic differential:

$$\eta_f^{\mathrm{ah}} := \frac{\bar{f}(z) d\bar{z}}{\langle f, f \rangle_{2,N}}.$$

Now we shall compute $\mathrm{reg}_{\mathbb{C}}\{u_\chi, u(\chi_1,\chi_2)\}(\eta_f^{\mathrm{ah}})$. Letting

$$u = u_\chi,\ v = u(\chi_1,\chi_2),\ \omega_f = \eta_f^{\mathrm{ah}}$$

we get:

$$\mathrm{reg}_{\mathbb{C}}\{u_\chi, u(\chi_1,\chi_2)\}(\eta_f^{\mathrm{ah}}) = \frac{\int_{X_1(N)(\mathbb{C})} \bar{f} \cdot \log|u_\chi| \cdot \mathrm{dlog}(u(\chi_1,\chi_2)(z)) dx dy}{\langle f, f \rangle_{2,N}}.$$

More about complex regulator can be found in [Bru07].

11 | Beilinson's formula

Beilinson's formula is given by:

$$L^*(f,\chi_1,2)\cdot L^*(f,\chi_2,1) = C_{f,\chi_1,\chi_2}\cdot \mathrm{reg}_{\mathbb{C}}\{u_\chi, u(\chi_1,\chi_2)\}(\eta_f^{\mathrm{ah}}),$$

where C_{f,χ_1,χ_2} is a constant.

Rankin method. The key to prove this formula is proposition by Shimura ([Hid93, 10.2]).

Theorem 13. *For a weight k modular form f, $s\in\mathbb{C}$, $Re(s) >> 0$*

$$\langle f^*(z), \tilde{E}_{k-l,\chi}(z,s)\cdot g(z)\rangle_{k,N} = 2\frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}L(\chi^{-1},2s+k-l)D(f,g,s+k-1).$$

This is an application of Rankin method. If we let $s = s-k+1$ we obtain

$$\langle f^*(z), \tilde{E}_{k-l,\chi}(z,s-k+1)\cdot g(z)\rangle_{k,N} = 2\tfrac{\Gamma(s)}{(4\pi)^s}L(\chi^{-1},2s-k-l+2)D(f,g,s)$$

$$L(\chi^{-1},2s-k-l+2)D(f,g,s) = \tfrac{(4\pi)^s}{2\Gamma(s)}\langle f^*(z), \tilde{E}_{k-l,\chi}(z,s-k+1)\cdot g(z)\rangle_{k,N}$$

Recall also:

$$L(f\otimes g,s) = L(\chi^{-1},2s-k-l+2)D(f,g,s).$$

Combining previous results we get

$$L(f\otimes g,s) = \frac{(4\pi)^s}{2\Gamma(s)}\langle f^*(z), \tilde{E}_{k-l,\chi}(z,s-k+1)\cdot g(z)\rangle_{k,N}.$$

Let $k = l+m+2t, k,l,m,t$ integers, and further $c := \frac{k+l+m-2}{2} = k-t-1$. Recall that:

$$\tilde{E}_{k,\chi}(z,-t) = \frac{(k-2t-1)!}{(k-t-1)!}(-4\pi)^t\delta_{k-2t}^t\tilde{E}_{k-2t,\chi}(z).$$

With our definition of k assuming $m \geq 1,\ t \geq 0$ we get

$$\tilde{E}_{k-l,\chi}(z,-t) = \frac{(m-1)!}{(m+t-1)!}(-4\pi)^t \delta_m^t \tilde{E}_{m,\chi}(z).$$

Therefore

$$L(f \otimes g, c) = \frac{1}{2}(-1)^t (4\pi)^{c+t} \frac{(m-1)!}{(m+t-1)!(c-1)!} \langle f^*, \delta_m^t \tilde{E}_{m,\chi}(z) \cdot g(z) \rangle_{k,N}.$$

Recall:

$$\tilde{E}_{k,\chi}(z) = 2N^{-k}\tau(\chi^{-1}) \frac{(-2\pi i)^k}{(k-1)!} E_{k,\chi}(z).$$

If we insert this identity to the formula above, using properties of scalar product we obtain:

$$L(f \otimes g, c) = \frac{(-1)^t 2^{k-1} (2\pi)^{k+m-1} (iN)^{-m} \tau(\chi^{-1})}{(m+t-1)!(c-1)!} \langle f^*(z), \delta_m^t E_{m,\chi}(z) \cdot g(z) \rangle_{k,N}.$$

Other approach is to recall that

$$L(E_k(\chi_1,\chi_2), s) = L(\chi_1, s) L(\chi_2, s-k+1)$$

and then we get

$$L(f \otimes E_l(\chi_1,\chi_2), c) = L(f,\chi_1,c) \cdot L(f,\chi_2,c-l+1).$$

So:

$$L(f,\chi_1,c) \cdot L(f,\chi_2,c-l+1) =$$
$$= \frac{(-1)^t 2^{k-1} (2\pi)^{k+m-1} (iN)^{-m} \tau(\chi^{-1})}{(m+t-1)!(c-1)!} \langle f^*(z), \delta_m^t E_{m,\chi}(z) \cdot E_l(\chi_1,\chi_2) \rangle_{k,N}.$$

From now on we will assume:

- $l = m$
- $\chi_f = 1,\ f^* = f,\ \chi = \bar{\chi}_1 \bar{\chi}_2$
- $|\tau(\chi)|^2 = N$
- $(N_1, N_2) = 1$ so

$$\tau(\chi) = \tau(\bar{\chi}_1)\tau(\bar{\chi}_2)\chi_1(N_2)\chi_2(N_1) = \overline{\tau(\chi_1)\tau(\chi_2)}\chi(-1)\chi_1(N_2)\chi_2(N_1).$$

We should treat separately the critical points.

Critical point. We want to deduce Beilinson's formula for critical values in the sense of Deligne (see [Del79]) of L-functions. Let then: $c = \frac{k}{2} + l - 1$. It turns out it is a critical value for all components of:

$$L(f \otimes E_l(\chi_1, \chi_2), c) = L(f, \chi_1, c) \cdot L(f, \chi_2, c - l + 1).$$

Let's gather previous results to obtain:

$$L^*(f, \chi_1, \frac{k}{2} + l - 1) \cdot L^*(f, \chi_1, \frac{k}{2}) =$$

$$= \frac{(k/2 + l - 2)!(k/2 - 1)!\tau(\bar{\chi}_1)(\tau(\bar{\chi}_2))}{(-2\pi i)^{k+l-1}\langle f, f\rangle_k, N} L(f, \chi_1, \frac{k}{2} + l - 1) \cdot L(f, \chi_1, \frac{k}{2}) =$$

$$= \frac{(k/2 + l - 2)!(k/2 - 1)!\tau(\bar{\chi}_1)(\tau(\bar{\chi}_2))}{(-2\pi i)^{k+l-1}\langle f, f\rangle_k, N} \cdot$$

$$\cdot \frac{(-1)^t 2^{k-1}(2\pi)^{k+m-1}(iN)^{-m}\tau(\chi^{-1})}{(m + t - 1)!(c - 1)!} \langle f^*, \delta_l^{k/2-l} E_{l,\chi} \cdot E_l(\chi_1, \chi_2)\rangle_{k,N}$$

so recalling our assumption we get

$$L^*(f, \chi_1, \frac{k}{2} + l - 1) \cdot L^*(f, \chi_1, \frac{k}{2}) = C_{f,\chi_1,\chi_2} \frac{\langle f, \delta_l^{k/2-l} E_{l,\chi} \cdot E_l(\chi_1, \chi_2)\rangle_{k,N}}{\langle f, f\rangle_{k,N}},$$

where $C_{f,\chi_1,\chi_2} = \frac{i 2^{k-1}}{N^{l-1}}$ is a constant.

Beilinson's formula for s=2. Form now on we set: $k = l = 2$ (note that with previous assumptions it gives us $c = 2, t = -1$). Recall the definition of normalised Eisenstein series $\tilde{E}_{k,\chi}(z)$:

$$\tilde{E}_{k,\chi}(z) = 2N^{-k}\tau(\chi^{-1}) \frac{(-2\pi i)^k}{(k - 1)!} E_{k,\chi}(z)$$

and the following formula:

$$\delta_k \tilde{E}_{k,\chi}(z, s) = -\frac{(s + k)}{4\pi} \tilde{E}_{k+2,\chi}(z, s - 1).$$

Letting $k = 0, s = 1$ we get:

$$\delta_0 \tilde{E}_{0,\chi}(z, 1) = \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}z} \tilde{E}_{0,\chi}(z, 1) = \frac{-1}{4\pi} \tilde{E}_{2,\chi}(z) =$$

$$\frac{-1}{4\pi} 2N^{-2}\tau(\chi^{-1})(-4\pi^2)E_{2,\chi}(z) = 2\pi N^{-2}\tau(\chi^{-1})E_{2,\chi}(z).$$

Recalling

$$\mathrm{dlog}(u_\chi) = E_{2,\chi},\ \mathrm{dlog}(u(\chi_1, \chi_2)) = E_2(\chi_1, \chi_2)$$

we get:

$$L(f \otimes E_2(\chi_1, \chi_2), 2) = \frac{1}{2}(4\pi)^2 \langle f(z), \tilde{E}_{0,\chi}(z, 1) \cdot E_2(\chi_1, \chi_2)(z) \rangle_{2,N} =$$

$$\frac{1}{2} \cdot (4\pi)^2 \cdot 2\pi N^{-2} \tau(\chi^{-1}) \langle f(z), \log|u_\chi(z)| \cdot \operatorname{dlog}(u(\chi_1, \chi_2)(z)) \rangle_{2,N} =$$

$$16\pi^3 N^{-2} \tau(\chi^{-1}) \langle f(z), \log|u_\chi(z)| \cdot \operatorname{dlog}(u(\chi_1, \chi_2)(z)) \rangle_{2,N}.$$

Comparing again two ways of expressing $L(f \otimes E_2(\chi_1, \chi_2), 2)$ we get:

$$L^*(f, \chi_1, 2) L^*(f, \chi_2, 1) = \frac{C_{f,\chi_1,\chi_2}}{\langle f, f \rangle_{2,N}} \langle f(z), \log|u_\chi(z)| \cdot \operatorname{dlog}(u(\chi_1, \chi_2)(z)) \rangle_{2,N} =$$

$$\frac{C_{f,\chi_1,\chi_2}}{\langle f, f \rangle_{2,N}} \int_{X_1(N)(\mathbb{C})} \bar{f} \cdot \log|u_\chi| \cdot \operatorname{dlog}(u(\chi_1, \chi_2)(z)) dxdy,$$

where C_{f,χ_1,χ_2} is a constant. Now recall the definition of an anti-holomorphic differential:

$$\eta_f^{\mathrm{ah}} = \frac{\bar{f}(z) d\bar{z}}{\langle f, f \rangle_{2,N}}$$

and what we have said about the complex regulator:

$$\operatorname{reg}_{\mathbb{C}}\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ah}}) = \frac{\int_{X_1(N)(\mathbb{C})} \bar{f} \cdot \log|u_\chi| \cdot \operatorname{dlog}(u(\chi_1, \chi_2)(z)) dxdy}{\langle f, f \rangle_{2,N}}$$

and we get Beilinson formula:

$$L^*(f, \chi_1, 2) \cdot L^*(f, \chi_2, 1) = C_{f,\chi_1,\chi_2} \cdot \operatorname{reg}_{\mathbb{C}}\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ah}}).$$

As it has been mentioned in the abstract, the motivation to write this thesis was the second chapter of article by Henri Darmon and Massimo Bertolini (see [BD]). The reasoning presented in this chapter follows [BD]. Therefore readers with special interest in Beilinson's formula should get familiar with this article, which later part describes p-adic case.

Bibliography

[BD] Massimo Bertolini and Darmon, *Kato's euler systems and rational points on elliptic curves I: A p-adic beilinson fomula*, Israel Journal of Mathematics, to appear.

[Bru07] François Brunault, *Valeur en 2 de fonctions L de formes modulaires de poids 2: théorème de Beilinson explicite*, Bull. Soc. Math. France **135** (2007), no. 2, 215–246. MR 2430191 (2009e:11096)

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, With an appendix by N. Koblitz and A. Ogus, pp. 313–346. MR 546622 (81d:12009)

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196 (2006f:11045)

[Hid93] Haruzo Hida, *Elementary theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, Cambridge, 1993. MR 1216135 (94j:11044)

[Kob93] Neal Koblitz, *Introduction to elliptic curves and modular forms*, second ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1993. MR 1216136 (94a:11078)

[Shi77] Goro Shimura, *On the periods of modular forms*, Math. Ann. **229** (1977), no. 3, 211–221. MR 0463119 (57 #3080)

[Ste85] Glenn Stevens, *The cuspidal group and special values of L-functions*, Trans. Amer. Math. Soc. **291** (1985), no. 2, 519–550. MR 800251 (87a:11056)