A FINITE AXIOMATIZATION OF THE MODEL COMPANION OF SEMILATTICES

1. Preliminaries

Let $\mathcal{SL}$ be the class of meet semilattices $\mathbf{A} = (A, \wedge)$, and let $\mathcal{SL}^{\text{ec}}$ be the subclass of existentially closed members of $\mathcal{SL}$. Since $\mathcal{SL}$ has the AP and JEP, $\mathcal{SL}^{\text{ec}}$ forms an elementary class, axiomatizable by $\forall\exists$-sentences.

For $\mathbf{A} \in \mathcal{SL}$, one has $\mathbf{A} \in \mathcal{SL}^{\text{ec}}$ iff for every finite $\mathbf{P} \leq \mathbf{A}$ and for every finite $\mathbf{Q} \in \mathcal{SL}$ that extends $\mathbf{P}$, that is, $\mathbf{P} < \mathbf{Q}$, the extension by $\mathbf{Q}$ can be realized in $\mathbf{A}$, that is, there is an embedding $\alpha : \mathbf{Q} \hookrightarrow \mathbf{A}$ that leaves $\mathbf{P}$ fixed.

In $\mathcal{SL}$ every finite extension $\mathbf{P} < \mathbf{Q}$ (meaning that $Q \smallsetminus P$ is finite) can be realized by a sequence of 1-element extensions

$$\mathbf{P} \;=\; \mathbf{S}_0 \;\leq\; \mathbf{S}_1 \;\leq\; \cdots \;\leq\; \mathbf{S}_n \;=\; \mathbf{Q},$$

that is, each $S_{i+1} \smallsetminus S_i$ has exactly one element in it. To see this, let q be a minimal element of $Q \smallsetminus P$. Then $S_1 := P \cup \{q\}$ is a subuniverse of $\mathbf{Q}$, so we have $\mathbf{P} < \mathbf{S}_1 \leq \mathbf{Q}$, and $\mathbf{S}_1$ is a 1-element extension of $\mathbf{P}$. Etc.

When working with semilattices, the following abbreviations are commonly used:

$a \leq b$	means	$a \wedge b = a$
$a \not\leq b$	means	it is not the case that $a \leq b$
$a < b$	means	$(a \leq b)$ & $(a \neq b)$
$a \not< b$	means	it is not the case that $a < b$.

Likewise there are definitions for $\geq$ and $>$. The relation $\leq$ defines a partial order on a semilattice from which one can recapture the meet operation by $a \wedge b = \text{glb}(a, b)$.

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A 1-*extension property* $\varepsilon(x_1,\dots,x_n)$ is a first-order formula in one of the two forms

$$(\exists\sigma)\omega(x_1,\dots,x_n,\sigma), \qquad \omega_1(x_1,\dots,x_n) \;\to\; (\exists\sigma)\omega_2(x_1,\dots,x_n,\sigma),$$

where $\omega(\vec{x},\sigma),\omega_1(\vec{x}),\omega_2(\vec{x},\sigma)$ are quantifier-free formulas, and $\omega(\vec{x},\sigma)$, $\omega_2(\vec{x},\sigma)$ just (partially) describe how σ is related to $x_1,\dots,x_n$. To say that a 1-extension property $\varepsilon(\vec{x})$ *holds* in a semilattice $\mathbf{A}$ means $\mathbf{A}\models(\forall\vec{x})\varepsilon(\vec{x})$.

$\mathcal{SL}^{\mathrm{ec}}$ will be axiomatized by adding finitely many 1-extension properties to the axioms for semilattices.

2. The 1-extension properties

E1: $\varepsilon_1(x,y)$ is $(\exists\sigma)(x<\sigma \;\&\; y<\sigma)$

This says there is an element σ above both x and y (x and y need not be distinct).

E2: $\varepsilon_2(x)$ is $(\exists\sigma)(\sigma<x)$

This says there is an element σ below x.

E3: $\varepsilon_3(x,y,z)$ is $(x<z \;\&\; y<z) \;\to\; (\exists\sigma)(x<\sigma \;\&\; y<\sigma \;\&\; \sigma<z)$.

This says that for elements x,y below z, there is an element σ above both x and y, and below z. (x,y need not be distinct.)

E4: $\varepsilon_4(x,y,z)$ is $(x<y \;\&\; x<z \;\&\; z\not\leq y)\to\; (\exists\sigma)(x<\sigma<z \;\&\; x=y\wedge\sigma)$

This says that if x is less than both y and z, and z is not $\leq y$, then there is an element σ properly between x and z whose meet with y is x.

E5: $\varepsilon_5(x,y)$ is $(x<y)\to\; (\exists\sigma)(x<\sigma \;\&\; x=y\wedge\sigma)$

This says that if x is less than y, then there is an element σ above x whose meet with y is x.

E6: $\varepsilon_6(x,y,z,u,v)$ is $(x<y \;\&\; x<z<u \;\&\; x=y\wedge z \;\&\; v<u \;\&\; y\wedge v\leq x)\to$ $(\exists\sigma)(z<\sigma<u \;\&\; v<\sigma \;\&\; x=y\wedge\sigma)$

This says that if x is less than both y and z, the meet of y and z is x, z and v are both less than u, and the meet of y and v is $\leq x$, then there is an element σ above both z and v, but below u, whose meet with y is x.

Remark 2.1. Thanks to Mick Adams for noting that E2 was missing from the paper [1], and E6 (as Axiom 3 of [1]) was not formulated correctly. Also Axiom 4 of [1] is not needed.

Proposition 2.2. *$\mathcal{SL}^{ec}$ satisfies the 1-extension properties* E1, ..., E6.

Proof. Each $\mathbf{A}$ in $\mathcal{SL}$ can be embedded in some power of the 2-element semilattice $\mathbf{2} = (\{0,1\}, \wedge)$, so we only need to consider existentially closed semilattices $\mathbf{A}$ which are subalgebras of $\mathbf{2}^I$. Without loss of generality, we can assume that

> (**) every homomorphism $\alpha : \mathbf{A} \to \mathbf{2}$ appears more than once as a projection from $\pi_i : \mathbf{2}^I \to \mathbf{2}$,

that is, for more than one choice of i, for all $a \in A$ we have $\alpha(a) = \pi_i(a)$. In particular this means that:

(φ1) For $a \in A$ there are $i \in I$ with $a_i = 0$.

(φ2) For $a \in A$ there are $i \in I$ with $a_i = 1$ and, for every $b \in A$ with $a \nleq b$, $b_i = 0$.

Let $\mathbf{A}$ be an existentially closed semilattice. We can assume $\mathbf{A}$ is sitting inside some $\mathbf{2}^I$ as just described. The following method of proof is based on noting that if $\varepsilon(x, y, \ldots)$ is a 1-extension property, if $a, b, \ldots$ are elements of A such that we can find a $\sigma \in 2^I$ to witness the fact that $\mathbf{2}^I \models \varepsilon(a, b, \ldots)$, then $\mathbf{A} \models \varepsilon(a, b, \ldots)$ since $\mathbf{A}$ is existentially closed and a subalgebra of $\mathbf{2}^I$.

For E1: Let $a, b \in A$. By (φ1), for some i we have $a_i = 0$, and for some j we have $b_j = 0$. Thus $\sigma = \vec{1}$ witnesses the fact that $\mathbf{2}^I \models \varepsilon_1(a, b)$, where $\vec{1}$ is the largest element of $\mathbf{2}^I$.

For E2: Let $a \in A$. By (φ2), for some i we have $a_i = 1$. Thus $\sigma = \vec{0}$ witnesses the fact that $\mathbf{2}^I \models \varepsilon_2(a)$, where $\vec{0}$ is the smallest element of $\mathbf{2}^I$.

For E3: Let $a, b < c \in A$. Since c is a maximal element among a, b, c, from (φ2) it follows that for some $i \neq j$ in I we have the values of a, b, c at the indices i, j and $t \neq i, j$ given by the following table:

	i	j	$t \neq i, j$
a	0	0	a_t
b	0	0	b_t
c	1	1	c_t
σ	1	0	c_t

The last line of the table defines an element σ of 2^I that witnesses the fact that $\mathbf{2}^I \models \varepsilon_3(a, b, c)$ since:

- $a < \sigma$ follows from $a_i < \sigma_i$, $a_j = \sigma_j$ and $a_t \leq c_t = \sigma_t$ (since $a < c$).
- Likewise $b < \sigma$.
- $\sigma < c$ follows from $\sigma_i = c_i$, $\sigma_j < c_j$ and $\sigma_t = c_t$.

For E4: Suppose $a < b, c$ in $\mathbf{A}$ with $c \not\leq b$. Again, since c is a maximal element among a, b, c, from (φ2) it follows that for some $i \neq j$ in I we have the values of a, b, c at the indices i, j and $t \neq i, j$ given by the table:

	i	j	$t \neq i, j$
a	0	0	a_t
b	0	0	b_t
c	1	1	c_t
σ	1	0	a_t

The last line of the table defines an element σ of 2^I that witnesses the fact that $\mathbf{2}^I \models \varepsilon_4(a, b, c)$ since:

- $a < \sigma$ follows from $a_i < \sigma_i$, $a_j = \sigma_j$, and $a_t \leq c_t = \sigma_t$ (since $a < c$).
- $\sigma < c$ follows from $\sigma_i = c_i$, $\sigma_j < c_j$, and $\sigma_t = c_t$.
- $a = b \wedge \sigma$ follows from $a_i = b_i \wedge \sigma_i$, $a_j = b_j \wedge \sigma_j$, and $a_t = b_t \wedge a_t = b_t \wedge \sigma_t$ (since $a < b$).

For E5: Suppose $a < b$ in $\mathbf{A}$. Since $a, b < \vec{1}$ and $\vec{1} \not\leq b$ hold in $\mathbf{2}^I$, there must be an element c of $\mathbf{A}$ such that $a, b < c$ and $c \not\leq b$ hold in $\mathbf{A}$ (since $\mathbf{A}$ is existentially closed). By E4, $\mathbf{A} \models \varepsilon_4(a, b, c)$, so there is a $\sigma \in A$ such that $a = b \wedge \sigma$ and $a < \sigma$. Thus $\mathbf{A} \models \varepsilon_5(a, b)$.

For E6: Suppose $a < b$, $a < c < d$, $a = b \wedge c$, and $e < d$ with $b \wedge e \leq c$ in $\mathbf{A}$. Since d is maximal among the elements a, b, c, d, e, from $(\varphi 2)$ it follows that for some $i \neq j$ in I we have the values of a, b, c, d, e at the indices i, j and $t \neq i, j$ given by the following table:

	i	j	$t \neq i, j$
a	0	0	a_t
b	0	0	b_t
c	0	0	c_t
d	1	1	d_t
e	0	0	e_t
σ	1	0	$\max(c_t, e_t)$

The last line of the table defines an element σ of 2^I that witnesses the fact that $\mathbf{2}^I \models \varepsilon_6(a, b, c, d, e)$ since:

- $c < \sigma$ follows from $c_i < \sigma_i$, $c_j = \sigma_j$ and $c_t \leq \max(c_t, e_t) = \sigma_t$.
- Likewise $e < \sigma$.
- $\sigma < d$ follows from $\sigma_i = d_i$, $\sigma_j < d_j$ and $\sigma_t = \max(c_t, e_t) \leq d_t$ since $c, e < d$.
- $a = b \wedge \sigma$ follows from $a_i = b_i \wedge \sigma_i$, $a_j = b_j \wedge \sigma_j$ and $a_t = b_t \wedge c_t = \max(b_t \wedge c_t, b_t \wedge e_t) = b_t \wedge \max(c_t, e_t) = b_t \wedge \sigma_t$ (since $b \wedge e \leq a = b \wedge c$).

□

3. Axioms for $\mathcal{SL}^{\mathrm{EC}}$

Theorem 3.1. *A semilattice* $\mathbf{A}$ *is existentially closed iff each of the 1-extension properties* $\varepsilon(x, y, \ldots)$ *in the list* E1,...E6 holds in $\mathbf{A}$.

Proof. The previous section showed the $(\Rightarrow)$ direction, so now we assume $\mathbf{A}$ satisfies the list of 1-extension properties and proceed to show that it must be existentially closed. Let $\mathbf{P}$ be a finite subalgebra of $\mathbf{A}$, and let $\mathbf{Q}$ be a 1-element extension of $\mathbf{P}$, say $Q = P \cup \{q\}$, $q \notin A$. We want to show that the extension $\mathbf{Q}$ of $\mathbf{P}$ can be realized in $\mathbf{A}$.

Define three subsets of P by:

$$\begin{aligned} M &:= \{a \in P : q < a\} \\ L &:= \{a \in P : a < q\} \\ K &:= P \smallsetminus (L \cup M). \end{aligned}$$

The goal is

(Γ): to find an element a of $\mathbf{A}$ that satisfies the following three conditions:

(γ1) every element of L is below a,

(γ2) a is below every element of M, and

(γ3) for $k \in K$ one has $k \wedge a = k \wedge q$.

Then $S := P \cup \{a\}$ is a subuniverse of $\mathbf{A}$, and $\mathbf{P} \leq \mathbf{S} \leq \mathbf{A}$ is a realization of the extension $\mathbf{Q}$ of $\mathbf{P}$ in $\mathbf{A}$.

It suffices to verify (Γ) for the case that each of K, L, M is nonempty because of the following. Let $p_0 = \bigwedge P$. Since $\mathbf{A} \models \varepsilon_2(p_0)$, there is an $a_0 \in A$ with $a_0 < p_0$, so a_0 is less than every member of P. Also by E1, using repeated applications if needed, there is an element $a_1 \in A$ that is above every element of P. Since $a_0 < a_1$, from $\mathbf{A} \models \varepsilon_5(a_0, a_1)$ there is an element $a_2 \in A$ with $a_0 = a_1 \wedge a_2$. This implies that a_2 is incomparable to every element of P. Note that $P' := P \cup \{a_0, a_1, a_2\}$ is a subuniverse of $\mathbf{A}$, and let $\mathbf{P}'$ be the corresponding semilattice. Let $Q' := Q \cup \{a_0, a_1, a_2\}$, define the relations $a_0 < q < a_1$, and define $a_2 \wedge q = a_0$. Then $\mathbf{Q}'$ is a semilattice that is a 1-element extension of $\mathbf{P}'$; indeed, $Q' = P' \cup \{q\}$. By realizing the 1-element extension $\mathbf{Q}'$ of $\mathbf{P}'$ in $\mathbf{A}$ one also realizes the 1-element extension $\mathbf{Q}$ of $\mathbf{P}$ in $\mathbf{A}$. The subsets K', L', M' of P' are all nonempty.

So now we assume K, L, M are all nonempty. Then the following hold:

(V1) M is an upper segment of $\mathbf{P}$.

(V2) L is a lower segment of $\mathbf{P}$.

(V3) Every element of L is below every element of M.

(V4) $\mu_0 := \bigwedge M$ is an element of M.

(V5) For $k \in K$ let $\lambda_k := k \wedge q$, and let $L_k := \{\ell \in L : \ell < k\}$. Then $\lambda_k \in L_k$, so L_k is nonempty, and $\lambda_k = \max(L_k)$.

(V6) For $k \in K$ we have $k \wedge \mu_0 = \lambda_k$.

We break the proof of Γ into two cases:

(Case 1): Suppose L has a largest element $\ell^\star$. Let $k_1, \ldots, k_m$ denote the distinct elements of K, and for $1 \leq i \leq m$ let $\ell_i := \lambda_{k_i}$. (The ℓ_i need not be distinct.) Clearly $\ell_i = k_i \wedge \ell^\star$, for $1 \leq i \leq m$. We will show, by induction on n, that

> (ξ): for $1 \leq n \leq m$, there are elements $a_1, \ldots, a_n \in A$ such that $\ell^\star < a_n \leq \cdots \leq a_1 < \mu_0$ and for $1 \leq i \leq n$ we have $\ell_i = k_i \wedge a_n$.

For $n = 1$ we want to show that there is an $a_1 \in A$ such that $\ell^\star < a_1 < \mu_0$ and $\ell_1 = k_1 \wedge a_1$.

> $\boxed{\text{If } \ell_1 = \ell^\star}$, note that from $\ell^\star < \mu_0$ and $\mathbf{A} \models \varepsilon_3(\ell^\star, \mu_0, \mu_0)$ there is an element $\sigma \in A$ such that $\ell^\star < \sigma < \mu_0$. Then
>
> $$\ell_1 = k_1 \wedge \ell^\star \leq k_1 \wedge \sigma \leq k_1 \wedge \mu_0 = \ell^\star = \ell_1,$$
>
> so $\ell_1 = k_1 \wedge \sigma$ and $\ell^\star < \sigma < \mu_0$. Thus we can choose $a_1 = \sigma$.
>
> $\boxed{\text{If } \ell_1 < \ell^\star}$ note that from $\ell_1 < k_1$, $\ell_1 < \ell^\star < \mu_0$, $\ell_1 = k_1 \wedge \ell^\star$ and $\mathbf{A} \models \varepsilon_6(\ell_1, k_1, \ell^\star, \mu_0, \ell_1)$, there is an element $\sigma \in A$ such that $\ell^\star < \sigma < \mu_0$ and $\ell_1 = k_1 \wedge \sigma$. We can let $a_1 = \sigma$.

For $1 < n \leq m$, assume we have found the desired $a_1, \ldots, a_{n-1}$. Then we have $\ell_n = k_n \wedge \ell^\star \leq k_n \wedge a_{n-1}$. If $\ell_n = k_n \wedge a_{n-1}$ then we can choose $a_n = a_{n-1}$. Otherwise, after noting that $\ell_n < k_n$, $\ell_n < \ell^\star < a_{n-1}$ and (again) $\ell_n = k_n \wedge \ell^\star$, from $\mathbf{A} \models \varepsilon_6(\ell_n, k_n, \ell^\star, a_{n-1}, \ell_n)$ it follows that there exists an element $\sigma \in A$ such that $\ell_n = k_n \wedge \sigma$ and $\ell^\star < \sigma < a_{n-1}$. Also, for $1 \leq i < n$ we have

$$\ell_i = k_i \wedge \ell^\star \leq k_i \wedge \sigma \leq k_i \wedge a_{n-1} = \ell_i,$$

so $\ell_i = k_i \wedge \sigma$. Thus we can choose $a_n = \sigma$.

Having established (ξ), we can choose $a := a_m$ to show (Γ) holds in Case 1.

(Case 2): Next we look at the case where L does not have a largest element. For each $k \in K$ we will first show that there is an $a_k \in A$ with $a_k < \mu_0$, $\lambda_k = k \wedge a_k$, and every element of L below a_k.

So fix an element $k \in K$, and let $L = \{\ell_1, \ldots, \ell_n\}$ with $\ell_1 = \lambda_k$. Since $\ell_1 < k$, $\ell_1 < \mu_0$, $\mu_0 \not\leq k$ and $\mathbf{A} \models \varepsilon_4(\ell_1, k, \mu_0)$, there is an element $\sigma_1 \in A$ with $\ell_1 < \sigma_1 < \mu_0$ and $k \wedge \sigma_1 = \ell_1$. If $n = 1$, then we can choose $a_k = \sigma_1$. Otherwise note that for any $\ell \in L$ we have, by V5, $k \wedge \ell \leq \ell_1$, since $k \wedge \ell \in L$ and $k \wedge \ell \leq k$. Then from $\mathbf{A} \models \varepsilon_6(\ell_1, k, \sigma_1, \mu_0, \ell_2)$ it follows that there is an element $\sigma_2 \in A$ with $\sigma_1, \ell_2 < \sigma_2 < \mu_0$ and $k \wedge \sigma_2 = \ell_1$. Thus $\ell_1, \ell_2 < \sigma_2 < \mu_0$ and $k \wedge \sigma_2 = \ell_1$. If $n = 2$ the we can choose $a_k = \sigma_2$. Otherwise we repeat this application of E6 until one has an element $\sigma_n \in A$ that is greater than every element of L, that is below μ_0, and such that $k \wedge \sigma_n = \ell_1 = \lambda_k$; and then let $a_k = \sigma_n$.

Now that we have found an a_k for each $k \in K$, let $a := \bigwedge_{k \in K} a_k$. Clearly every member of L is $\leq a$, and $a < \mu_0$. a cannot belong to L since we have assumed L has no largest element. Thus every member of L is less than a. Also, for $k \in K$, we have $k \wedge a \leq k \wedge a_k = \lambda_k \leq k \wedge a$, so $\lambda_k = k \wedge a$, proving (Γ).

□

References

[1] Michael H. Albert and Stanley N. Burris, *Finite axiomatizations for existentially closed posets and semilattices.* Order **3** (1986), 169–178.