# A DOMINATED CONVERGENCE THEOREM FOR EISENSTEIN SERIES

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Abstract. Basing on the new approach to modular forms presented in [6] that uses rational functions we prove a dominated convergence theorem for certain modular forms in the Eisenstein space. It states that certain rearrangements of the Fourier series will converge very fast near the cusp $\tau = 0$. As an application, we consider $L$-functions associated to products of Eisenstein series and present natural generalized Dirichlet series representations, that converge in an expanded half plane.

## Introduction

We recall that an elliptic modular form $f$ of weight $k \in \mathbb{Z}$ for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with multiplier system $v : \Gamma \to \mathbb{C}^\times$ is a holomorphic function on the extended upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \cup \mathbb{Q} \cup \{\infty\}$, which satisfies the transformation law

$$f|_k M(\tau) = v(M) f(\tau).$$

Here $f|_k M$ denotes the usual Petersson slash operator

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (ad - bc)^{\frac{k}{2}} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

One can show that there are no non-constant modular forms for $k \leqslant 0$ and that the spaces $M_k(\Gamma, v)$ are finite-dimensional. A useful tool for computing the exact value of the dimensions is the Riemann-Roch formula, for more explicit details see for example [4]. Modular forms play an extraordinary important role in many fields of mathematics and physics such as number theory, geometry and string theory. Also many generalizations of the classical modular forms have been found, such as Siegel modular forms (see also [1] and [8]) for matrix valued arguments that transform under congruence subgroups of the symplectic group $Sp_n$; and Hilbert modular forms (for a great introduction, the reader may wish to consult [7]) that transform under congruence subgroups of $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a number field $K$.

Basically, two elementary ideas for constructing modular forms dominate in literature. One of them uses the so called Poincare series, which give in the simplest case Eisenstein series. The other one goes via Fourier analysis and quadratic forms. This leads to theta functions. In [6] a third elementary approach to modular forms was presented. It grounds on a class of very simple functions which we will call *weak functions*. A weak function $\omega$ is a 1-periodic meromorphic function in the entire plane, which has the following properties:

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(i) All poles of $\omega$ are simple and lie in $\mathbb{Q}$.

(ii) The function $\omega$ tends to 0 rapidly as the absolute value of the imaginary part increases, so
$$\omega(x+iy) = O(|y|^{-M})$$
for all $M > 0$ as $|y| \to \infty$.

By Liouville's theorem one quickly sees that each weak $\omega$ is essentially just a rational function $R \in \mathbb{C}(X)$ with (only simple) poles only in roots of unity, such that $R(0) = R(\infty) = 0$. Here we put $\omega(z) := R(e(z))$, where $e(z) := e^{2\pi i z}$. One defines $W_N$ to be the space of weak functions with the property, that $\omega(z/N)$ only has poles in $\mathbb{Z}$. We associate to $\omega$ a periodic divisor function $\beta_\omega(x) := -2\pi i \mathrm{res}_{z=x}\omega(z)$. Now one can show the following construction theorem for modular forms for the congruence subgroup
$$\Gamma(N_1N_2) \subset \Gamma_1(N_1,N_2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_1,N_2) \middle| a \equiv d \equiv 1 \pmod{N_1N_2} \right\}.$$

**Theorem 0.1.** *Let $k \geqslant 3$ and $N_1, N_2 > 1$ be integers. There is a homomorphism*
$$W_{N_1} \otimes W_{N_2} \longrightarrow M_k(\Gamma_1(N_1,N_2))$$
$$\omega \otimes \eta \longmapsto \vartheta_k(\omega \otimes \eta; \tau) := \sum_{x \in \mathbb{Q}^\times} x^{k-1}\beta_\eta(x)\omega(x\tau).$$
*In the case that $k = 1$ and $k = 2$ the map stays well-defined under the restriction that the function $z \mapsto z^{k-1}\eta(z)\omega(z\tau)$ is removable in $z = 0$.*

The main tools for the proof are Weil's converse theorem and the following observation.

**Theorem 0.2.** *We define the involution $\omega \mapsto \hat{\omega}$ by $\hat{\omega}(z) := \omega(-z)$. Let $k \in \mathbb{Z}$ be an integer. For all weak $\omega$ and $\eta$ we have the following transformation property.*
$$\vartheta_k(\omega \otimes \eta; \tau)|_k S = \vartheta_k(\eta \otimes -\hat{\omega}; \tau) + g_{\omega,\eta}(\tau)|_k S,$$
*where $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $g_{\omega,\eta}$ is a rational function which can be evaluated explicitly by*
$$g_{\omega,\eta}(\tau) = 2\pi i \mathrm{res}_{z=0}\left(z^{k-1}\eta(z)\omega(z\tau)\right).$$

In this paper, we continue the study of this new perspective to modular forms and apply it to Dirichlet series. We first want to investigate the space $\vartheta_k(W_{N_1} \otimes W_{N_2})$ and it will turn out, that it is generated by Eisenstein series.

**Theorem 0.3** (cf. 1.1)**.** *Let $k \geqslant 3$. The image space $\vartheta_k(W_{N_1} \otimes W_{N_2})$ is generated by the elements $E_k(\chi, \psi; \frac{N_1 d_2}{N_2 d_1}\tau)$ where $\chi$ and $\psi$ run over all non-trivial characters modulo $d_1|N_1$ and $d_2|N_2$, respectively, such that $\chi(-1)\psi(-1) = (-1)^k$.*

The cases $k = 1$ and $k = 2$ can be treated similarly. We want to apply the series representations of $\vartheta_k$ in terms of rational functions to Dirichlet series. To every modular form $f(\tau) = \sum_{n \geqslant 0} a(n) q^{n/N}$ of weight $k$ for some congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ we can associate an $L$-function $L(f, s)$ given by

$$L(f, s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

One can show that this function converges absolutely on the half plane $\mathrm{Re}(s) > k$, has meromorphic continuation to the entire plane and satisfies a certain functional equation. The complete $L$-function $\Lambda(f, s) = (2\pi/N)^{-s}\Gamma(s)L(f, s)$ can be written as an integral

$$\Lambda(f, s) = \int_0^{\infty} (f(ix) - a(0)) x^{s-1} \mathrm{d}x.$$

In the case that $f$ is a cuspidal Hecke eigenform its $L$-function is entire, has an Euler product expansion and encodes deep arithmetic information.

We give a proof for a dominated convergence theorem for Eisenstein series arising from rational functions. In order to formulate it, we need the concept of the degree of a $N$-periodic function $\beta$. The degree of such a $\beta$ is defined as the largest integer $0 \leqslant d$ such that

$$\sum_{n=1}^{N} \beta(n) n^r = 0, \qquad \text{for all } 0 \leqslant r \leqslant d.$$

**Theorem 0.4** (cf. 2.9)**.** *Let $\omega \otimes \eta \in W_{N_1} \otimes W_{N_2}$ be a pair of weak functions such that $\omega$ is removable in $z = 0$ and $\kappa_{N_2}\beta_\eta$ has degree $d$. Then for all $\alpha \in \mathbb{N}_0$ there is a constant $C_{\beta,\omega,\alpha} > 0$ such that uniformly for all $T \in \mathbb{N}$ and $y \in [0, 1]$*

$$\left| \sum_{n=1}^{N_2 T} n^{\alpha} \beta_\eta(n/N_2) \omega(niy) \right| \leqslant C_{\beta,\omega,\alpha} y^{d-\alpha}.$$

An application of this theorem is a new, in some sense more natural, representation of $L$-functions associated to products of Eisenstein series in terms of a generalized Dirichlet series. Modifying the sum a bit leads to convergence in a much wider region. In particular, we have the following theorem.

**Theorem 0.5** (cf. 3.16)**.** *Let $l \in \mathbb{N}$, $\boldsymbol{k} = (k_1, ..., k_l) \in \mathbb{N}^l$ a vector of positive integers and $\chi_j, \psi_j$, $j = 1, ..., l$, be non-principal, primitive characters modulo $M$ and $N$, respectively, such that $\chi_j(-1)\psi_j(-1) = (-1)^{k_j}$. Then, if we put $|\boldsymbol{k}| = k_1 + \cdots + k_l$, we have for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > |\boldsymbol{k}| - l - \frac{1}{2}\sum_{j=1}^{l}(\psi_j(-1) + 1)$*

$$L\left( \prod_{j=1}^{l} E_{k_j}(\chi_j, \psi_j; \tau), s \right) = \left( -\frac{2\pi i}{N} \right)^{|\boldsymbol{k}|} \prod_{j=1}^{l} \frac{\mathcal{G}(\psi_j)}{(k_j - 1)!} \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^l \times \mathbb{N}^l} \Pi_{\boldsymbol{k}}(\boldsymbol{u}) \overline{\psi}(\boldsymbol{u}) \chi(\boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v} \rangle^{-s},$$

*where $\Pi_{\boldsymbol{k}}(\boldsymbol{u}) = u_1^{k_1} \cdots u_l^{k_l}$, $\psi(\boldsymbol{u}) = \psi_1(u_1) \cdots \psi_l(u_l)$ and $\chi(\boldsymbol{v}) = \chi_1(v_1) \cdots \chi_l(v_l)$. For convergence in the extended half plane the sum has to be modified slightly, this is explained below.*

Note that this representation of the $L$-function of the considered product is more natural since it is a direct generalization of the formula in the case $l = 1$, where the series directly splits into a product of two Dirichlet $L$-functions. An important question, which is still unsolved in the very general case, is that which modular forms can be written as sums of products of Eisenstein series. But there is a lot of progress in this field. Dickson and Neururer have shown in [5], that, if $k \geqslant 4$, $N = p^a q^b N'$ where $p^a, q^b$ are powers of primes and $N'$ is square free, the space $M_k(\Gamma_0(N))$ is generated by $E_k(\Gamma_0(N), \chi_{0,N})$ and a subspace containing products of two Eisenstein series. A similar result for $M_k(p)$ and $k \geqslant 4$, where $p$ is prime, is due to Imamoḡlu and Kohnen [9]. For a correspondence between values of $L$-functions for products of pairs of different Eisenstein series see [3].

The paper is organized as follows. In the first section we identify generators for the space of modular forms that arise from rational functions. In the second section we prove a Dominated convergence theorem for Eisenstein series, which provides an upper bound for several partial sums of the series involving weak functions for modular forms near the cusp $\tau = 0$. In the last section we apply this theorem to $L$-functions associated to products of Eisenstein series.

**Notation.** We use the introduced notations $W_N$ for the vector space of weak functions of level $d|N$, and $W_N^{\pm}$ for the odd and even function part.

Throughout $l$ is a positive integer. We briefly define $\boldsymbol{k} = (k_1, ..., k_l) \in \mathbb{N}^l$ to be a vector of positive integers. We write $|\boldsymbol{k}| = k_1 + \cdots + k_l$.

For real valued vectors $\boldsymbol{u} = (u_1, ..., u_l) \in \mathbb{R}^l$ we briefly write $\max(\boldsymbol{u}) := \max\{u_1, ..., u_l\}$.

We sometimes use the notation $\operatorname{sgn}(f) = \pm 1$ to indicate that $f$ is an even or odd function, respectively.

We define for any set $L$ to be $L^{\mathbb{C}_0}$ the space of all functions $f : L \to \mathbb{C}$, that are zero everywhere except *finitely* many $x \in L$. The subspace $L_0^{\mathbb{C}_0} \subset L^{\mathbb{C}_0}$ is given by all $f$ satisfying $\sum_{x \in L} f(x) = 0$. For positive integers $N$ we abbreviate $\mathbb{F}_N := \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{F}_{\frac{1}{N}} := \mathbb{Z}[\frac{1}{N}]/\mathbb{Z}$. Especially when going over to Fourier series it will be useful to identify functions in $\mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_0}$ with those in $\mathbb{F}_N^{\mathbb{C}_0}$ via the obvious map

$$\kappa_N : \mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_0} \xrightarrow{\sim} \mathbb{F}_N^{\mathbb{C}_0}$$

$$(\kappa_N f)(x) := f\left(\frac{x}{N}\right).$$

We will identify functions $f \in \mathbb{F}_N^{\mathbb{C}_0}$ with $N$-periodic functions $f : \mathbb{Z} \to \mathbb{C}$. For integers $M$ we will set $f[M](x) := f(Mx)$ when $f : \mathbb{Z} \to \mathbb{C}$.

For any Dirichlet character $\psi$ modulo $N$ we define the Gauss sum $\mathcal{G}(\psi) := \sum_{n=0}^{N-1} \psi(n) e^{2\pi i n/N}$. For the generalized Gauss sum it will be more convenient to use the more general notion of a discrete Fourier transform

$$\mathcal{F}_N : \mathbb{F}_N^{\mathbb{C}_0} \xrightarrow{\sim} \mathbb{F}_N^{\mathbb{C}_0}.$$

$$(\mathcal{F}_N f)(j) := \sum_{n=0}^{N-1} f(n) e^{-2\pi i j n/N}.$$

Note that we have an inverse transformation

$$(\mathcal{F}_N^{-1} g)(j) := \frac{1}{N} \sum_{n=0}^{N-1} g(n) e^{2\pi i j n/N}.$$

We use the same notation for functions $f \in \mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_0}$ and have $\kappa_N \mathcal{F}_N f = \mathcal{F}_N \kappa_N f$. For $d|N$ we also use the trivial injection

$$\iota_N^d : (\mathbb{F}_d)^{\mathbb{C}_0} \longrightarrow (\mathbb{F}_N)^{\mathbb{C}_0}$$

$$(\iota_N^d f)(x) := \begin{cases} f(\frac{xd}{N}), & x \equiv 0 \mod \frac{N}{d} \\ 0, & \text{else} \end{cases}$$

for purposes of notation. Note that if $f \in \mathbb{F}_d^{\mathbb{C}_0}$ and $d|N$ we have $\mathcal{F}_N \iota_N^d f = \mathcal{F}_d f$.

For the complex variable $z = x + iy$ we write $e(z) := e^{2\pi i z}$ and for the complex variable $\tau$ we define $q := e^{2\pi i \tau}$.

We denote $\mathfrak{C}_N$ as the group of all characters modulo $N$. Also we write $\overline{\mathfrak{C}_N}$ for the set of all characters modulo $d$, where $d$ divides $N$. We write $\chi_{0,d}$ for the principal character modulo $d$. In particular, $\chi_{0,1}$ denotes the *trivial* character.

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## 1. The space of weak modular forms

For Dirichlet characters $\chi$ and $\psi$ modulo positive integers $M$ and $N$, respectively, and some integer $k \geqslant 3$ one defines the corresponding Eisenstein series for $\tau \in \mathbb{H}$ (= upper half plane) via

$$E_k(\chi, \psi; \tau) := \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \chi(m)\psi(n)(m\tau + n)^{-k}. \tag{1.1}$$

This series converges absolutely and uniformly on compact subsets of the upper half plane and defines a holomorphic function in that region. One can show that (1.1) leads to a non-zero function if and only if $\chi(-1)\psi(-1) = (-1)^k$ and that the $E_k$ are modular forms of weight $k$ for the congruence subgroups

$$\Gamma_0(M, N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| b \equiv 0 \pmod{M}, c \equiv 0 \pmod{N} \right\}$$

with Nebentypus character $\chi\overline{\psi}$ of $\Gamma_0(M, N)$. The cases $k = 1, 2$ are treated differently, see also [10] on p. 274 ff. or [4].

Every Eisenstein series admits a Fourier series. The coefficients are well-known and given by

$$2L(\psi,k)\chi(0)+\frac{2(-2\pi i)^k}{N^k(k-1)!}\sum_{m=1}^{\infty}\left(\sum_{d|m}d^{k-1}(\mathcal{F}_N\psi)(-d)\chi\left(\frac{m}{d}\right)\right)q^{m/N}, \tag{1.2}$$

where as usual $q := e^{2\pi i\tau}$ and $L(\psi,s)$ is the Dirichlet $L$-function. Note that in the case that $\psi$ is primitive one has $(\mathcal{F}_N\psi)(a) = \overline{\psi(a)}(\mathcal{F}_N\psi)(1)$ and one obtains the simpler expression $\sum_{d|n} d^{k-1}\overline{\psi}(d)\chi(n/d)$ for the coefficients up to a constant.
It is clear that every $\vartheta_k(\omega\otimes\eta;\tau)$ admits a Fourier expansion. Since we only focus on the non-trivial cases we assume $\omega\otimes\eta\in(W_M\otimes W_N)^{\pm}$ if $(-1)^k=\pm1$. It is given by

$$\vartheta_k(\omega\otimes\eta;\tau) = 2N^{1-k}\sum_{m=1}^{\infty}\sum_{d|m}\left(d^{k-1}(\kappa_N\beta_\eta)(d)\,(\mathcal{F}_M\kappa_M\beta_\omega)\left(\frac{m}{d}\right)\right)q^{m/N}. \tag{1.3}$$

According to (1.2) we conclude for non-principal characters

$$E_k(\chi,\psi;\tau) = \frac{\psi(-1)(-2\pi i)^k}{N(k-1)!}\vartheta_k(\omega_{\mathcal{F}_M^{-1}(\chi)}\otimes\omega_{\mathcal{F}_N(\psi)};\tau). \tag{1.4}$$

In particular, if $\chi$ and $\psi$ are primitive and hence conjugate up to a constant under the Fourier transform, this simplifies to

$$E_k(\chi,\psi;\tau) = \frac{\chi(-1)(-2\pi i)^k\mathcal{G}(\psi)}{N(k-1)!\mathcal{G}(\overline{\chi})}\vartheta_k(\omega_{\overline{\chi}}\otimes\omega_{\overline{\psi}};\tau). \tag{1.5}$$

Already here the connection between Eisenstein series and weak functions is intuitively clear.

In this section we want to find generators for the space $\vartheta_k(W_{N_1}\otimes W_{N_2})$. We call their elements weak modular forms. In other words, the vector space $V_k(\Gamma_1(N_1,N_2))$ of all weak modular forms is the image of the linear map

$$W_{N_1}\otimes W_{N_2}\longrightarrow M_k(\Gamma_1(N_1,N_2)).$$

Let $(\mathbb{F}_N)^{\mathbb{C}_0}_{*,0}\subset\mathbb{F}_N^{\mathbb{C}_0}$ be the subspace of all functions with $f(0)=0$. It is an easy exercise to verify that the discrete Fourier transform defines an isomorphism

$$\mathcal{F}_N : (\mathbb{F}_N)^{\mathbb{C}_0}_{*,0}\xrightarrow{\sim}(\mathbb{F}_N)^{\mathbb{C}_0}_0.$$

With this we conclude that $(\omega_{\mathcal{F}_{N_1}^{-1}\chi}\otimes\omega_{\mathcal{F}_{N_2}\psi})_{(\chi,\psi)\in\overline{\mathfrak{C}_{N_1}}\backslash\{\chi_{0,1}\}\times\overline{\mathfrak{C}_{N_2}}\backslash\{\chi_{0,1}\}}$ is a basis for $W_{N_1}\otimes W_{N_2}$. The next theorem provides generators for the space $V_k(\Gamma_1(N_1,N_2))$.

**Theorem 1.1.** *Let $k\geqslant 3$. The space $V_k(\Gamma_1(N_1,N_2))$ is generated by the elements $E_k(\chi,\psi;\frac{N_1d_2}{N_2d_1}\tau)$ where $\chi$ and $\psi$ run over all non-trivial characters modulo $d_1|N_1$ and $d_2|N_2$, respectively, such that $\chi(-1)\psi(-1)=(-1)^k$.*

*Proof.* It is clear that the Fourier transform preserves the subspaces of odd and even functions. Hence, for characters $\chi(-1)\psi(-1) = (-1)^k$, we have the Fourier expansion

$$\vartheta_k(\omega_{\mathcal{F}_{N_1}^{-1}\chi} \otimes \omega_{\mathcal{F}_{N_2}\psi}; \tau) = 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left( d^{k-1} (\mathcal{F}_{N_2} \iota_N^{d_2} \psi)(d) \left(\mathcal{F}_{N_1} \mathcal{F}_{N_1}^{-1} \iota_{N_1}^{d_1} \chi\right) \left(\frac{m}{d}\right)\right) q^{m/N_2}$$

$$= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left( d^{k-1} (\mathcal{F}_{d_2} \iota_{N_2}^{d_2} \psi \left[\tfrac{N_2}{d_2}\right])(d) \iota_{N_1}^{d_1} \chi \left(\frac{m}{d}\right)\right) q^{m/N_2}$$

$$= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left( d^{k-1} (\mathcal{F}_{d_2} \psi)(d) \iota_{N_1}^{d_1} \chi \left(\frac{mN_1}{dd_1}\right)\right) q^{mN_1/N_2 d_1}$$

$$= 2N_2^{1-k} \sum_{m=1}^{\infty} \sum_{d|m} \left( d^{k-1} (\mathcal{F}_{d_2} \psi)(d) \chi \left(\frac{m}{d}\right)\right) q^{m(N_1 d_2/N_2 d_1)/d_2}.$$

This proves the theorem. ∎

For our investigations we are especially interested in a subspace of $V_k$ which we will denote by $U_k$ and which contains all weak modular forms which arise from weak functions that are removable in $z = 0$. In the following we shall give generators for $U_k$. Let $H_{N_i} \subset W_{N_i}$ be the subspace of weak functions that are removable in $z = 0$. Then we have

$$W_{N_i} = \mathbb{C}\omega_{\mathcal{F}_{N_i}^{\pm 1}\chi_{0,N_i}} \oplus H_{N_i}.$$

In other words, the space $H_N$ is given by weak elements $\omega(z)$ such that $\beta_\omega(0) = 0$. On the periodic function side, we define the subspace of these coefficients by $(\mathbb{F}_{\frac{1}{N_i}})_{0,0}^{\mathbb{C}_0}$. Note the the Fourier transform $\mathcal{F}_{N_i}$ defines an automorphism on the subspace $(\mathbb{F}_{N_i})_{0,0}^{\mathbb{C}_0}$. So firstly, consider the basis $(\omega_{\mathcal{F}_{N_1}^{-1}\chi} \otimes \omega_{\mathcal{F}_{N_2}\psi})_{\chi,\psi}$ of $H_{N_1} \otimes H_{N_2}$, where $\chi$ and $\psi$ are either non-principal characters modulo $d_1|N_1$ and $d_2|N_2$ or functions $\frac{\varphi(N_i)}{\varphi(d_i)} \iota_{N_i}^{d_i} \chi_{0,d_i} - \chi_{0,N_i}$ for $i = 1, 2$.

**Theorem 1.2.** *Let $k \geqslant 1$. The space $U_k = \vartheta_k(H_{N_1} \otimes H_{N_2})$ is generated by the elements $E_k(\chi, \psi; \frac{N_1 d_2}{N_2 d_1}\tau)$ and the linear combinations*

$$\frac{\varphi(N_1)}{\varphi(d_1)} E_k(\chi_{0,d_1}, \psi; \tfrac{N_1 d_2}{N_2 d_1}\tau) - E_k(\chi_{0,N_1}, \psi; \tfrac{d_2}{N_2}\tau),$$

$$\frac{\varphi(N_2)}{\varphi(d_2)} E_k(\chi, \chi_{0,d_2}; \tfrac{N_1 d_2}{N_2 d_1}\tau) - \left(\frac{N_2}{d_2}\right)^k E_k(\chi, \chi_{0,N_2}; \tfrac{N_1}{d_1}\tau),$$

*and*

$$\frac{\varphi(N_1)}{\varphi(d_1)} \frac{\varphi(N_2)}{\varphi(d_2)} E_k(\chi_{0,d_1}, \chi_{0,d_2}; \tfrac{N_1 d_2}{N_2 d_1}\tau) - \frac{\varphi(N_1)}{\varphi(d_1)} \left(\frac{N_2}{d_2}\right)^k E_k(\chi_{0,d_1}, \chi_{0,N_2}; \tfrac{N_1}{d_1}\tau)$$
$$- \frac{\varphi(N_2)}{\varphi(d_2)} E_k(\chi_{0,N_1}, \chi_{0,d_2}; \tfrac{d_2}{N_2}\tau) + \left(\frac{N_2}{d_2}\right)^k E_k(\chi_{0,N_1}, \chi_{0,N_2}; \tau),$$

*where $1 < d_i < N_i$ and $\chi, \psi$ are non-principal characters modulo $d_1$ and $d_2$, respectively, such that* $\operatorname{sgn}(\chi\psi) = (-1)^k$.

*Proof.* Since all considered weak functions are removable in $z = 0$, we can apply the theorem to all positive weights $k = 1, 2, ....$ The proof works similar as the one of 1.1 and we omit it. $\square$

**Theorem 1.3.** *We have the following.*

*(i) The space of weak modular forms of weight $k = 1$ is given by $V_1(\Gamma_1(N_1, N_2)) = \vartheta_1(H_{N_1} \otimes H_{N_2})$. In particular, it is generated by the elements given in 1.2 for $k = 1$.*

*(ii) The space of weak modular forms of weight $k = 2$ is given by $V_2(\Gamma_2(N_1, N_2)) = \vartheta_2(H_{N_1} \otimes H_{N_2} \oplus \mathbb{C}\omega_{\mathcal{F}_{N_1}^{-1}\chi_{0,N_1}} \otimes H_{N_2} \oplus H_{N_1} \otimes \mathbb{C}\omega_{\mathcal{F}_{N_2}\chi_{0,N_2}})$. In particular, it is generated by the elements in 1.2 for $k = 2$ and $E_2(\chi_{0,N_1}, \psi; \frac{d_2}{N_2}\tau), E_2(\chi, \chi_{0,N_2}; \frac{N_1}{d_1}\tau)$, where $\chi$ and $\psi$ are non-principal characters modulo $d_1|N_1$ and $d_2|N_2$, respectively.*

In the last section we would like to investigate $L$-functions of products of weak functions. To formalize this, we give the following final definition.

**Definition 1.4.** *Let $\boldsymbol{k} = (k_1, ..., k_l)$ a vector of weights. We then define $V_{\boldsymbol{k}}(\Gamma_1(N_1, N_2))$ as the vector space of all modular forms that can be written as a sum $\sum_j c_j f_{1,j} \cdots f_{l,j}$, where each $f_{r,j}$ is an element of $V_{k_r}(\Gamma_1(N_1, N_2))$. Analogously, we define the subspace $U_{\boldsymbol{k}}(\Gamma_1(N_1, N_2)) \subset V_{\boldsymbol{k}}(N_1, N_2)$ by demanding $f_{r,j} \in U_{k_r}(\Gamma_1(N_1, N_2))$.*

## 2. A dominated convergence theorem

**Definition 2.1.** *Let $N$ be a positive integer and $g : \mathbb{Z} \to \mathbb{C}$ an $N$-periodic function. We define the degree of $g$ to be the largest positive integer $d$ such that for all $0 \leqslant \alpha \leqslant d$:*

$$\sum_{j=1}^{N} g(j)j^\alpha = 0.$$

*The degree of the zero function is defined to be $\infty$. In the case $\sum_{j=1}^{N} g(j) \neq 0$ we say that $g$ has degree $-\infty$. We denote $[N, d]$ as the vector space of $N$-periodic functions with degree at least $d$.*

We have $[N, N-1] = 0$ (Vandermonde-Matrix).

**Proposition 2.2.** *Let $\beta : \mathbb{Z} \to \mathbb{C}$ have period $N$ and degree $d$. Then we have for all $T \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant d$:*

$$\sum_{j=1}^{NT} \beta(j)j^\alpha = 0.$$

*Proof.* It is sufficient to prove

$$\sum_{j=NT+1}^{N(T+1)} \beta(j)j^\alpha = 0.$$

for arbitrary $T \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant d$. This follows from

$$\sum_{j=NT+1}^{N(T+1)} \beta(j)j^\alpha = \sum_{j=1}^{N} \beta(NT+j)(NT+j)^\alpha = \sum_{u=0}^{\alpha} \binom{\alpha}{u} (NT)^{\alpha-u} \sum_{j=1}^{N} \beta(j)j^u = 0.$$

□

In the following we will abbreviate

$$S_\beta(u;q) := \sum_{j=1}^{q} \beta(j)j^u.$$

**Example 2.3.** *Each non-principal Dirichlet character mod $N$ has degree 0 and each (non-principal) even character has degree 1.*

**Proposition 2.4.** *Let $\beta : \mathbb{Z} \to \mathbb{C}$ be in $[N,d]$. Then the polynomial*

$$W_\beta(u;x) = \sum_{p=1}^{N} S_\beta(u;p)x^p$$

*has a zero of degree at least $d-u$ in $X=1$.*

*Proof.* Let $0 \leqslant \ell \leqslant d-u-1$ be an integer. Then we obtain

$$\begin{aligned} W_\beta^{(\ell)}(u;1) &= \sum_{p=1}^{N} \left( \sum_{r=1}^{p} \beta(r)r^u p(p-1)\cdots(p-\ell+1) \right) \\ &= \sum_{r=1}^{N} \beta(r)r^u \sum_{p=r}^{N} \left[ p^\ell + b_{\ell-1}p^{\ell-1} + \cdots + b_1 p \right] \\ &= \sum_{r=1}^{N} \beta(r)r^u \left( Q_\ell(N) - Q_\ell(r-1) \right) \end{aligned}$$

for some polynomial $Q_\ell$ of degree $\ell+1 \leqslant d-u$

$$= 0.$$

□

We will write

$$W_\beta(u;x) = (1-x)^{\alpha-u}\widetilde{W}_\beta(u;x)_\alpha$$

in further applications when $0 \leqslant u \leqslant \alpha \leqslant d$.
Our investigations foot on the properties of some explicit polynomials. For a fixed non-negative integer $\alpha$ we define a sequence by

$$p_T(\alpha;x) = (1-x)^{\alpha+1} \sum_{\ell=1}^{T} \ell^\alpha x^\ell, \qquad T = 1,2,3,...$$

For example we have $p_T(0;x) = x - x^{T+1}$ for $T = 1,2,....$

**Lemma 2.5.** *The sequence $(p_T(\alpha;x))_{T\in\mathbb{N}}$ converges to some polynomial function on the interval $[0,1)$ from below for all $\alpha \geqslant 0$. In particular the terms $p_T$ are uniformly bounded in the sense*

$$\sup_{T\in\mathbb{N}} \sup_{x\in[0,1]} |p_T(\alpha;x)| \leqslant C_\alpha$$

*for some constant $C_\alpha > 0$.*

This uniform boundedness is a very important property as we will see later.

*Proof.* It is clear that $p_T(\alpha;x)$ is increasing for fixed $x$. The power series

$$\sum_{\ell=1}^{\infty} \ell^\alpha x^\ell$$

converges for $x \in [0,1)$ to a rational function $\frac{Q_\alpha(x)}{(1-x)^{\alpha+1}}$ where $Q_\alpha(x)$ is some polynomial which is non-negative in $[0,1]$. This follows inductively by $\sum_{\ell=1}^{\infty} x^\ell = \frac{x}{1-x}$ and the fact that

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q_{\alpha-1}(x)}{(1-x)^\alpha}\right) = \frac{Q_\alpha(x)}{(1-x)^{\alpha+1}}$$

with polynomials $Q_{\alpha-1}$ and $Q_\alpha$. Put $C_\alpha = \sup_{x\in[0,1]} Q_\alpha(x)$. □

**Remark 2.6.** *In fact, one can give an explicit formula for the $Q_\alpha$ in terms of Eulerian numbers, but we will not need such a precise description for our applications.*

**Lemma 2.7.** *For each $T \geqslant 1$ there is some number $0 < \xi_{\alpha,T} < 1$ such that $p_T(\alpha;x)$ is increasing in the interval $[0,\xi_{\alpha,T}]$ and decreasing in the interval $[\xi_{\alpha,T},1]$.*

*Proof.* Since we have $p_T(\alpha;x) \geqslant 0$ for $0 \leqslant x \leqslant 1$ (with equality if $x = 0$ or $x = 1$) it is sufficient to show that $p_T'(\alpha;x) = 0$ has exactly one solution $0 < \xi_{\alpha,T} < 1$. For values $0 < x < 1$ we obtain

$$\begin{aligned}
p_T'(\alpha;x) = &-(\alpha+1)(1-x)^\alpha \sum_{\ell=1}^{T} \ell^\alpha x^\alpha + (1-x)^{\alpha+1} \sum_{\ell=1}^{T} \ell^{\alpha+1} x^{\ell-1} = 0 \\
\iff &\sum_{\ell=1}^{T} \left(-(\alpha+1)x^\ell + \ell^{\alpha+1}x^{\ell-1} - \ell^{\alpha+1}x^\ell\right) = 0 \\
\iff &\frac{1}{x^T} + \sum_{\ell=1}^{T-1} \left(\sum_{j=2}^{\alpha+1} \binom{\alpha+1}{j} \ell^{\alpha+1-j}\right) x^{\ell-T} = (\alpha+1)T^\alpha + T^{\alpha+1}.
\end{aligned}$$

Since the right hand side is greater than the left hand side for $x = 1$ and the left hand side is unbounded and monotonically decreasing in the interval $(0,1]$, there is exactly one solution for the above equation in this area and the claim follows. □

Before we can go on to the main theorem of this section we recall

**Lemma 2.8.** *Let $a_k$ be a sequence of complex numbers and $b_k$ and $c_k$ sequences of positive real numbers such that $0 \leqslant b_{k+1} \leqslant b_k$ and $c_{k+1} \geqslant c_k \geqslant 0$ for all $k$. Then we have for all $n \geqslant 1$:*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leqslant b_1 \cdot \max_{r=1,\ldots,n} \left|\sum_{k=1}^{r} a_k\right|$$

*and*

$$\left|\sum_{k=1}^{n} a_k c_k\right| \leqslant (2c_n - c_1) \cdot \max_{r=1,\ldots,n} \left|\sum_{k=1}^{r} a_k\right|.$$

**Theorem 2.9.** *Let $\beta$ be a function in $[N,d]$ and $\omega \in W_M$ be a weak function which is holomorphic around $z = 0$. Then for all $\alpha \in \mathbb{N}_0$ there is a constant $C_{\beta,\omega,\alpha} > 0$ such that uniformly for all $T \in \mathbb{N}$ and $y \in [0,1]$*

$$\left|\sum_{n=1}^{NT} n^{\alpha}\beta(n)\omega(niy)\right| \leqslant C_{\beta,\omega,\alpha} y^{d-\alpha}.$$

**Remark 2.10.** *Note that in the case $\alpha \leqslant d$ the left hand side is bounded uniformly for values $T$ and $y \in [0,1]$. Since the series converges absolutely and uniformly on compact subsets $K \subset [\varepsilon,\infty]$ for all $\varepsilon > 0$ dominant convergence is clear for these areas of $y$ (of course the number $\varepsilon = 1$ was arbitrary chosen in the theorem).*

*Proof.* For $y = 0$ the inequality holds since in the case $\alpha \leqslant d$ the left hand side is always zero and otherwise the right hand side is $+\infty$ from the right. Let $y > 0$. We then have

$$\sum_{n=1}^{NT} n^{\alpha}\beta(n)\omega(niy) = \sum_{j\in\mathbb{F}_N} \beta_\omega(j) \lim_{L\to\infty} \sum_{k=1}^{L}\sum_{n=1}^{NT} n^{\alpha}\beta(n)\zeta_M^{kj} e^{-2\pi kny}, \qquad \zeta_M := e^{2\pi i/M}.$$

In the first step we will only deal with the inner sums. For reasons of simplicity we ignore the scalars $2\pi k$. We obtain with partial summation

$$\sum_{n=1}^{NT} n^{\alpha}\beta(n)e^{-ny} = e^{-NTy}\sum_{n=1}^{NT} n^{\alpha}\beta(n) + \sum_{n=1}^{NT}\left(\sum_{r=1}^{n}\beta(r)r^{\alpha}\right)\left(e^{-ny} - e^{-(n+1)y}\right)$$

and since the first term vanishes

$$= (1 - e^{-y})\sum_{n=1}^{NT}\left(\sum_{r=1}^{n}\beta(r)r^{\alpha}\right)e^{-ny}.$$

Now we have for $n = N\ell + q$ with $0 \leqslant \ell \leqslant T-1$ and $1 \leqslant q \leqslant N$

$$\sum_{r=0}^{n}\beta(r)r^{\alpha} = \sum_{r=N\ell+1}^{N\ell+q}\beta(r)r^{\alpha} = \sum_{u=0}^{\alpha}\binom{\alpha}{u}(N\ell)^{\alpha-u}S_\beta(u;q).$$

Hence

$$\begin{aligned}
&(1-e^{-y})\sum_{n=1}^{NT}\left(\sum_{r=1}^{n}\beta(r)r^{\alpha}\right)e^{-ny}\\
&=(1-e^{-y})\sum_{q=1}^{N}\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^{\alpha-u}S_\beta(u;q)e^{-yq}\sum_{\ell=0}^{T-1}\ell^{\alpha-u}e^{-yN\ell}\\
&=(1-e^{-y})\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^{\alpha-u}W_\beta(u;e^{-y})\frac{p_{T-1}(\alpha-u;e^{-yN})}{(1-e^{-yN})^{\alpha-u+1}}\\
&=(1-e^{-y})^{d-\alpha}\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^{\alpha-u}\widetilde{W}_\beta(u;e^{-y})_d\frac{p_{T-1}(\alpha-u;e^{-yN})}{(1+e^{-y}+e^{-2y}+\cdots+e^{-y(N-1)})^{\alpha-u+1}}.
\end{aligned}$$

Let $L$ be a positive integer. It follows

$$\begin{aligned}
&\left|\sum_{k=1}^{L}\sum_{n=1}^{NT}n^{\alpha}\beta(n)\zeta_M^{kj}e^{-kny}\right|\\
&=(1-e^{-y})^{d-\alpha}\left|\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^{\alpha-u}\sum_{k=1}^{L}\widetilde{W}_\beta(u;e^{-ky})_d\frac{p_{T-1}(\alpha-u;e^{-kNy})\zeta_M^{kj}}{(1+e^{-ky}+e^{-2ky}+\cdots+e^{-k(N-1)y})^{\alpha-u+1}}\right|
\end{aligned}$$

and when writing $\widetilde{W}_\beta(u;e^{-ky})_d=\sum_{a=1}^{b_u}\gamma_u(a)e^{-aky}$ (note that this polynomial vanishes at zero and only depends on $u$):

$$\leqslant 2^{|d-\alpha|}y^{d-\alpha}\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^{\alpha-u}\sum_{a=1}^{b_{\alpha,u}}|\gamma_{\alpha,u}(a)|\left|\sum_{k=1}^{L}\frac{p_{T-1}(\alpha-u;e^{-kNy})(\zeta_M^{j}e^{-ay})^k}{(1+e^{-ky}+e^{-2ky}+\cdots+e^{-(N-1)ky})^{\alpha-u+1}}\right|$$

since $(1-e^{-y})^m\leqslant 2^{|m|}y^m$ for all $m\in\mathbb{Z}$ and $y\in[0,1]$. For each $y$ the sequence $c_k=1/(1+e^{-ky}+e^{-2ky}+\cdots+e^{-(N-1)ky})^{\alpha-u+1}$ satisfies $1\geqslant c_{k+1}\geqslant c_k\geqslant 0$ for all $k\geqslant 1$ and hence using 2.8 we find

$$\leqslant 2^{|d-\alpha|+1}y^{d-\alpha}\sum_{u=0}^{\alpha}\binom{\alpha}{u}N^{\alpha-u}\sum_{a=1}^{b_{\alpha,u}}|\gamma_{\alpha,u}(a)|\max_{1\leqslant I\leqslant L}\left|\sum_{k=1}^{I}p_{T-1}(\alpha-u;e^{-kNy})(\zeta_M^{j}e^{-ay})^k\right|$$

Now let $T > 1$ and $y > 0$ be arbitrary chosen. We can split the set $\{1, 2, ..., I\}$ in elements $1 \leqslant 2 \leqslant \cdots \leqslant I(T, y)$ such that $e^{-Nky} \geqslant \xi_{\alpha-u,T-1}$ and $I(T, y) < k \leqslant I$ with $e^{-Nky} < \xi_{\alpha-u,T-1}$. In this sections the function $p_{T-1}(\alpha - u; x)$ is increasing and then decreasing, respectively. Hence

$$\leqslant 2^{|d-\alpha|+1} y^{d-\alpha} \sum_{u=0}^{\alpha} \binom{\alpha}{u} N^{\alpha-u} \sum_{a=1}^{b_{\alpha,u}} |\gamma_{\alpha,u}(a)| \left( \max_{1\leqslant I\leqslant L} \left( \left| \sum_{\substack{1\leqslant k\leqslant I(T,y) \\ e^{-Nky}\geqslant \xi_{\alpha-u,T-1}}} p_{T-1}(\alpha-u; e^{-kNy}) \right.\right.\right.$$

$$\left.\left.\left. \times (\zeta_M^j e^{-ay})^k \right| + \left| \sum_{\substack{I(T,y)<k\leqslant I \\ e^{-Nky}<\xi_{\alpha-u,T-1}}} p_{T-1}(\alpha-u; e^{-kNy}) (\zeta_M^j e^{-ay})^k \right| \right)\right)$$

and by applying 2.8 again using $|p_{T-1}(\alpha - u; x)| \leqslant m_{\alpha-u}$

$$\leqslant 2^{|d-\alpha|+1} y^{d-\alpha} \sum_{u=0}^{\alpha} \binom{\alpha}{u} N^{\alpha-u} \sum_{a=1}^{b_{\alpha,u}} |\gamma_{\alpha,u}(a)|$$

$$\times \max_{1\leqslant I\leqslant L} \left( 2m_{\alpha-u} \max_{1\leqslant k\leqslant I(T,y)} \left| \sum_{h=1}^{k} (\zeta_M^j e^{-ay})^h \right| + m_{\alpha-u} \max_{I(T,y)+1\leqslant k\leqslant I} \left| \sum_{h=I(T,y)+1}^{k} (\zeta_M^j e^{-ay})^h \right| \right)$$

Since $\zeta_M^j \neq 1$ (this is because $\omega$ is removable in $z = 0$) we obtain

$$\leqslant 2^{|d-\alpha|+1} y^{d-\alpha} \sum_{u=0}^{\alpha} \binom{\alpha}{u} N^{\alpha-u} \sum_{a=1}^{b_{\alpha,u}} |\gamma_{\alpha,u}(a)| \times \max_{1\leqslant I\leqslant L} (2m_{\alpha-u}M + m_{\alpha-u}M)$$

$$\leqslant 2^{|d-\alpha|+3} M y^{d-\alpha} \sum_{u=0}^{\alpha} m_{\alpha-u} \binom{\alpha}{u} N^{\alpha-u} \sum_{a=1}^{b_{\alpha,u}} |\gamma_{\alpha,u}(a)|.$$

This is independent of $L$ and remains true when $L \to \infty$. Since the constant in the last expression besides $y^{d-\alpha}$ does not depend on $T$ and $y$ the theorem is proved. ☐

## 3. Application to $L$-functions of modular forms

Let $S = \{t_1, t_2, ...\}$ be a countable, totally ordered set (the direction is simply given by $t_i \leqslant t_j \iff i \leqslant j$) equipped with an *integer map* $|\cdot|_S : S \to \mathbb{N}$ such that for some $L \geqslant 0$:

$$\#\{t \in S \mid |t|_S = n\} = O(n^L). \tag{3.1}$$

In the case the set $S$ is clear we simply write $|\cdot|$. For example, $S$ could be the set of integral ideals of a number field and $|\cdot|$ their norm. Let $a(t_m)_{m\in\mathbb{N}}$ a sequence of complex numbers. We define the corresponding formal Dirichlet series by

$$F(s) := \sum_{t\in S} a(t)|t|^{-s} := \sum_{m=1}^{\infty} a(t_m)|t_m|^{-s}.$$

In the case that the series
$$\sum_{n=1}^{\infty}\left|\frac{1}{|t_n|^s}-\frac{1}{|t_{n+1}|^s}\right|$$
converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ one can check using partial summation that such Dirichlet series converge (if they do) on half planes and represent holomorphic functions in these regions. This is for example the case, if the $|t_n|$ increase monotonously. Since we have (3.1), one can show that $F(s)$ will converge in some point $s_0$ if and only if $a(t) = O(|t|^{\nu})$ for some $\nu \in \mathbb{R}$.

**Definition 3.1.** *Let $F(s) = \sum_{t\in S} a(t)|t|^{-s}$ a Dirichlet series, $Q$ a totally ordered countable set together with a surjective map $w : Q \to S$ with finite fibres. We also assume that $F$ converges to a holomorphic function on some half plane $\{\mathrm{Re}(s) > \sigma_0\}$. The order of $Q$ shall respect the order of $S$, this means $u_1 \leqslant u_2 \implies w(u_1) \leqslant w(u_2)$ for all $u_1, u_2 \in Q$. We define an integer map on $Q$ via $|u|_Q := |w(u)|_S$. In other words, all elements in the same fibre of a $t \in S$ are associated to the same integer. By a splitting of $F$ we mean a Dirichlet series $\widetilde{F}(s) = \sum_{u\in Q} b(u)|u|_Q^{-s}$ that has the following properties:*

*(i) $\widetilde{F}(s)$ converges to a holomorphic function in some half plane $\{\mathrm{Re}(s) > \tilde{\sigma}_0\}$.*

*(ii) We have for all $t \in S$ the summation formula $\sum\limits_{u\in w^{-1}(t)} b(u) = a(t)$.*

We may think of splittings in the following way: we have $Q = \bigcup_{t\in S} \sigma^{-1}(t)$ and therefore
$$\sum_{t\in S} a(t)|t|^{-s} = \sum_{t\in S}\sum_{u\in w^{-1}(t)} b(u)|u|_Q^{-s}.$$
The next definition provides kind of an inverse concept for splittings.

**Definition 3.2.** *Let $S = \bigcup_{j=1}^{\infty} S_j$ be a disjoint covering with finite $S_j$. We say that a Dirichlet series $F(s) = \sum_{t\in S} a(t)|t|^{-s}$ respects the rearrangement $(S_j)_{j\in\mathbb{N}}$, if the series is given by the partial sums*
$$F_n(s) = \sum_{j=1}^{n}\sum_{t\in S_j} a(t)|t|^{-s}.$$
*If there might be danger with confusion we simply write*
$$(F, (S_j)_{j\in\mathbb{N}})(s) = \sum_{j=1}^{\infty}\sum_{t\in S_j} a(t)|t|^{-s}.$$

Obviously, $F(s)$ and $(F, (S_j)_{j\in\mathbb{N}})(s)$ coincide in all regions of absolute convergence. In the case of $S_j = \{t \in S \mid |t| = j\}$, $(F, (S_j)_{j\in\mathbb{N}})(s)$ is an ordinary Dirichlet series $\sum b(n)n^{-s}$ – we call this the standard rearrangement. The next proposition makes clear why rearrangements makes splitting undone in some situations.

**Proposition 3.3.** *Let $\widetilde{F}$ be a splitting of $F$ over $Q$. Define the disjoint union $Q_j := \sigma^{-1}(t_j)$. If we now sum $\widetilde{F}$ with respect to $(Q_j)_{j\in\mathbb{N}}$ we obtain $F$.*

*Proof.* This follows directly from the definitions. □

**Definition 3.4.** *We call $(T_j)_{j\in\mathbb{N}}$ a sub-rearrangement of $(S_j)_{j\in\mathbb{N}}$, if there is a sequence $0 < k_1 < k_2 < k_3 < \cdots$ of integers such that $T_1 = S_1 \cup \cdots \cup S_{k_1}$, $T_2 = S_{k_1+1} \cup \cdots \cup S_{k_2}$ and so on.*

In the following we define for any rearrangement the abscissa of convergence $\sigma((F,(S_j)_{j\in\mathbb{N}}))$ to be the infimum real value $\sigma_0$, such that for all complex values $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_0$ the series converges and represents a holomorphic function in this region.

**Remark 3.5.** *One easily checks $\sigma((F,(T_j)_{j\in\mathbb{N}})) \leqslant \sigma((F,(S_j)_{j\in\mathbb{N}}))$. Hence 3.3 shows that splitting does not improve the area of convergence. However, when rearranging a split series the situation might look different.*

Let $\mathfrak{R}(F)$ the set of all rearrangements of $F$. We define an equivalence relation on $\mathfrak{R}(F)$ by putting two coverings in the same class if the resultant series have the same abscissa of convergence. We collect this data in $\mathfrak{R}(F)/\sim$. We would like to study $\mathfrak{R}(F)/\sim$, in particular, we are interested in the following question:

**Question 3.6.** *What is the value $\widetilde{\sigma}(F) := \inf_{G\in\mathfrak{R}(F)/\sim}\sigma(G)$?*

There is no simple answer to this question. It rather strongly depends on the Dirichlet series itself, as the next examples demonstrate.

(i) If $a(t) \geqslant 0$ globally, the region of convergence can not be improved by rearranging the Dirichlet series. Hence $|\mathfrak{R}(F)/\sim| = 1$ and $\widetilde{\sigma}(F) = \sigma(F)$.

(ii) Although the set $\mathfrak{G}$ is large, $-\widetilde{\sigma}(F)$ does not have to be unbounded even in the case that $F$ is entire. If $\chi$ is an *even* real non-principal character modulo $M$, one can show that $\widetilde{\sigma}(L(\chi,s)) = -1$ if $L(\chi,-1) \notin \mathbb{Z}$. In this case the "best" rearrangement of $L(\chi,s)$ is given by $\mathbb{N} = \bigcup_{j\in\mathbb{N}}\{M(j-1)+k \mid 1 \leqslant k \leqslant M\}$ and we have
$$L(\chi,s) = \sum_{j=1}^{\infty}\left(\sum_{m=1}^{M}\chi(m)(M(j-1)+m)^{-s}\right), \qquad \mathrm{Re}(s) > -1.$$
We conclude $L(\chi,0) = 0$. Since all inner summands in the rearrangements are integers when $s = -1$, there is indeed no better choice if $L(\chi,-1) \notin \mathbb{Z}$, as the reader may easily check.
A similar argument shows $\widetilde{\sigma}(L(\chi,s)) = \sigma_0 = 0$ if $\chi$ is real, odd and $L(\chi,0) \notin \mathbb{Z}$.

(iii) The identity $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ for $\mathrm{Re}(s) > 1$ is well-known and elementary. Here $\mu(n)$ is the Möbius function. Since $\mu(n)$ has sign changes, it makes sense to look at possible rearrangements. However it seems extremely difficult to find improvements of $\sigma = 1$, since there is no progress in this area until today! We have $\frac{1}{2} \leqslant \widetilde{\sigma}(1/\zeta) \leqslant 1$ and $\widetilde{\sigma}(1/\zeta) = \frac{1}{2}$ implies the RH.

**Remark 3.7.** *In the case of (ii), where the coefficients are well-studied, there are of course even more powerful tools for analytic continuation using series transformations,* *that can be seen as generalized rearrangements in the sense that we allow the splitting sets $S_n$ to have infinite order. For example, when using Euler summation, we find the right hand series*

$$L(\chi, s) \equiv \sum_{n=0}^{\infty} 2^{-n-1} \sum_{\nu=0}^{n} \binom{n}{\nu} \chi(\nu+1)(\nu+1)^{-s},$$

*will converge globally for non-principal characters $\chi$.*

Let $\boldsymbol{k} = (k_1, ..., k_l)$ and $f \in U_{\boldsymbol{k}}(\Gamma_1(M, N))$ be a weak modular form. In the following we give a natural splitting for $L(f, s)$ in terms of the overset $Q = \mathbb{N}^l \times \mathbb{N}^l$. After this, when applying the dominated convergence theorem from the last section we can find good rearrangements of these splittings to give estimates for the size defined in 3.6.

Let $G_N^{(l)} = \mathbb{F}_N^\times \times \cdots \times \mathbb{F}_N^\times$ be the $l$-fold product of the residue class groups modulo $N$. Then $G_N^{(l)}$ is a multiplicative group and there are $\varphi(N)^l$ characters $\psi : G_N^{(l)} \to \mathbb{C}^\times$ given by $\psi(\boldsymbol{n}) = \prod_{j=1}^{l} \psi_j(n_j)$, where $\psi_1, ..., \psi_l$ are characters modulo $N$. We further call a character $\psi : G_N^{(l)} \to \mathbb{C}^\times$ non-principal, if no component $\psi_j$ with $1 \leqslant j \leqslant l$ is principal and principal else. Analogously we say that $\psi$ is primitive if and only if all components are primitive.
Note that each $\psi$ extends multiplicatively to a map $\psi : \mathbb{Z}^l \to \mathbb{C}^\times$. For $\boldsymbol{k} \in \mathbb{N}^l$ also define the (multiplicative) map $\Pi_{\boldsymbol{k}}(\boldsymbol{n}) = n_1^{k_1-1} \cdots n_l^{k_l-1}$.

**Lemma 3.8.** *Let $t_1, ..., t_l$ be functions in $\left(\mathbb{F}_{\frac{1}{N}}\right)_0^{\mathbb{C}_0}$ such that the associated weak functions $\omega_{t_j}$ are removable in $z = 0$. Then we have for all vectors $\boldsymbol{u} \in \mathbb{N}^l$ and $s$ with* $\mathrm{Re}(s) > 0$*:*

$$\int_0^\infty \omega_{t_1}(u_1 ix/N) \cdots \omega_{t_l}(u_l ix/N) x^{s-1} \mathrm{d}x = \Gamma(s) \left(\frac{N}{2\pi}\right)^s \sum_{\boldsymbol{v} \in \mathbb{N}^l} \mathcal{F}_N^{(l)} t(\boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v} \rangle^{-s},$$

*where $\mathcal{F}_N^{(l)} t(\boldsymbol{v}) = (\mathcal{F}_N t_1)(v_1) \cdots (\mathcal{F}_N t_l)(v_l)$ is the vector valued Fourier transform. Here, the order of summation respects the maximum values of $\boldsymbol{v}$.*

*Proof.* We have

$$\omega_{t_1}(u_1 ix/N) \cdots \omega_{t_l}(u_l ix/N) = \sum_{\boldsymbol{q} \in G_N^{(l)}} t_1(q_1) \cdots t_l(q_l) \prod_{j=1}^{l} \frac{e(u_j ix/N)}{e(q_j/N) - e(u_j ix/N)}$$

$$= \sum_{\boldsymbol{q} \in G_N^{(l)}} t_1(q_1) \cdots t_l(q_l) \prod_{j=1}^{l} \sum_{v_j=1}^{\infty} e^{-2\pi u_j v_j x/N - 2\pi i q_j v_j/N}$$

$$= \lim_{T \to \infty} \sum_{\boldsymbol{q} \in G_N^{(l)}} t_1(q_1) \cdots t_l(q_l) \sum_{v_1=1}^{T} \cdots \sum_{v_l=1}^{T} e^{-2\pi(v_1 u_1 + \cdots + v_l u_l)x/N - 2\pi i(v_1 q_1 + \cdots + v_l q_l)/N}.$$

The multisum is in the sense of a summation which respects the maximum of the vectors $\boldsymbol{v}$. Since for $0 < \theta < 2\pi$ the geometric sums $\sum_{j=1}^{n}(ye^{i\theta})^j$ are bounded uniformly for $0 \leqslant y \leqslant 1$ and $n \in \mathbb{N}$ we may switch integration and summation and the claim follows. $\square$

**Proposition 3.9.** *Let $f \in U_{\boldsymbol{k}}(\Gamma_1(M,N))$ be a modular form, such that*

$$f = \sum_{\alpha=1}^{R} \mu_\alpha \vartheta_{k_1}(\omega_{h_{\alpha,1}} \otimes \omega_{t_{\alpha,1}}) \cdots \vartheta_{k_l}(\omega_{h_{\alpha,l}} \otimes \omega_{t_{\alpha,l}}).$$

*Here we assume that* $\operatorname{sgn}(h_{\alpha,j}t_{\alpha,j}) = (-1)^{k_j}$ *for all* $j = 1, ..., l$. *Then, for all complex numbers $s$ with* $\operatorname{Re}(s) > |\boldsymbol{k}|$, *we have*

$$L(f,s) = \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^l \times \mathbb{N}^l} a(\boldsymbol{u},\boldsymbol{v}) \left\langle \boldsymbol{u},\boldsymbol{v} \right\rangle^{-s}, \tag{3.2}$$

*where the coefficients $a(\boldsymbol{u},\boldsymbol{v})$ are given by*

$$a(\boldsymbol{u},\boldsymbol{v}) = N^{l-k}\Pi_{\boldsymbol{k}}(\boldsymbol{u}) \sum_{\alpha=1}^{R} \mu_\alpha \prod_{j=1}^{l} t_{\alpha,j}(u_j)(\mathcal{F}_M h_{\alpha,j})(v_j).$$

*Proof.* The series on the right of (3.2) clearly converges absolutely on the half plane $\{s \in \mathbb{C} \mid \sigma > |\boldsymbol{k}|\}$. Since $t_j(0) = h_j(0) = 0$ for all $1 \leqslant j \leqslant l$, all involved weak functions are removable in $z = 0$ and so is their product. We have for all $s \in \mathbb{C}$

$$\Lambda(f,s) = \left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) L(f,s) = \int_0^\infty \sum_{\alpha=1}^{R} \mu_\alpha b_\alpha(ix) x^{s-1} \mathrm{d}x.$$

Hence, due to absolute convergence, we obtain for all $s$ with $\sigma > k$

$$L(f,s) = \lim_{T\to\infty} \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N}\right)^{s} N^{l-k} \sum_{\substack{u_j=1\\ 1\leqslant j\leqslant l}}^{T} \sum_{\alpha=1}^{R} \mu_\alpha u_1^{k_{\alpha,1}-1} \cdots u_l^{k_{\alpha,l}-1} t_{\alpha,1}(u_1) \cdots t_{\alpha,l}(u_l)$$

$$\times \int_0^\infty \omega_{h_1}(u_1 ix/N) \cdots \omega_{h_l}(u_l ix/N) x^{s-1} \mathrm{d}x.$$

Together with 3.8 we obtain

$$= N^{l-k} \sum_{\substack{u_j,v_j=1\\ 1\leqslant j\leqslant l}}^{\infty} \sum_{\alpha=1}^{R} \mu_\alpha \left( \prod_{j=1}^{l} u_j^{k_j-1} t_{\alpha,j}(u_j)(\mathcal{F}_M h_{\alpha,j})(v_j) \right) (u_1v_1 + \cdots + u_lv_l)^{-s}$$

$$= N^{l-k} \sum_{\substack{u_j,v_j=1\\ 1\leqslant j\leqslant l}}^{\infty} \prod_{j=1}^{l} u_j^{k_j-1} \sum_{\alpha=1}^{R} \mu_\alpha \left( \prod_{j=1}^{l} t_{\alpha,j}(u_j)(\mathcal{F}_M h_{\alpha,j})(v_j) \right) (u_1v_1 + \cdots + u_lv_l)^{-s}.$$

$\square$

3.9 provides us coefficients $a(\boldsymbol{u},\boldsymbol{v})$ that belong to splittings of $L(f,s)$ over $Q=\mathbb{N}^l\times\mathbb{N}^l$. However, as we have already pointed out, this is not a well-defined linear map in general, since the coefficients are not uniquely determined. We are especially interested in a reasonable subset of the large set of all possible splittings of $L(f,s)$, this becomes clear in the following discussion.
Put

$$U_{\boldsymbol{k}}^{\otimes}(\Gamma_1(M,N)) := \bigotimes_{j=1}^{l} U_{k_j}(\Gamma_1(M,N))$$

and

$$A_{M,N}^l := \{g:\mathbb{F}_M^l\times\mathbb{F}_N^l\to\mathbb{C} \mid (*)\}.$$

Here the condition $(*)$ means that $g$ is zero whenever a component of the argument is zero. Note that $A_{M,N}^l$ is clearly isomorphic to the corresponding space $\widetilde{A}_{M,N}^l$ of periodic maps $g:\mathbb{Z}^l\times\mathbb{Z}^l\to\mathbb{C}$. Consider the commutative diagram

$$\begin{tikzcd}
U_{\boldsymbol{k}}^{\otimes}(\Gamma_1(M,N)) \arrow[r, "\sim", "\iota"'] \arrow[d, "\varphi_{\boldsymbol{k}}"'] & \widetilde{A}_{M,N}^l \arrow[dl, "\varphi_{\boldsymbol{k}}\circ\iota^{-1} =: \psi_{\boldsymbol{k}}"] \\
U_{\boldsymbol{k}}(\Gamma_1(M,N)) &
\end{tikzcd}$$

where $\iota$ and $\varphi_{\boldsymbol{k}}$ are determined by

$$\vartheta_{k_1}(\omega_{h_1}\otimes\omega_{t_1})\otimes\cdots\otimes\vartheta_{k_l}(\omega_{h_l}\otimes\omega_{t_l}) \longmapsto \prod_{j=1}^{l} t_j(\mathcal{F}_M h_j)$$

and

$$\vartheta_{k_1}(\omega_{h_1}\otimes\omega_{t_1})\otimes\cdots\otimes\vartheta_{k_l}(\omega_{h_l}\otimes\omega_{t_l}) \longmapsto \prod_{j=1}^{l} \vartheta_{k_j}(\omega_{h_j}\otimes\omega_{t_j}),$$

respectively. The linear map $\varphi_{\boldsymbol{k}}:U_{\boldsymbol{k}}^{\otimes}(\Gamma_1(M,N))\to U_{\boldsymbol{k}}(\Gamma_1(M,N))$ is surjective by construction, however, no isomorphism in general (indeed, this is the case if and only if $l=1$). We shall write $\Lambda_{\boldsymbol{k}}$ for the kernel of $\varphi_{\boldsymbol{k}}$. 3.9 now gives us a map that equips a modular form $f\in U_{\boldsymbol{k}}(\Gamma_1(M,N))$ with a pre-image of $\psi_{\boldsymbol{k}}$, but of course, this pre-image is in general not uniquely determined. Therefore it is more reasonable to consider a map

$$f\longmapsto v_0(f)+\iota(\Lambda_{\boldsymbol{k}})$$

that sends $f$ to a complete family of coefficients $a(\boldsymbol{u},\boldsymbol{v})$ in the sense of 3.9. Here, $v_0(f)$ is any fixed pre-image of $f$ under $\psi_{\boldsymbol{k}}$.

**Proposition 3.10.** *For every $f\in U_{\boldsymbol{k}}(\Gamma_1(M,N))$ consider the family $\mathfrak{a}(f)=\Pi_{\boldsymbol{k}}(\boldsymbol{u})(v_0(f)+\iota(\Lambda_{\boldsymbol{k}}))(\boldsymbol{u},\boldsymbol{v})$ of coefficients with arguments $(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^l\times\mathbb{N}^l$. All of them define splittings of $L(f,s)$ $(S=\mathbb{N})$ over $Q=\mathbb{N}^l\times\mathbb{N}^l$ equipped with the integer map $|(\boldsymbol{u},\boldsymbol{v})|:=\langle\boldsymbol{u},\boldsymbol{v}\rangle$.*

Of course, when using 3.3 one could reconstruct the original ordinary Dirichlet series with a standard rearrangement. However, in the following we study a completely different rearrangement $(U_{N,m})_{m\in\mathbb{N}}$ that arises from the results in the previous section. With this we want to extend the region of convergence of the series $L(f,s) = \sum a(\boldsymbol{u},\boldsymbol{v}) \langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s}$ naturally. Fix an integer $N$. We define for $p,q \in \mathbb{N}$

$$T_{N,p,q} = \{t = (\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^l \times \mathbb{N}^l \mid N(p-1) < \max(\boldsymbol{u}) \leqslant Np, \max(\boldsymbol{v}) = q\}.$$

Note that the $T_{N,p,q}$ define a disjoint covering of $\mathbb{N}^l \times \mathbb{N}^l$. We then define

$$\begin{aligned} U_{N,1} &:= T_{N,1,1} \\ U_{N,2} &:= T_{N,1,2} \cup T_{N,2,1} \cup T_{N,2,2} \\ U_{N,3} &:= T_{N,1,3} \cup T_{N,2,3} \cup T_{N,3,1} \cup T_{N,3,2} \cup T_{N,3,3} \end{aligned}$$

and so on. After 3.10 provided us some natural splittings (in fact, all Dirichlet series of $L(f,s)$ arising from products of weak functions for $\boldsymbol{k}$ and not from the usual Fourier series) we show that we can improve the region of convergence by rearranging the splittings by $U_{N.m}$.

**Theorem 3.11.** *Let $N > 1$ and $l \geqslant 1$ be integers and $h_j \in [M,0]$, $t_j \in [N,d_j]$ be even or odd functions for $1 \leqslant j \leqslant l$ and some non-negative integers $d_j$. We further assume that we have $\operatorname{sgn}(h_j t_j) = (-1)^{k_j}$ for every $1 \leqslant j \leqslant l$. Consider the modular form*

$$f(\tau) = \prod_{j=1}^{l} \vartheta_{k_j}(\omega_{h_j} \otimes \omega_{t_j}; \tau) \in U_{\boldsymbol{k}}(\Gamma_1(M,N)).$$

*For all values $s \in \mathbb{C}$ with $\operatorname{Re}(s) > |\boldsymbol{k}| - l - d$, where $d = \sum_{j=1}^{l} d_j$, we have the series representation*

$$L(f,s) = N^{l-|\boldsymbol{k}|} \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^l \times \mathbb{N}^l} \Pi_{\boldsymbol{k}}(\boldsymbol{u}) t(\boldsymbol{u}) (\mathcal{F}_N^{(l)} h)(\boldsymbol{v}) \langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s},$$

*where $t(\boldsymbol{u}) := t_1(u_1)\cdots t_l(u_l)$ and $(\mathcal{F}_N^{(l)} h)(\boldsymbol{v}) := (\mathcal{F}_N h_1)(v_1)\cdots(\mathcal{F}_N h)(v_l)$ is the multidimensional Fourier transform. The summation respects the rearrangement $(U_{N,m})_{m\in\mathbb{N}}$. In particular, we have*

$$\inf_{b \in \mathfrak{a}(f)} \widetilde{\sigma}\left( \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^l \times \mathbb{N}^l} b(\boldsymbol{u},\boldsymbol{v}) \langle \boldsymbol{u},\boldsymbol{v}\rangle^{-s} \right) \leqslant |\boldsymbol{k}| - l - d.$$

The short notation $\mathfrak{a}(f)$ was introduced in 3.10.

*Proof.* The series on the right of (3.2) converges absolutely for all $s$ with $\operatorname{Re}(s) > |\boldsymbol{k}|$. Since $t_j(0) = h_j(0) = 0$ for all $1 \leqslant j \leqslant l$, all involved weak functions are removable in $z = 0$ and so is their product. We have for all $s \in \mathbb{C}$

$$\Lambda(f,s) = \left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) L(f,s) = \int_0^\infty f(ix) x^{s-1} \mathrm{d}x = \int_0^\infty \prod_{j=1}^{l} \vartheta_{k_j}(\omega_{h_j} \otimes \omega_{t_j}; xi) x^{s-1} \mathrm{d}x.$$

The functions $t_1, ..., t_l$ have degrees $d_1, ..., d_l$ which means by 2.9 that there is a constant $C > 0$ such that for all $T \in \mathbb{N}$ and $0 \leqslant x \leqslant 1$:

$$\begin{aligned}
&\left| \sum_{\substack{u_j=1 \\ 1\leqslant j\leqslant l}}^{NT} u_1^{k_1-1} \cdots u_l^{k_l-1} t_1(u_1) \cdots t_l(u_l) \omega_{h_1}(u_1 ix/N) \cdots \omega_{h_l}(u_l ix/N) x^{s-1} \right| \\
=& x^{\sigma-1} \prod_{j=1}^{l} \left| \sum_{u_j=1}^{NT} u_j^{k_j-1} t_j(u_j) \omega_{h_j}(u_j ix/N) \right| \\
\leqslant& C x^{\sigma+d-1-(|\boldsymbol{k}|-l)}
\end{aligned}$$

and the right hand side is an integrable majorant for $\sigma > |\boldsymbol{k}| - l - d$. For these values we therefore have dominated convergence on the interval $[0, 1]$ and uniform convergence on the interval $[1, \infty)$, hence we obtain for $\mathrm{Re}(s) > |\boldsymbol{k}| - l - d$

$$\begin{aligned}
L(f, s) =& \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N}\right)^s \int_0^\infty \lim_{T\to\infty} N^{l-|\boldsymbol{k}|} \sum_{\substack{u_j=1 \\ 1\leqslant j\leqslant l}}^{NT} u_1^{k_1-1} \cdots u_l^{k_l-1} t_1(u_1) \cdots t_l(u_l) \\
&\times \omega_{h_1}(u_1 ix/N) \cdots \omega_{h_l}(u_l ix/N) x^{s-1} \mathrm{d}x \\
=& \lim_{T\to\infty} \frac{1}{\Gamma(s)} \left(\frac{2\pi}{N}\right)^s N^{l-|\boldsymbol{k}|} \sum_{\substack{u_j=1 \\ 1\leqslant j\leqslant l}}^{NT} u_1^{k_1-1} \cdots u_l^{k_l-1} t_1(u_1) \cdots t_l(u_l) \\
&\times \int_0^\infty \omega_{h_1}(u_1 ix/N) \cdots \omega_{h_l}(u_l ix/N) x^{s-1} \mathrm{d}x.
\end{aligned}$$

In the proof of the dominated convergence theorem the upper bound was independent of the choice of the partial sums for the series of $\omega$. Hence, together with 3.8 we obtain

$$= \lim_{T\to\infty} N^{l-|\boldsymbol{k}|} \sum_{\substack{u_j=1 \\ 1\leqslant j\leqslant l}}^{NT} u_1^{k_1-1} \cdots u_l^{k_l-1} t_1(u_1) \cdots t_l(u_l) \sum_{\substack{v_j=1 \\ 1\leqslant j\leqslant l}}^{T} \frac{\mathcal{F}_N^{(l)} h(\boldsymbol{v})}{(u_1 v_1 + \cdots + u_l v_l)^s}.$$

Since the order of summation in the partial sums respects the rearrangement $(U_{N,m})_{m\in\mathbb{N}}$ the theorem is proved. $\square$

From this we obtain a much more general result as (ii) presented in the above examples.

**Corollary 3.12.** *Let $t \neq 0$ be in $[N, d]$. Then the series*

$$\sum_{r=0}^{\infty} \sum_{\ell=1}^{N} t(\ell)(Nr + \ell)^{-s}$$

*converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -d$ to a holomorphic function $L(t, s)$. In particular, $L(t, -\alpha) = 0$ for all $0 \leqslant \alpha < d$.*

*Proof.* Put $k = d + 1$. Choose $h \neq 0$ such that $\operatorname{sgn}(t \cdot h) = (-1)^k$. Then we obtain with 3.11 that the series

$$\lim_{T\to\infty} N^{1-k} \sum_{u=1}^{NT} \sum_{\nu=1}^{\infty} \frac{u^{k-1} t(u)(\mathcal{F}_N h)(\nu)}{(u\nu)^s} = \lim_{T\to\infty} N^{1-k} \left( \sum_{u=1}^{NT} u^{d-s} t(u) \right) \left( \sum_{\nu=1}^{\infty} \frac{(\mathcal{F}_N h)(\nu)}{\nu^s} \right)$$

converges for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$ to a holomorphic function. Since

$$\lim_{T\to\infty} \sum_{u=1}^{NT} u^{d-s} t(u) = \sum_{r=0}^{\infty} \sum_{\ell=1}^{N} t(\ell)(Nr + \ell)^{-s+d}$$

the claim follows. □

One consequence of this observation is an application to infinite products.

**Example 3.13.** *Consider the function*

$$a_4(n) = \begin{cases} -1, & n \equiv \pm 1 \pmod 4, \\ 2, & n \equiv 2 \pmod 4, \\ 0, & n \equiv 0 \pmod 4. \end{cases}$$

*Then $a_4$ has degree 1, since obviously $\sum_{j=1}^4 a_4(j) = \sum_{j=1}^4 a_4(j)j = -1 + 4 - 3 = 0$. One sees quickly that*

$$f(s) = \sum_{n=1}^{\infty} a_4(n) n^{-s} = \left(3 \cdot 2^{-s} - 2 \cdot 4^{-s} - 1\right) \zeta(s),$$

*where $\zeta(s)$ denotes the Riemann zeta function. Together with 3.12 we conclude that*

$$\sum_{n=0}^{\infty} \sum_{j=1}^{4} a_4(4n + j)(4n + j)^{-s}$$

*converges to a holomorphic function for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > -1$ and we find*

$$\sum_{n=0}^{\infty} (\log(4n + 1) - 2\log(4n + 2) + \log(4n + 3)) = f'(0).$$

*Since $\zeta(0) = -\frac{1}{2}$ we obtain*

$$\prod_{n=0}^{\infty} \frac{(4n + 1)(4n + 3)}{(4n + 2)^2} = \frac{1}{\sqrt{2}}.$$

**Remark 3.14.** *With a rearranged splitting*

$$\sum_{n=1}^{\infty} ((2n - 1)^{-s} - 2(2n)^{-s} + (2n + 1)^{-s}) = 2(1 - 2^{1-s})\zeta(s) - 1,$$

*that converges for $\operatorname{Re}(s) > -1$, we similarly conclude (when using $\zeta'(0) = -\frac{1}{2}\log(2\pi)$) the Wallis product*

$$\prod_{n=1}^{\infty} \frac{(2n - 1)(2n + 1)}{(2n)^2} = \frac{2}{\pi}.$$

**Example 3.15.** *Let $\chi$ be a non-principal even character modulo $N$. Then, using the well-known Weierstraßproduct expansion*

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n},$$

*we find*

$$\prod_{n=0}^{\infty}(Nn+1)(Nn+2)^{\chi(2)}(Nn+\neg 3)^{\chi(3)}\cdots(Nn+N-1)^{\chi(N-1)} = \prod_{m=1}^{N}\Gamma\left(\frac{m}{N}\right)^{\chi(m)}.$$

*As a consequence, we obtain the following well-known identity*

$$e^{L'(\chi,0)} = \prod_{m=1}^{N}\Gamma\left(\frac{m}{N}\right)^{\chi(m)}.$$

The next final corollary provides natural generalized Dirichlet series representations for $L$-functions associated to products of Eisenstein series for non-principal primitive Dirichlet characters.

**Corollary 3.16.** *Let $\chi, \psi : \mathbb{Z}^l \to \mathbb{C}^\times$ be non-principal, primitive characters modulo $M$ and $N$, respectively, such that $\chi_j(-1)\psi_j(-1) = (-1)^{k_j}$ for all $j = 1, ..., l$. For all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > |\boldsymbol{k}| - l - \frac{1}{2}\sum_{j=1}^{l}(\psi_j(-1)+1)$ *we have*

$$L\left(\prod_{j=1}^{l}E_{k_j}(\chi_j,\psi_j;\tau), s\right) = \left(-\frac{2\pi i}{N}\right)^{|\boldsymbol{k}|}\prod_{j=1}^{l}\frac{\mathcal{G}(\psi_j)}{(k_j-1)!}\sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^l\times\mathbb{N}^l}\Pi_{\boldsymbol{k}}(\boldsymbol{u})\overline{\psi}(\boldsymbol{u})\chi(\boldsymbol{v})\left\langle\boldsymbol{u},\boldsymbol{v}\right\rangle^{-s},$$

*where the summation respects the rearrangement $(U_{N,m})_{m\in\mathbb{N}}$.*

*Proof.* Since all characters are primitive we have

$$E_{k_j}(\chi_j,\psi_j;\tau) = \frac{\chi_j(-1)(-2\pi i)^{k_j}\mathcal{G}(\psi_j)}{N(k_j-1)!\mathcal{G}(\overline{\chi_j})}\vartheta_{k_j}(\omega_{\overline{\chi_j}}\otimes\omega_{\overline{\psi_j}};\tau).$$

Hence we obtain with 3.11

$$L\left(\prod_{j=1}^{l}E_{k_j}(\chi_j,\psi_j;\tau), s\right) = \lambda_1\cdots\lambda_l N^{l-|\boldsymbol{k}|}\sum_{(\boldsymbol{u},\boldsymbol{v})\in\mathbb{N}^l\times\mathbb{N}^l}\Pi_{\boldsymbol{k}}(\boldsymbol{u})\overline{\psi}(\boldsymbol{u})(\mathcal{F}_N^{(l)}\overline{\chi})(\boldsymbol{v})\left\langle\boldsymbol{u},\boldsymbol{v}\right\rangle^{-s},$$

where

$$\lambda_j = \frac{\chi_j(-1)(-2\pi i)^{k_j}\mathcal{G}(\psi_j)}{N(k_j-1)!\mathcal{G}(\overline{\chi_j})}.$$

We can simplify the expression $(\mathcal{F}_N^{(l)})(\overline{\chi})$ by

$$(\mathcal{F}_N^{(l)})(\overline{\chi})(\boldsymbol{v}) = \chi(\boldsymbol{v})(\mathcal{F}_N^{(l)}\overline{\chi})(\mathbf{1}) = \chi(\boldsymbol{v})\prod_{j=1}^{l}\chi_j(-1)\mathcal{G}(\overline{\chi_j}),$$

so we obtain

$$\lambda_1\cdots\lambda_l N^{l-|\boldsymbol{k}|}(\mathcal{F}_N^{(l)})(\overline{\chi})(\boldsymbol{v}) = \left(-\frac{2\pi i}{N}\right)^{|\boldsymbol{k}|}\prod_{j=1}^{l}\frac{\mathcal{G}(\psi_j)}{(k_j-1)!}\chi(\boldsymbol{v}).$$

The extended domain of convergence follows, because of the rearrangement, with 3.11 and the fact that the degree of $\psi_j$ is given by $\frac{1}{2}(\psi_j(-1)+1)$. $\square$

Finally, we give an example.

**Example 3.17.** *Let $\chi$ be a primitive even Dirichlet character modulo $N > 1$. We then look on the Eisenstein series $E_2(\chi,\chi;\tau)$ of weight $k = 2$ and define*

$$f(\tau) = E_2(\chi,\chi;\tau)^2.$$

*Then $f$ is a modular form of weight 4 for the group $\Gamma(N^2)$ and vanishes in the cusps $z = 0$ and $z = i\infty$, hence its $L$-function $L(f,s)$ is entire. We are especially interested in the critical value $L(f,1)$. With 3.16 we obtain*

$$L(f,1) = C_1 \sum_{r_1,r_2,\nu_1,\nu_2=1}^{\infty} \left( \sum_{q_1,q_2=0}^{N-1} \chi(q_1)\chi(q_2)\overline{\chi}(\nu_1)\overline{\chi}(\nu_2) \frac{(Nr_1-q_1)(Nr_2-q_2)}{(Nr_1-q_1)\nu_1 + (Nr_2-q_2)\nu_2} \right),$$

*where the constant $C_1$ is given by*

$$C_1 = \frac{\chi(-1)(2\pi i)^4}{N^3}.$$

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