SYMMETRIC SPACES, PENROSE LIMITS AND MAXIMAL SUPERSYMMETRY

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Abstract. This is the written version of a talk given at the Center for Advanced Mathematical Sciences of the American University of Beirut. The aim of the talk was to describe my recent work on supersymmetric solutions of supergravity theories in the broadest possible terms and to make this topic accessible to an audience unfamiliar with string theory and/or supergravity.

The fundamental tenet of General Relativity can be paraphrased as follows: "*space tells matter how to move and matter tells space how to curve.*" A precise mathematical restatement of this mantra are the Einstein equations relating the Ricci tensor Ric of the *spacetime* (M, g) to the energy-momentum tensor T of the matter. The Ricci tensor measures a certain average of the curvature of the metric g of the spacetime. One way to write the Einstein equations is

$$\operatorname{Ric} - \tfrac{1}{2} R g = T \ , \tag{1}$$

where R is the scalar curvature of the metric g and where we have absorbed Newton's constant and some geometrical factors in the definition of T. The left-hand side of the Einstein equations defines the *Einstein tensor*, which, like the right-hand side, is covariantly conserved; that is, its (covariant) divergence vanishes. The energy-momentum tensor typically depends on the matter and hence is not generally geometrical. However for a very particular type of matter, T is purely geometrical. In this case it can only be proportional to the metric g, and the Einstein equations rearrange themselves into

$$\operatorname{Ric} = \Lambda g \ , \tag{2}$$

where $\Lambda \in \mathbb{R}$ is the *cosmological constant*. A manifold (M, g) obeying (2) is called an *Einstein manifold*.

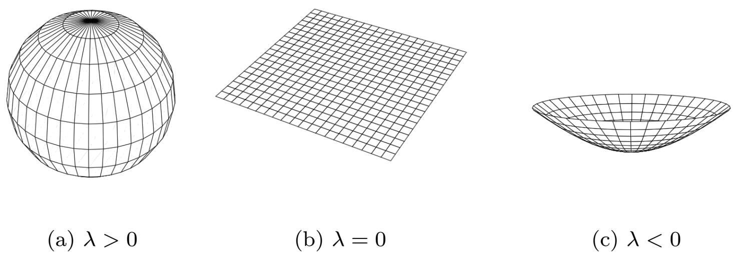
The study of Einstein manifolds is still an active area of differential geometry. There is no general classification, but many methods of construction are known. The state of the art in the mid-to-late 1980s is contained in the classic book [2] by the Besse collective, and some of the progress that has been made since then is collected in the book of essays [14]. Finding explicit Einstein metrics – that is, solutions of the Einstein equations (2) – is not an easy task, as the equations form a system of nonlinear second-order partial differential equations for g; but by imposing enough symmetry we can simplify the equations and hope to find solutions. For example, we could demand that (M, g) be a homogeneous space; that is, that there should be a Lie group G acting transitively and isometrically on M. In other words, M consists of a single orbit of G and the action of G preserves the metric. This renders the Einstein equations (2) algebraic. We could relax this hypothesis somewhat by demanding that G act *not* transitively but with generic orbits of codimension 1 (the so-called cohomogeneity-one case) while still preserving the metric. In this case there is a one-parameter family of generic orbits and the Einstein equations (2) become ordinary differential equations in that parameter. There is no classification for cohomogeneity-one Einstein manifolds, although there is for homogeneous Einstein manifolds in the riemannian case; although in the lorentzian case even this is lacking.

The only extant classification is that of Einstein manifolds of maximal symmetry. It is a classic result that isometries of euclidean space are given by affine orthogonal transformations: that is, transformations consisting of an orthogonal transformation followed by a translation. In n-dimensional euclidean space, there are n independent translations and $n(n-1)/2$ independent orthogonal transformations, generating an $n(n+1)/2$-dimensional isometry group. The same is true in any other signature, for example in Minkowski spacetime, where Lorentz transformations replace the orthogonal transformations.

Something similar happens in any curved space M. The tangent space at any given point is a copy of a euclidean space whose inner product is given by the value of the metric at that point. Every infinitesimal isometry (i.e., every Killing vector) has a translational component and an orthogonal component relative to this euclidean space, and moreover it is uniquely determined by the values of these components. Therefore there are at most $n(n+1)/2$ linearly independent Killing vectors, where $n = \dim M$, whence the dimension of the isometry group is at most $n(n+1)/2$. Again a similar story holds in lorentzian signature, where the tangent space at every point is a copy of Minkowski spacetime.

It is known that maximally symmetric spaces have constant (sectional) curvature. In the riemannian case (and up to quotients) we have spheres, hyperbolic spaces and flat space, fitting into a one-parameter (say, λ) family of solutions, illustrated for two dimensions in Figure 1:

- if $\lambda > 0$, then we have the sphere S^n of radius $1/\lambda^2$ in $\mathbb{E}^{n+1}$; that is, the set of points $(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that
$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1/\lambda^2 \ .$$
- if $\lambda = 0$, then we have the euclidean space $\mathbb{E}^n$ itself; and
- if $\lambda < 0$, then we have hyperbolic space H^n: one sheet of the hyperboloid in $\mathbb{E}^{n,1}$ consisting of those points $(x_0, x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1}$ such that
$$-x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2 = -1/\lambda^2 \ .$$
Notice that although the metric in $\mathbb{E}^{n,1}$ is lorentzian, the one induced on the hyperboloid is riemannian.



(a) $\lambda > 0$ (b) $\lambda = 0$ (c) $\lambda < 0$

FIGURE 1. Riemannian spaces of constant curvature λ.

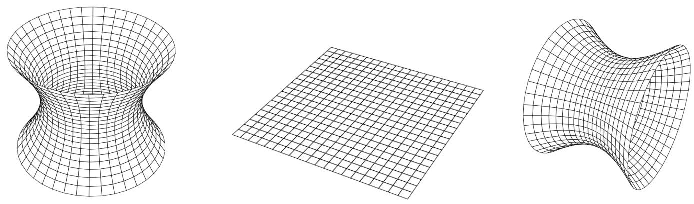
In the lorentzian case, we have de Sitter and anti de Sitter spacetimes and Minkowski spacetime, also fitting into a one-parameter family and illustrated for two dimesions in Figure 2:

- if $\lambda > 0$, then we have de Sitter spacetime dS_n: the quadric in $\mathbb{E}^{n,1}$ consisting of points $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ such that
$$-x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2 = 1/\lambda^2 \ .$$
- if $\lambda = 0$, then we have Minkowski spacetime $\mathbb{E}^{n-1,1}$ itself; and

- if $\lambda < 0$, then we have anti de Sitter spacetime AdS_n: the hyperboloid in $\mathbb{E}^{n-1,2}$ consisting of points $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ such that

$$-x_0^2 + x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - x_n^2 = -1/\lambda^2 \ .$$

Again we notice that although the embedding space has two "times", the metric induced on the hyperboloid is lorentzian.



(a) $\lambda > 0$ (b) $\lambda = 0$ (c) $\lambda < 0$

FIGURE 2. Lorentzian spaces of constant curvature λ.

Notice that spheres, hyperbolic spaces, de Sitter and anti de Sitter spaces are all quadrics in some euclidean or lorentzian space one dimension higher, and the flat spaces (euclidean and Minkowski) correspond to degenerations of these quadrics. Indeed, in both cases, λ is a continuous *modulus* which interpolates between the different solutions. In particular, euclidean space is the limit $\lambda \to 0$ of the sphere (for $\lambda > 0$) and of hyperbolic space (for $\lambda < 0$). Similarly, Minkowski spacetime is the limit $\lambda \to 0$ of de Sitter (for $\lambda > 0$) and anti de Sitter (for $\lambda < 0$) spacetimes.

Now General Relativity is a classical theory: it explains weak field, large scale gravitational interactions from the large scale structure of the universe to the motion of planets (and satellites, as used in the Global Posititioning System!) and even an apple falling from a tree (a process already adequately described by the newtonian limit); but it insists on treating classically other forces in nature which we know to be explained by quantum field theories. This does not represent a major problem provided we restrict ourselves to scales where we can treat the gravitational interactions classically or ignore them altogether. However in regimes where quantum gravitational effects cannot be neglected, one needs a theory of quantum gravity. One such theory is superstring/M-theory, as explained to us earlier this week by Mohab Abou-Zeid. String theories have low-energy limits called supergravity theories, to which Ali Chamseddine made important pioneering contributions. The study of supergravity is still a very active field of research and is one of the areas of expertise of both Ali Chamseddine and Wafic Sabra here in CAMS.

For the purposes of this talk we will think of supergravity theories as extensions of General Relativity describing the dynamics of other fields besides the metric: *bosonic* fields such as scalars, gauge fields and their p-form generalisations and their supersymmetric partners, *fermionic* fields such as gravitini, dilatini, gaugini. The field content depends among other things on the dimension of the spacetime, but one thing that all supergravity theories have in common (apart from the metric and at least one gravitino) is the invariance of the equations of motion under *supersymmetry* transformations with an arbitrary spinor parameter. Supersymmetry is a nontrivial extension of the familiar spacetime symmetries (translations and lorentz transformations) which mixes particles of different spins and hence of different statistics.

With fermions put to zero, the classical equations of a supergravity theory consist of generalised Einstein equations (1) where the energy-momentum tensor T receives contributions from all the bosonic fields in the theory, together with field equations for the other fields which generalise the Klein-Gordon and Maxwell equations.

For example, let us consider eleven-dimensional supergravity. This theory started life as a promising candidate for Kaluza–Klein unification in the 1980s, and it is now interpreted as a low-energy description of the strong coupling limit of IIA superstring theory. It is in fact an eleven-dimensional corner of M-theory. Its bosonic field content is (g, F) where g is a lorentzian metric and F is a closed four-form. The equations of motion are a version of equation (1) with an explicit right-hand side $T(g, F)$ depending on g, F, together with a nonlinear generalisation of the Maxwell equation (written in differential form language)

$$d \star F = \tfrac{1}{2} F \wedge F \ , \tag{3}$$

where $\star$ is the Hodge star operator.

As with the Einstein equations (2), these equations admit a plethora of solutions (with new solutions appearing on a daily basis) whose study continues to be vigorously pursued. The analogue of the strategy of looking for solutions with a large degree of symmetry is to look for solutions with a large amount of supersymmetry. In the present context, this means to look for solutions (g, F) which admit *Killing*[1] *spinors*; that is, solutions of a linear first-order partial differential equation

$$D\varepsilon = (\nabla + \Omega)\varepsilon = 0 \ , \tag{4}$$

where ∇ is the spin connection and $\Omega(g, F)$ is a zeroth order piece (strictly speaking a one-form with values in endomorphisms of the spinor bundle). This is a linear equation hence the space of solutions is a vector space of (real) dimension ≤ 32, which is the dimension of the spinor representation. The normalised dimension $\nu = \frac{1}{32} \dim \ker D$ defines a fraction which measures the amount of supersymmetry preserved by the solution.

For example if $F = 0$, then the solutions are Ricci-flat lorentzian manifolds and Killing spinors are parallel. The existence of a Killing spinor then reduces the holonomy group of the spacetime. Since there are two (maximal) spinor isotropy groups in eleven dimensions: $\mathrm{SU}(5)$ or $\mathrm{Spin}(7) \ltimes \mathbb{R}^9$, there are two types of supersymmetric solutions, which generalise the Kaluza–Klein monopole and the pp-wave, respectively [7]. If $F \neq 0$ things are more complicated; although some partial results along these lines have been obtained recently [10].

To this date, the only complete classification is that of the maximally supersymmetric solutions, arrived at in collaboration with George Papadopoulos [9]. There is again one parameter which can be chosen to be the constant scalar curvature R of the spacetime. Up to local isometry, we have the following classification:

- If $R > 0$ then the metric is that of $S^4 \times \mathrm{AdS}_7$, where S^4 has scalar curvature $8R$ and AdS_7 has scalar curvature $-7R$, and the four-form F is proportional to the volume form on S^4: $F = \sqrt{6R}\, \mathrm{dvol}(S^4)$;
- If $R < 0$ then the metric is that of $\mathrm{AdS}_4 \times S^7$, where AdS_4 has scalar curvature $8R$, S^7 has scalar curvature $-7R$ and $F = \sqrt{-6R}\, \mathrm{dvol}(\mathrm{AdS}_4)$; and
- If $R = 0$ then there is a one-parameter (μ, say) family of symmetric plane waves [6]. For $\mu \neq 0$, all solutions are isometric, whereas for $\mu = 0$ the solution is flat with $F = 0$.

The maximally supersymmetric plane wave is an example of a *symmetric plane wave*, whose metric is generally given by

$$g = 2du dv + \sum_{i,j=1}^{9} A_{ij} x^i x^j du^2 + \sum_{i=1}^{9} dx^i dx^i \ , \tag{5}$$

[1] so called because they are square roots of Killing vectors.

where A_{ij} is a constant symmetric matrix. The four-form is null (that is, it has zero norm) and has the generic form

$$F = du \wedge \Theta \ , \qquad \text{where} \quad \Theta = \tfrac{1}{6} \sum_{i,j,k=1}^{9} \Theta_{ijk} dx^i \wedge dx^j \wedge dx^k \ , \tag{6}$$

where Θ_{ijk} are constants.

We can give an explicit description of a symmetric plane wave as the intersection of two quadrics in a flat space with two time coordinates $\mathbb{E}^{11,2}$. Let $(U^1, V^1, U^2, V^2, X^i)$, for $i = 1, \ldots, 9$, be coordinates in $\mathbb{E}^{11,2}$, where U^a, V^a are lightcone coordinates so that the flat metric takes the form

$$dU^1 dV^1 + dU^2 dV^2 + \sum_{i=1}^{9} dX^i dX^i \ . \tag{7}$$

Then the symmetric plane wave is the induced metric on the intersection of the two quadrics

$$(U^1)^2 + (U^2)^2 = 4 \qquad \text{and} \qquad U^1 V^1 + U^2 V^2 = \sum_{i,j=1}^{9} A_{ij} X^i X^j \ . \tag{8}$$

Notice that the solutions for $R \neq 0$ are also intersection of two quadrics in the same space.

In the maximally supersymmetric case [13, 8], A_{ij} is given by

$$A_{ij} = \begin{cases} -\mu^2 \delta_{ij} & \text{for } i,j = 1,2,3, \\ -\frac{1}{4}\mu^2 \delta_{ij} & \text{for } i,j = 4,5,\ldots,9; \end{cases} \tag{9}$$

and $\Theta = \mu dx^1 \wedge dx^2 \wedge dx^3$.

Notice that taking the limit $R \to 0$ in either the $\mathrm{AdS}_4 \times S^7$ or $S^4 \times \mathrm{AdS}_7$ solutions, we get the flat solution (corresponding to $\mu = 0$). It is therefore natural to ask whether one can also obtain the symmetric plane wave (with $\mu \neq 0$) as a limit.

In the 1970s Roger Penrose wrote a paper [16] titled "Any space-time has a plane wave as a limit". This limit consists in blowing up the neighbourhood of a null geodesic (that is, the trajectory of light ray) while simultaneously boosting along the direction of the geodesic. It was Güven [12] who extended Penrose's limit to supergravity theories. The most important property of this limit is that it takes solutions to solutions and part of its beauty is that it exploits the different symmetries of the supergravity equations of motion. This limit can be placed in the broader context of "limits of spacetimes" due to Geroch [11], and this allowed Matthias Blau, George Papadopoulos and myself [5] to derive other important properties of the Penrose limit: a covariance property which is useful in classifying the possible Penrose limits and the fact that in the limit symmetries and supersymmetries are preserved (or often enhanced). It follows that the plane wave limit of a maximally supersymmetric solution will be maximally supersymmetric. It turns out that there are two non-isometric plane wave limits of $\mathrm{AdS}_4 \times S^7$ and $S^4 \times \mathrm{AdS}_7$: the maximally supersymmetric plane wave and flat space [4, 5].

The situation is similar for ten-dimensional IIB supergravity. There is a one-parameter family of solutions, where the parameter R' now takes nonnegative values [9]:

- If $R' > 0$ then the metric is that of $\mathrm{AdS}_5 \times S^5$, where AdS_5 has scalar curvature $-R'$ and S^5 has scalar curvature R', the self-dual five-form F is proportional (depending on the value of the constant dilaton) to the sum of the volume forms

$$F \propto \mathrm{dvol}(\mathrm{AdS}_5) - \mathrm{dvol}(S^5) \ . \tag{10}$$

  The other fields are not turned on.

- If $R' = 0$ then there is a one-parameter (μ, say) family of symmetric plane waves. Again for $\mu \neq 0$, all solutions are isometric, whereas for $\mu = 0$ the solution is flat with $F = 0$.

The maximally supersymmetric plane wave, discovered in collaboration with Matthias Blau, Chris Hull and George Papadopoulos [3], has metric of the form (5), except that i, j run from 1 to 8 and $A_{ij} = -\mu^2\delta_{ij}$. The self-dual five-form takes the form

$$F \propto \mu du \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8) \ , \tag{11}$$

where the proportionality constant depends on the value of the dilaton.

These solutions are again related by plane wave limits. This fact, aided by the fact that string theory on symmetric plane waves are exactly solvable [15], has led to new insights into the gravity/gauge theory correspondence thanks to the observations in [1] and much subsequent work by a large number of individuals.

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References

1. D Berenstein, J Maldacena, and H Nastase, *Strings in flat space and pp waves from N=4 Super Yang Mills*, `arXiv:hep-th/0202021`, 2002.
2. AL Besse, *Einstein manifolds*, Springer-Verlag, 1987.
3. M Blau, J M Figueroa-O'Farrill, C M Hull, and G Papadopoulos, *A new maximally supersymmetric background of type IIB superstring theory*, J. High Energy Phys. **01** (2002), 047, `arXiv:hep-th/0110242`.
4. ———, *Penrose limits and maximal supersymmetry*, Class. Quant. Grav. **19** (2002), L87–L95, `arXiv:hep-th/0201081`.
5. M Blau, J M Figueroa-O'Farrill, and G Papadopoulos, *Penrose limits, supergravity and brane dynamics*, Class. Quant. Grav. **19** (2002), 4753–4805, `arXiv:hep-th/0202111`.
6. M Cahen and N Wallach, *Lorentzian symmetric spaces*, Bull. Am. Math. Soc. **76** (1970), 585–591.
7. J M Figueroa-O'Farrill, *Breaking the M-waves*, Class. Quant. Grav. **17** (2000), 2925–2947, `arXiv:hep-th/9904124`.
8. J M Figueroa-O'Farrill and G Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, J. High Energy Phys. **06** (2001), 036, `arXiv:hep-th/0105308`.
9. ———, *Maximal supersymmetric solutions of ten- and eleven-dimensional supergravity*, J. High Energy Phys. **03** (2003), 048, `arXiv:hep-th/0211089`.
10. J P Gauntlett and S Pakis, *The geometry of $D = 11$ Killing spinors*, J. High Energy Phys. **04** (2003), 039, `arXiv:hep-th/0212008`.
11. R Geroch, *Limits of spacetimes*, Comm. Math. Phys. **13** (1969), 180–193.
12. R Güven, *Plane wave limits and T-duality*, Phys. Lett. **B482** (2000), 255–263, `arXiv:hep-th/0005061`.
13. J Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, Phys. Lett. **134B** (1984), 194–196.
14. C R LeBrun and M Wang (eds.), *Essays on Einstein manifolds*, International Press, 1999.
15. R R Metsaev, *Type IIB Green–Schwarz superstring in plane wave Ramond-Ramond background*, `arXiv:hep-th/0112044`, 2002.
16. R Penrose, *Any space-time has a plane wave as a limit*, Differential geometry and relativity, Reidel, Dordrecht, 1976, pp. 271–275.

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