Word length statistics and Lyapunov exponents for Fuchsian groups with cusps

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Abstract. Given a Fuchsian group with at least one cusp, Deroin, Kleptsyn and Navas define a Lyapunov expansion exponent for a point on the boundary, and ask if it vanishes for almost all points with respect to Lebesgue measure. We give an affirmative answer to this question, by considering the behavior of word metric along typical geodesic rays and their excursions into cusps. We also consider the behavior of word length along rays chosen according to harmonic measure on the boundary, arising from random walks with finite first moment. We show that the excursions have different behavior in the Lebesgue measure and harmonic measure cases, which implies that these two measures are mutually singular.

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1. Introduction

Let G be a Fuchsian group, i.e. a discrete subgroup of $SL(2, \mathbb{R})$, and suppose the quotient $G \backslash \mathbb{H}^2$ has finite volume but is not compact (such a group is also called a *non-uniform lattice* in $SL(2, \mathbb{R})$).

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For any finitely generated group G of circle diffeomorphisms and any point $p \in S^1$, Deroin-Kleptsyn-Navas [3] define the *Lyapunov expansion exponent* of G at p as

$$\lambda_{exp}(p) := \limsup_{R\to\infty} \max_{g\in B(R)} \frac{1}{R} \log |g'(p)| \tag{1}$$

where $B(R)$ is a ball of radius R in G with respect to a word metric for some finite generating set.

Theorem 1.1. *For a non-uniform lattice in $SL(2,\mathbb{R})$, we have*

$$\lambda_{exp}(p) = 0$$

for almost every $p \in S^1$ with respect to Lebesgue measure.

This answers a question of Deroin-Kleptsyn-Navas [3, Question 3.3] in the affirmative. The essential idea is that, given $p \in S^1$, the group elements realizing the maximum of the derivative in definition (1) are the closest ones to the geodesic ray from the basepoint to p, and we show that their derivative grows subexponentially.

We shall consider two different metrics on the group G. As G is finitely generated, we can endow it with a *word metric* d_G with respect to a finite set of generators. On the other hand, the group G is hyperbolic relatively to the parabolic subgroups, in the sense of Farb [4]. Thus, G can be also equipped with a *relative metric* d_{rel}, in which any distance in a subgroup fixing a cusp has constant length (see section 2; note that this metric is usually not proper).

Given a basepoint $x_0 \in \mathbb{H}^2$, we may identify the unit tangent space at x_0 with the circle $S^1 = \partial\mathbb{H}^2$ at infinity, and the measure induced on the boundary is absolutely continuous with respect to Lebesgue measure on the unit circle.

Let γ be a geodesic ray from the basepoint x_0, and γ_t a point at distance t from the basepoint along γ. For each time t, let h_t be a group element such that $h_t x_0$ is a closest element of the G-orbit of x_0 to γ_t. A way to measure the penetration into the cusp of the geodesic γ_t is to consider the ratio d_G/d_{rel} between the word and relative metrics, since consecutive powers of parabolic elements increase the numerator but not the denominator. We thus define the quantity

$$\rho(\gamma) := \lim_{t\to\infty} \frac{d_G(1,h_t)}{d_{rel}(1,h_t)},$$

which we shall refer to as the *word length ratio*. In Section 2, we shall show that the limit is infinite for almost all geodesics in visual measure.

Theorem 1.2. *Let G be a non-uniform lattice in $SL_2(\mathbb{R})$. Then the word length ratio $\rho(\gamma) = \infty$ for almost all geodesics chosen according to Lebesgue measure on the circle at infinity.*

In Section 3, we shall use Theorem 1.2 to prove Theorem 1.1. Finally, in Section 4, we show an analogous result for random walks. Furstenberg [5] showed that the image of a random walk on G in $\mathbb{H}^2$ under the orbit map $g \to gx_0$ converges almost surely to the boundary, defining a harmonic measure ν on S^1. We show that the word length ratio is finite for almost all rays chosen according to harmonic measure.

Theorem 1.3. *Let G be a non-uniform lattice in $SL_2(\mathbb{R})$, and let ν be a harmonic measure on the boundary $\partial\mathbb{H}^2$ determined by a probability distribution μ with finite first moment in the word metric, whose support generates G as a semigroup. Then there is a constant $c > 0$ such that the word length ratio $\rho(\gamma) = c$, for ν-almost all geodesic rays.*

Comparing Theorems 1.2 and 1.3 shows that geodesics chosen at random with respect to the Lebesgue measure penetrate more deeply into the cusps than geodesics chosen at random with respect to harmonic measure.

In particular, this shows that Lebesgue measure and hitting measure are mutually singular. This was previously known for the congruence subgroup $\Gamma(2)$ of $PSL(2,\mathbb{Z})$ by Guivarc'h and Le Jan [9,10], for $SL(2,\mathbb{Z})$ and μ with finite first moment, by Deroin, Kleptsyn and Navas [3], and for non-uniform lattices in $SL(2,\mathbb{R})$ and μ with finite support, by Blachère, Haïssinsky and Mathieu [1]. It is worth remarking that the assumption of finite first moment is essential, as Furstenberg showed that it is possible to construct random walks (with infinite first moment in the word metric) whose harmonic measure is in the Lebesgue class [6].

Theorems 1.2 and 1.3 are extended to the case of the mapping class group acting on Teichmüller space in [7].

1.1. Notation. We will write $f(x) \lesssim g(x)$ to mean that the inequality holds up to additive and multiplicative constants, i.e. there are constants K and c such that

$$f(x) \leqslant Kg(x) + c,$$

and similarly $f(x) \asymp g(x)$ will mean that there exist constants K, c such that

$$\frac{1}{K}g(x) - c \leqslant f(x) \leqslant Kg(x) + c.$$

2. The word length ratio for Lebesgue measure

Let G be a non-uniform lattice in $SL(2,\mathbb{R})$, and $X = \mathbb{H}^2/G$ the corresponding hyperbolic surface with cusps. Given $\epsilon > 0$, the *thick part* of X is the set of points $x \in X$ with injectivity radius larger than ϵ (i.e. such that the ball of radius $\epsilon/2$ in X centered at x has fundamental group with finite image in the (orbifold) fundamental group of X, which we may identify with G). We shall denote the thick part, which is compact, by N, and its complement is called the *thin part*. If ϵ is sufficiently small, then the thin part is the union of disjoint neighborhoods $c_1, \cdots, c_p$ of the cusps of X. The universal cover of X is the hyperbolic plane $\mathbb{H}^2$, and the lift of the union $c_1 \cup \cdots \cup c_p$ of the cusp neighborhoods in the universal cover is the union of countably many disjoint horoballs, which we shall denote by $\mathcal{H}$.

The group G is finitely generated, and a finite choice of generators $\mathcal{A}$ for G defines a proper *word metric* on G. Different choices of generators produce quasi-isometric metrics. For each cusp neighborhood c_i in X, let us choose a lift $\tilde{c}_i$ in the universal cover, and denote by G_i the stabiliser of $\tilde{c}_i$. The group G_i is infinite cyclic and is a maximal parabolic subgroup; let g_i be a generator of G_i. We may also define a *relative metric* on G by taking the word metric with respect to the larger (infinite) generating set

$$\mathcal{A}' := \mathcal{A} \cup G_1 \cup \cdots \cup G_p;$$

that is, along with the generators of G, the set $\mathcal{A}'$ includes all powers of all the parabolic generators g_i. The metric space (G, d_{rel}) is not proper, but it is Gromov hyperbolic. In fact, as proven by Farb, G is strongly hyperbolic relative to the parabolic subgroups G_i [4, Theorem 4.11].

The unit tangent bundle $T^1\mathbb{H}^2$ carries a natural $SL(2, \mathbb{R})$-invariant measure, which in the upper half-plane model is given by $d\ell = \frac{dx\ dy\ d\theta}{y^2}$. This measure descends to a measure on the unit tangent bundle to $X = G \backslash \mathbb{H}^2$ which is invariant for the geodesic flow, and is called *Liouville measure.* Moreover, it is a classical result due to Hopf [11] that this flow is ergodic, and indeed mixing. The Haar measure on the unit circle in the tangent space at any point is the pullback via the visual map of the standard Lebesgue measure on $\partial\mathbb{H}^2 = S^1$.

By studying the collection $\mathcal{H}$ of horoballs, Sullivan [15] showed that a generic geodesic ray with respect to Lebesgue measure is recurrent to the thick part of X, and ventures into the cusps infinitely often with maximum depth in the cusps of about $\log t$, where t is the time along the geodesic ray.

Given a horoball H and a geodesic γ that enters and leaves H, we define the excursion $E(\gamma, H)$ to be the distance in the path metric on ∂H between the entry and exit points (see section 2.2). Sullivan's theorem implies that a lift in $\mathbb{H}^2$ of a Lebesgue-typical geodesic ray enters and leaves infinitely many horoballs in the packing. We use this setup to estimate from below the word length along a Lebesgue-typical geodesic in terms of the sum of the excursions in these horoballs.

We say a basepoint $x_0 \in \mathbb{H}^2$ is generic if the stabilizer of x_0 in G is trivial. The G-orbit of the basepoint x_0 is called a lattice, and if x_0 is a generic basepoint, then each lattice point corresponds to a unique group element. We shall assume that we have chosen a generic basepoint, and then each point γ_t along the geodesic has at least one closest lattice point $h_t x_0$, and in fact this closest point is unique for almost all points along the geodesic.

2.1. Projected paths are quasigeodesic. Let us now fix some thick part N of X, and let $\widetilde{N}$ be its preimage in the universal cover. The space $\widetilde{N}$ is a geodesic metric space with the following path metric. Every two points

x, y in $\widetilde{N}$ are connected by some arc, and the *path metric* between x and y is defined as the infimum of the (hyperbolic) lengths of all rectifiable arcs connecting x and y. We shall denote this distance as $d_{\widetilde{N}}(x, y)$. Since the quotient $G\backslash\widetilde{N} = N$ is compact, then by the Švarc-Milnor lemma the space $\widetilde{N}$ with the path metric is quasi-isometric to the group G endowed with the word metric. A geodesic for the metric $d_{\widetilde{N}}$ will be called a *thick geodesic*.

In order to have a better control on the geometry of the thick part, we shall now define a canonical way to connect two points in the thick part, and prove that these canonical paths (which we call *projected paths*) are quasigeodesic for the path metric on $\widetilde{N}$.

Each point of $\mathbb{H}^2$ has a unique closest point in the thick part $\widetilde{N}$, hence we can define the closest point projection map $\pi_{\widetilde{N}} : \mathbb{H}^2 \to \widetilde{N}$. Any two points x, y in the thick part $\widetilde{N}$ are connected by a hyperbolic geodesic segment γ in $\mathbb{H}^2$, which may pass through a number of horoballs in $\mathcal{H}$. The *projected path* $p(x, y)$ between x and y is the closest point projection of the geodesic segment between x and y to the thick part:

$$p(x, y) := \pi_{\widetilde{N}}(\gamma).$$

More explicitly, the geodesic γ intersects a finite number r (possibly zero) of horoballs of the collection $\mathcal{H}$, which we denote as $H_1, \ldots, H_r$, and the intersection of γ with $\widetilde{N}$ is the union of $r + 1$ geodesic segments

$$[x, x_1] \cup [x_2, x_3] \cup \cdots \cup [x_{2r}, y].$$

The projected path $p(x, y)$ follows the geodesic segment $[x, x_1]$ in the thick part, then follows the boundary of the horoball H_1 from x_1 to x_2, then again the geodesic segment $[x_2, x_3]$ and so on, alternating paths on the boundary of the horoballs H_i with hyperbolic geodesic segments in the thick part until it reaches y. Given x and y in $\widetilde{N}$, we shall denote as $L(x, y)$ the length of the projected path $p(x, y)$ joining x and y.

The usefulness of projected paths arises from the fact that they are quasi-geodesic, as proven in the following lemma.

Lemma 2.1. *There are positive constants L, K and c, such that if the distance between the horoballs is at least L, then the projected path p is a (K, c)-quasigeodesic in the thick part $\widetilde{N}$.*

Proof. Let γ be a geodesic ray in $\mathbb{H}^2$, both of whose endpoints lie in the thick part $\widetilde{N}$. Let p be the projected path, and let q be the thick geodesic in $\widetilde{N}$ connecting the endpoints of γ. As q is a thick geodesic, the length of q is at most the length of the projected path p. We now show that the length of the thick geodesic q is at least the length of the projected path p, minus $2n$, where n is the number of horoballs the geodesic γ intersects. As long as the distance between the horoballs is at least 4, this implies that p is a $(2, 2)$-quasigeodesic.

Label the intersecting horoballs H_i, in the order in which they appear along γ. The hyperbolic geodesic γ intersects the boundary of each horoball twice, and we shall label these intersections $\gamma_{t_{2i-1}}$ and $\gamma_{t_{2i}}$, as illustrated below in Figure 1.

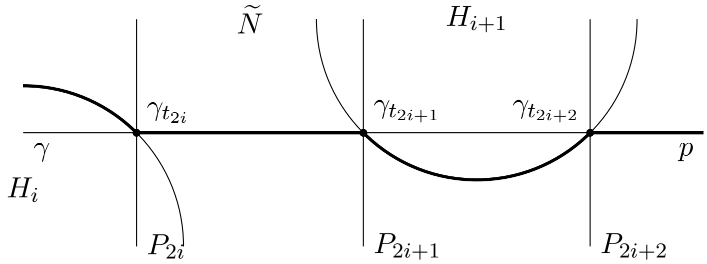

FIGURE 1. Perpendicular geodesics through intersections of γ and ∂H_i.

For each point of intersection γ_{t_i}, let P_i be the perpendicular geodesic to γ through γ_{t_i}. Each perpendicular geodesic P_i separates the endpoints of γ, so any path connecting the endpoints must pass through each perpendicular plane. Furthermore, the perpendicular geodesics are all disjoint, so they divide the hyperbolic plane into regions, each of which contains a subsegment of γ which is either entirely contained in the thick part $\widetilde{N}$, or else is entirely contained in a single horoball. As the regions are disjoint, the length of any path is the sum of the lengths of its intersections with each region. We now show that the length of the thick geodesic q in each region is bounded below by the length of the projected path in that region, up to a bounded additive error.

First consider a region between an adjacent pair P_{2i} and P_{2i+1} of perpendicular geodesics containing a segment of γ of length d_{2i} in the thick part $\widetilde{N}$. The length of the projected path p inside this region has length exactly d_{2i}. As nearest point projection onto the geodesic is distance decreasing in $\mathbb{H}^2$, any path from P_{2i} to P_{2i+1} has length at least d_{2i} in the hyperbolic metric, and hence also in the thick metric. Therefore the intersection of the thick geodesic q with this region has length at least d_{2i}, i.e. at least the length of the projected path.

Now consider a region between an adjacent pair P_{2i+1} and P_{2i+2} of perpendicular geodesics containing a segment of γ of length d_{2i+1} in the boundary of a horoball H_{i+1}. The length of the projected path p in this region has length exactly d_{2i+1}. The image of the part of the perpendicular geodesic P_{2i+1} in the thick part $\widetilde{N}$ projected onto the horoball H_{i+1} has diameter at most 1. Similarly, image of the part of the perpendicular geodesic P_{2i+2} in the thick part $\widetilde{N}$ projected onto the horoball H_{i+1} also has diameter at most 1. Therefore, as the nearest point projection from $\mathbb{H}^2 \setminus H_{i+1}$ onto the boundary of the horoball H_{i+1} is distance decreasing, the length of any path in $\widetilde{N}$ between P_{2i+1} and P_{2i+2} has length at least $d_{2i+1} - 2$.

This implies that the length of the thick geodesic q is at least the length of the projected path, minus $2n$, where n is the number of horoballs the geodesic γ passes through. If we assume that the horoballs are distance at least $L \geqslant 4$ apart, then the length of the thick geodesic is at least half the length of the projected path, up to an additive error of at most 2. □

2.2. The word metric for Fuchsian groups. We now show that word length is coarsely monotonic along geodesics. Recall that we write h_t to denote the closest lattice point to γ_t.

Proposition 2.2. *There are constants $c_1 > 0$ and c_2 such that for any geodesic γ and for any $0 \leqslant s \leqslant t$*

$$d_G(1, h_s) \leqslant c_1 d_G(1, h_t) + c_2.$$

Proof. Let $p_t := \pi_{\widetilde{N}}(\gamma_t)$ be the point on the projected path that is closest to γ_t. Recall that $L(x, y)$ is the length of the projected path joining x and y. The function $t \mapsto L(x_0, p_t)$ is continuous and for any $0 \leqslant s \leqslant t$ it satisfies $L(x_0, p_s) \leqslant L(x_0, p_t)$. The proposition then follows as the projected path is a (K, c)-quasi geodesic in the thick part $\widetilde{N}$, and the thick part with its path metric is quasi-isometric to G with the word metric. □

We shall define the *excursion* of γ_t with respect to the horoball H to be the length (in $\widetilde{N}$) of the intersection of the projected path $p(0, t)$ from p_0 to p_t with the horoball H, i.e.

$$E(\gamma_t, H) := L_{\widetilde{N}}(p(0, t) \cap H),$$

where $L_{\widetilde{N}}$ denotes the length of the path in the $\widetilde{N}$-metric. Similarly, we shall denote the excursion of the geodesic γ into H as

$$E(\gamma, H) := \lim_{t \to \infty} E(\gamma_t, H),$$

and this limit is finite for each horoball for almost all geodesic rays.

We now show that the sum of the excursions along the geodesic gives a lower bound on the word length, using the cutoff function $\lfloor x \rfloor_A$, defined by

$$\lfloor x \rfloor_A = \begin{cases} x & \text{if } x \geqslant A \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Proposition 2.3. *There are constants $A > 0, c > 0$ and d such that*

$$d_G(1, h_t) \geqslant \sum_{H \in \mathcal{H}} c \lfloor E(\gamma_t, H) \rfloor_A - d. \tag{3}$$

Proof. The excursion $E(\gamma_t, H)$ is the length of the horocyclic segment of the projected path in ∂H, and so the sums of the lengths of the excursions is a lower bound on the length of the projected path. The projected path p

is quasi-geodesic in $\widetilde{N}$, and $\widetilde{N}$ is quasi-isometric to the word metric, and so the result follows. □

2.3. The geodesic flow. Let $\mathcal{H}_n$ be the subset of the horoballs $\mathcal{H}$ consisting of those points which are at least distance $\log n$ from the boundary of the horoballs in the hyperbolic metric, i.e.

$$\mathcal{H}_n := \{x \in \mathbb{H}^2 \ : \ d(x, \partial\mathcal{H}) \geqslant \log n\}.$$

Let us denote as X_n the quotient of $\mathcal{H}_n$ under the action of G, so $X_n \subset X$. We will write T^1X for the unit tangent bundle to X, and T^1Y for the restriction of the unit tangent bundle to any subset $Y \subset X$. Given a geodesic ray γ, we will denote as v_t the unit tangent vector to γ at the point γ_t. Let ℓ denote the Liouville measure on T^1X. Since the geodesic flow on T^1X is ergodic, for any function $\psi \in L^1(T^1X, \ell)$, and for almost every geodesic ray γ, we have the equality

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(v_t)dt = \int_X \psi(v)d\ell.$$

In particular, the proportion of time that a geodesic ray spends in X_n is asymptotically the same as the volume of T^1X_n, and an elementary calculation in hyperbolic space shows that this volume is $1/n$, up to a multiplicative constant depending on the choice of cusp horoballs. Let χ_n be the characteristic function of $T^1X_{2^n}$, and let $\psi : T^1X \to \mathbb{R}$ be

$$\psi(v) := \sum_{n=1}^{\infty} 2^n \chi_n(v).$$

This function is not in $L^1(T^1X, \ell)$, but it is well defined, since each v lies in finitely many X_n. We now show that, as a consequence of the $1/n$ decay of volumes, the ergodic average of ψ is infinite.

Proposition 2.4. *For almost every tangent vector $v \in T^1X$ with respect to Liouville measure, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(v_t)dt = \infty. \tag{4}$$

Proof. Let $\psi_N : T^1X \to \mathbb{R}$ be the truncation

$$\psi_N(v) = \sum_{n=1}^{N} 2^n \chi_n(v),$$

which does lie in $L^1(T^1X, \ell)$, and is a lower bound for ψ. Up to a uniform multiplicative constant,

$$\int_{T^1X} \psi_N \ d\ell \asymp N.$$

By ergodicity, along ℓ-almost every geodesic ray γ we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \psi_N(v_t)dt = \int_{T^1X}\psi_N \; d\ell \asymp N$$

where v_t is the unit tangent vector to γ at the point γ_t. As a consequence, along ℓ-almost every geodesic ray γ the inequality

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T \psi(v_t)dt \geqslant \lim_{T\to\infty}\frac{1}{T}\int_0^T \psi_N(v_t)dt \asymp N$$

holds for all N, which yields the claim. □

Proposition 2.5. *Let H be a horoball in $\mathcal{H}$, and let $t_1 < t_2$ be the entry and exit times in H for a geodesic ray γ, and let $A > 0$ be a constant. Then up to uniform additive and multiplicative constants, which depend on A,*

$$\int_{t_1}^{t_2}\psi(v_t)\; dt \asymp \lfloor E(\gamma, H)\rfloor_A,$$

where $\lfloor x\rfloor_A$ is the cutoff function defined in (2)*.*

Proof. Let N be the smallest number such that $\psi(v_t) = \psi_N(v_t)$ for every $t \in [t_1, t_2]$, so that up to a uniform additive constant $2^N \leqslant E(\gamma, H) \leqslant 2^{N+1}$. We shall write H_n for the intersection of the horoball H with $\mathcal{H}_n$, so H_n consists of all points of H that are distance at least $\log n$ from ∂H. In the upper half-plane model for hyperbolic space, we may assume that the boundaries of the H_n are given by horizontal lines, and the geodesic γ is part of a circle perpendicular to the real line. The hyperbolic distance between H_{2^k} and $H_{2^{k+1}}$ is independent of k, and the shortest geodesic running between them is a vertical line, and the longest geodesic segment is given by a semicircle tangent to the upper horizontal line. This implies that for $k \leqslant N-1$, there are uniform lower and upper bounds independent of k and N for the amount of time s_k that the geodesic ray γ can spend in $H_{2^k}\setminus H_{2^{k+1}}$. There is also a uniform upper bound independent of N for the amount of time s_N that the ray γ can spend in $H_{2^N}\setminus H_{2^{N+1}}$. These bounds imply

$$\int_{t_1}^{t_2}\psi_N(v_t)\; dt \asymp \sum_{k=1}^{N} s_k\left(\sum_{j=1}^{k}2^j\right) \asymp 2^N \asymp E(\gamma, H).$$

Finally, we observe that the function x is equivalent to $\lfloor x\rfloor_A$, up to a suitably chosen additive constant, and so the result follows. □

Combining Propositions 2.3, 2.5 and Equation (4) we obtain the

Proposition 2.6. *For Lebesgue-almost every γ we have*

$$\lim_{T\to\infty}\frac{d_G(1, h_T)}{T} = \infty.$$

On the other hand, the relative length of h_T is up to a uniform multiplicative constant bounded above by T. In fact, by ergodicity, the ray γ spends a definite proportion of its time in the thick part of X. This implies that the relative length of h_T grows linearly in T. Combining this observation with the limit above completes the proof Theorem 1.2.

3. Lyapunov expansion exponent

We now use Theorem 1.2 to prove Theorem 1.1, that if G is a non-uniform lattice in $SL(2,\mathbb{R})$, then for Lebesgue-almost every $p \in S^1$ the Lyapunov expansion exponent is zero,

$$\lambda_{exp}(p) = 0.$$

Here is the rough idea of the proof of Theorem 1.1. Suppose p is a point in S^1 and let γ be the hyperbolic geodesic ray that connects the origin x_0 in $\mathbb{D}$ to p. Let h_T be the approximating group element for γ_T. We will show that for every group element in a ball of radius $R = d_G(1, h_T)/2K^2$ where K is some uniform constant, the derivative at p has a coarse upper bound of e^{2T}. As T increases, the word length of the approximating group elements is monotonically increasing with bounded jump size. Finally, for Lebesgue-almost every p, Proposition 2.6 says that the ratio T/R goes to zero, which proves Theorem 1.1.

3.1. Derivatives of isometries. We shall use the unit disc model $\mathbb{D}$ of hyperbolic plane. An isometry of $\mathbb{D}$ is of the form

$$f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

where $a \in \mathbb{D}$. Write a as $a = Ae^{i\phi}$ and suppose $f(e^{it}) = e^{ig(t)}$. Differentiation with respect to t, and an elementary calculation, shows that

$$|g'(t)| = \frac{1-A^2}{1+A^2-2A\ \mathrm{Re}(e^{i\phi}e^{-it})}. \tag{5}$$

It follows that $|g'(t)|$ is maximum with value $(1+A)/(1-A)$ when $t = \phi$. Denoting the origin in $\mathbb{D}$ as x_0, note that $(1+A)/(1-A) = e^{d_{\mathbb{H}^2}(x_0, f(x_0))}$ and so in particular, the calculation shows that the maximum value of the logarithm of the derivative on S^1 is equal to the hyperbolic distance that f moves the origin x_0. To summarize, we get

Lemma 3.1. *If g is an isometry of $\mathbb{D}$ such that $d_{\mathbb{H}^2}(x_0, gx_0) \leqslant T$ then for any $p \in S^1$,*

$$|g'(p)| \leqslant e^T.$$

Remark. *Note that the expression* (5) *for $|g'(t)|$ equals the Poisson kernel $P_A(\phi - t)$. In fact, if $u : \mathbb{D} \to \mathbb{R}$ is a harmonic function, then $u \circ f^{-1}$ is* *also harmonic, hence we have from the mean value property and a change of variables*

$$u(a) = u(f^{-1}(0)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(f^{-1}(e^{it}))\ dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{is})|g'(s)|\ ds$$

which is precisely Poisson's representation formula if one sets

$$|g'(s)| = P_A(\phi - s).$$

3.2. Bounding the derivative over a ball in the word metric. Let $p \in S^1$, and γ be the geodesic ray from the origin x_0 to p. Let $p_T = \pi_{\widetilde{N}}(\gamma_T)$ denote the point in the thick part closest to γ_T and let h_T be the approximating group element. Let

$$S(x_0, \gamma_{2T}) := \{x \in \mathbb{D} : d_{\mathbb{H}^2}(x_0, x) \geqslant d_{\mathbb{H}^2}(\gamma_{2T}, x)\}.$$

Thus, $S(x_0, \gamma_{2T})$ is the half-space with $\partial S(x_0, \gamma_{2T})$ orthogonal to γ at the point γ_T.

Proposition 3.2. *There exists constants K, K' such that, if gx_0 lies in $S(x_0, \gamma_{2T})$, then*

$$d_G(1, g) \geqslant \frac{1}{K} d_G(1, h_T) - K'.$$

Before proving Proposition 3.2, we state a basic lemma in hyperbolic geometry. If H is a horoball, we shall denote as π_H the closest point projection map onto the boundary of H; moreover, if x, y lie on ∂H, we denote as $d_{\partial H}(x, y)$ the length of the path along the boundary of H between x and y. Note that the hyperbolic distance between x and y is equal to $\log d_{\partial H}(x, y)$, up to a multiplicative error independent of x and y. We then have the following fact, whose proof we omit.

Lemma 3.3. *Fix a point $y \in \mathbb{D}$ and let H be a horoball that does not contain y. Let γ_0 be the hyperbolic geodesic that goes from y to the point at infinity of H. Let $\pi_H(y)$ denote the point of entry of γ_0 into H. Let γ be any geodesic ray from y that enters H, and let γ_u be its point of entry. Then*

$$d_{\partial H}(\gamma_u, \pi_H(y)) \leqslant 1.$$

Proof of Proposition 3.2. Let $x = gx_0$ and let δ be the hyperbolicity constant for the hyperbolic metric $d_{\mathbb{H}^2}$.

Case 1: Suppose γ_T is in the thick part. The hyperbolic geodesic from x_0 to x must pass through a 3δ neighborhood of γ_T (see Proposition 3.2 of [13]). This means that there is a point x' on the hyperbolic geodesic from x_0 to x that also lies in the thick part. So the projected path from x_0 to x necessarily passes through x'. Recall $L(y, y')$ is the distance along the projected path between the points y, y'. It follows that

$$L(x_0, x) = L(x_0, x') + L(x', x) \geqslant L(x_0, x')$$

hence passing to the word metric we get

$$d_G(1,g) \asymp L(x_0,x) \geqslant L(x_0,x') \asymp d_G(1,h_T).$$

Case 2: Suppose γ_T is in some horoball H and let γ_u and γ_v be the points where γ enters and leaves H. We may assume that a ball of hyperbolic radius 3δ about γ_T is contained in H. Then the hyperbolic geodesic γ' from x_0 to x must enter and leave H. Denote its entry and exit points by γ'_r and γ'_s. Moreover, let $p_T = \pi_{\widetilde{N}}(\gamma_T)$ be the projection of γ_T to the boundary of the horoball, and denote by $E := d_{\widetilde{N}}(\gamma_u,\gamma_v)$ the excursion of γ in H, and $D := d_{\widetilde{N}}(\gamma_u, p_T)$.

There are two sub-cases to consider.

Case 2a: If $D \geqslant E/2$, then we are in the situation of Figure 2 and x must lie in the shaded region.

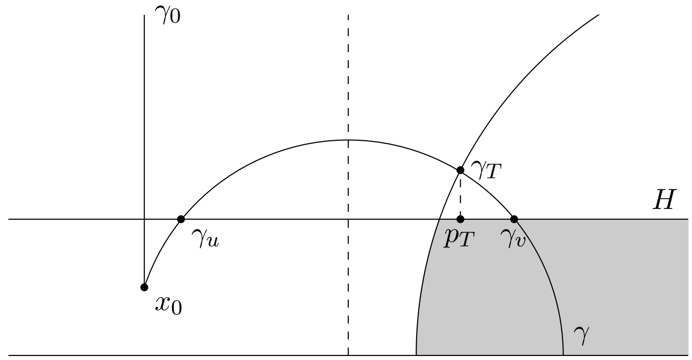

FIGURE 2. Perpendicular geodesics through intersections of γ and ∂H.

In this case, let $\pi_H(x)$ be the closest points projection of x onto the boundary of H; then by Lemma 3.3, the entry point γ'_r is within distance 1 of γ_u and the exit point γ'_s is within distance 1 of $\pi_H(x)$. So we get

$$d_{\partial H}(\gamma'_r,\gamma'_s) \geqslant d_{\partial H}(\gamma_u, \pi_H(x)) - 2 \geqslant \frac{E}{2} - 2.$$

On the other hand, $d_{\partial H}(\gamma_u, p_T) \leqslant E$, so we have

$$d_{\partial H}(\gamma'_r,\gamma'_s) \geqslant \frac{1}{2} d_{\partial H}(\gamma_u, p_T) - 2.$$

Moreover, by Lemma 3.3 and Lemma 2.1,

$$L(x_0,\gamma'_r) \asymp d_{\widetilde{N}}(x_0,\gamma'_r) \geqslant d_{\widetilde{N}}(x_0,\gamma_u) - 1 \asymp L(x_0,\gamma_u).$$

Consequently, the distances along respective projected paths satisfy

$$\begin{aligned} L(x_0,x) &\geqslant L(x_0,\gamma'_r) + d_{\partial H}(\gamma'_r,\gamma'_s) \\ &\gtrsim L(x_0,\gamma_u) + d_{\partial H}(\gamma_u, p_T) \\ &= L(x_0, p_T). \end{aligned}$$

Thus, passing to the word metric we get

$$d_G(1,g) \asymp L(x_0,x) \gtrsim L(x_0,p_T) \asymp d_G(1,h_T).$$

Case 2b: If $D \leqslant E/2$, then we are in the situation of Figure 3 and x must lie in the shaded regions.

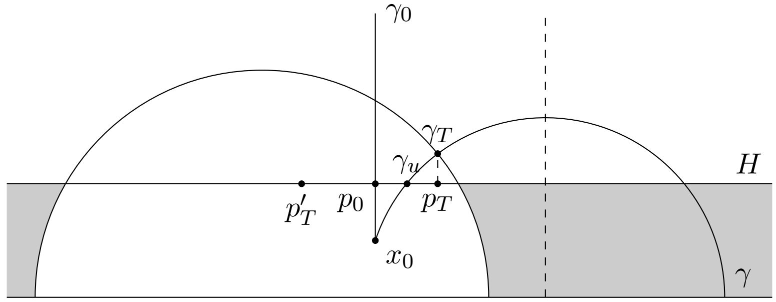

FIGURE 3. Perpendicular geodesics through intersections of γ and ∂H.

If x is in the shaded region on the right, then note that

$$d_{\partial H}(\gamma_u, \pi_H(x)) \geqslant d_{\partial H}(\gamma_u, p_T),$$

which by Lemma 3.3 implies

$$d_{\partial H}(\gamma'_r, \gamma'_s) \geqslant d_{\partial H}(\gamma_u, p_T) - 2,$$

and the required estimate for $d_G(1,g)$ then follows by estimates on distances along respective projected paths similar to *Case 2a*. If x is in the shaded region on the left, let p'_T be the point on ∂H such that p_T and p'_T are symmetric about γ_0, the geodesic ray from x_0 to the point at infinity for H, and denote $p_0 = \pi_H(x_0)$. Observe that $d_{\partial H}(p_0, \pi_H(x)) \geqslant d_{\partial H}(p_0, p'_T)$. Hence, by Lemma 3.3,

$$\begin{aligned} d_{\partial H}(\gamma'_r, \gamma'_s) &\geqslant d_{\partial H}(p_0, \pi_H(x)) - 2 \\ &\geqslant d_{\partial H}(p_0, p'_T) - 2 \\ &= d_{\partial H}(p_0, p_T) - 2 \\ &\geqslant d_{\partial H}(\gamma_u, p_T) - 3, \end{aligned}$$

and the required estimate for $d_G(1,g)$ then follows by estimates on distances along respective projected paths similar to *Case 2a*.

□

Let K, K' be the constants in Proposition 3.2, and for each T let

$$R_T := d_G(1,h_T)/K - K'.$$

Consider the ball $B(R_T)$ of radius R_T in G in the word metric; our goal is to prove an upper bound on the derivatives of the elements in the ball. Let us first establish another elementary lemma in hyperbolic geometry.

Lemma 3.4. *Let $L > T$, and y be a point on $\partial S(x_0, \gamma_{2T})$ such that $d_{\mathbb{H}^2}(y, \gamma_T) = L - T$. Let θ be the angle between γ and the ray γ' from x_0 to y. Then there exists a constant $C > 0$ such that, if T is sufficiently large (say when $\tanh T > 1/2$) and $L \geqslant 2T$, then*

$$\theta \geqslant Ce^{-T}.$$

Proof. It follows by hyperbolic trigonometry, applied to the right triangle $\Delta(x_0, \gamma_T, y)$. □

Proposition 3.5. *Any $g \in B(R_T)$ satisfies*

$$|g'(p)| \lesssim e^{2T}$$

for each $p \in S^1$.

Proof. Fix $p \in S^1$, and let γ be the geodesic ray from the origin x_0 of the unit disc to p. Fix $T > 0$ and $g \in B(R_T)$, and let $L := d_{\mathbb{H}^2}(x_0, gx_0)$. By Lemma 3.1, if $L \leqslant 2T$, then $|g'(p)| \leqslant e^{2T}$ which implies the proposition. Hence, we may assume $L \geqslant 2T$. Let y_1 and y_2 be points on $\partial S(x_0, \gamma_{2T})$, on opposite sides of γ_T, such that $d_{\mathbb{H}^2}(y_i, \gamma_T) = L - T$ for $i = 1, 2$, and let U be the sector subtended at x_0 by rays from x_0 passing through y_1 and y_2. We claim that the point gx_0 cannot be in U. Indeed:

- the point gx_0 cannot lie in $S(x_0, \gamma_{2T})$, because otherwise (by Proposition 3.2 and the definition of R_T) the word length of g satisfies $d_G(1, g) > R_T$, contradicting the fact that g is in $B(R_T)$;
- gx_0 cannot lie in $U \setminus S(x_0, \gamma_{2T})$, because otherwise it belongs to the geodesic triangle $\Delta(x_0, y_1, y_2)$, hence $d_{\mathbb{H}^2}(x_0, gx_0) < (L-T)+T = L$.

Now, by the derivative calculations (equation (5))

$$|g'(p)| = \frac{1 - A^2}{1 + A^2 - 2A\cos\phi}$$

where ϕ is the angle between γ and the geodesic ray joining x_0 with gx_0, and $A = (e^L - 1)/(e^L + 1)$. Hence,

$$|g'(p)| \quad = \quad \frac{4e^L}{2e^{2L}(1 - \cos\phi) + 2(1 + \cos\phi)} \leqslant \frac{e^{-L}}{\sin^2(\phi/2)}.$$

Now, by Lemma 3.4, the angle ϕ satisfies $\phi \geqslant Ce^{-T}$, so

$$|g'(p)| \lesssim e^{2T-L} \lesssim 1,$$

which completes the proof of the case $L \geqslant 2T$. □

3.3. Proof of Theorem 1.1. Before proving the theorem, we still need to show that the function $T \to d_G(1, h_T)$ has bounded jump size, in the following sense.

Lemma 3.6. *For γ a hyperbolic geodesic ray, let us define the set*

$$\mathcal{R}(\gamma) := \{r \in \mathbb{Z}_{\geqslant 0} : r = d_G(1, h_T) \text{ for some } T\}.$$

If γ is recurrent to the thick part, then the set $\mathcal{R}(\gamma)$ is infinite, and we can index its elements in increasing order $r_1 < r_2 < \dots$. Then there exists a constant $k > 0$ such that for any recurrent geodesic ray γ and any i, we have

$$r_{i+1} - r_i < k.$$

Proof of Lemma 3.6. For a geodesic ray γ, recall that $p_T = \pi_{\widetilde{N}}(\gamma_T)$ is the point on the projected path of γ that is the closest to γ_T. By Lemma 2.1, the image of the function $T \mapsto p_T$ is a continuous path which is (K, c)-quasigeodesic in $\widetilde{N}$. Let us choose times T_n along the geodesic such that $L(x_0, p_{T_n}) = n$. Since the thick part $\widetilde{N}$ is quasi-isometric to the group G, then, up to multiplicative constants which depend only on the quasi-isometry constants, we have

$$|d_G(1, h_{T_{n+1}}) - d_G(1, h_{T_n})| \lesssim d_{\widetilde{N}}(p_{T_n}, p_{T_{n+1}}) \lesssim 1.$$

$\square$

Let us now turn to the proof of Theorem 1.1. Recall that the Lyapunov expansion exponent is defined as

$$\lambda_{exp}(p) = \limsup_{R \to \infty} \max_{g \in B(R)} \frac{1}{R} \log |g'(p)|.$$

Lemma 3.6 implies that along geodesic rays recurrent to the thick part the corresponding values of R given by $R_T = d_G(1, h_T)/K - K'$ are infinite and have a bounded jump size. So the lim sup in the above definition can be replaced by a lim sup over values given by R_T. By Proposition 3.5, for almost every $p \in S^1$,

$$\max_{g \in B(R_T)} \frac{1}{R_T} \log |g'(p)| \leqslant \frac{1}{R_T} \log(e^{2T}) = \frac{2T}{R_T} \asymp \frac{2T}{d_G(1, h_T)}.$$

Hence, by Proposition 2.6 for Lebesgue-almost every p

$$\lambda_{exp}(p) = 0$$

proving Theorem 1.1.

4. Random walks

In this section we prove Theorem 1.3. Recall that a subgroup G of $SL(2, \mathbb{R})$ is called *non-elementary* if it contains a pair of hyperbolic isometries with disjoint fixed points. Let μ be a probability measure with finite first moment on G, i.e. such that

$$\int_G d_G(1, g)\, d\mu(g) < \infty$$

where d_G is a choice of word metric on G. Assume moreover that the support of μ generates a non-elementary subgroup of $SL(2, \mathbb{R})$ as a semigroup, and consider the random walk generated by μ. That is, the space $G^{\mathbb{N}}$ of sequences $(g_1, g_2, \ldots)$ is endowed with the product measure $\mu^{\mathbb{N}}$, and we define the random walk as the process $(w_n)_{n\in\mathbb{N}}$ with $w_0 = id$ and

$$w_{n+1} = w_n g_{n+1}.$$

Given a basepoint $x_0 \in \mathbb{H}^2$, one can consider the orbit map $G \to \mathbb{H}^2$ which sends $g \mapsto g(x_0)$, so each sample path in G projects to a sample path in $\mathbb{H}^2$. Furstenberg [5] showed that for almost all sequences the random walk converges to a point in the boundary $S^1 = \partial\mathbb{H}^2$, giving a boundary map $(w_n)_{n\in\mathbb{N}} \mapsto p_w$, defined for almost all sample paths. The harmonic measure ν on the boundary records the probability that the random walk hits a particular part of $\partial\mathbb{H}^2$, i.e.

$$\nu(A) = \text{Prob}\left(\lim_{n\to\infty} w_n(x_0) \in A\right).$$

We start by verifying the linear progress properties that we require. Since G is non-amenable, a random walk makes linear progress in the word metric as shown by Kesten [12], Day [2] and Guivarc'h [8]. Moreover, the random walk makes linear progress in the relative metric, too:

Proposition 4.1 (Maher-Tiozzo [14]). *Let μ be a probability distribution on a countable group G, which acts by isometries on a separable Gromov hyperbolic space (X, d). Assume moreover that μ has finite first moment, i.e. $\sum \mu(g) d(x_0, gx_0) < \infty$, and the semigroup generated by its support is a non-elementary subgroup of G. Then there is a constant $c > 0$ such that*

$$\lim_{n\to\infty} \frac{d(1, w_n)}{n} = c.$$

This applies to our situation, in which G is a non-uniform lattice acting on the group G, with the relative metric d_{rel}; it is well known that the relative metric is a (non-proper) hyperbolic metric on G. If μ has finite first moment with respect to the word length in G, then it also has finite first moment with respect to the relative metric. An earlier result, under the additional hypothesis of convergence to the boundary and finite support, is proven in [13].

We shall now prove Theorem 1.3. As the random walk makes linear progress in both the word metric and the relative metric, by taking the quotient, the limit

$$\lim_{n\to\infty} \frac{d_G(1, w_n)}{d_{rel}(1, w_n)}$$

exists and is finite along almost every sample path $w = (w_1, w_2, \ldots)$. We know that almost every sample path w converges to some boundary point $F^+(w) := \lim_{n\to\infty} w_n x_0 \in \partial\mathbb{H}^2$. We will denote by ρ_w the geodesic ray which joins the basepoint x_0 to the boundary point $F^+(w)$. We wish to obtain a limit for points along the geodesic ρ_w, and so we need to relate the sample path locations $w_n x_0$ to the geodesic ρ_w, which we will do using a sublinear tracking argument. The fundamental argument for sublinear tracking in [16] is the following lemma.

Lemma 4.2 (Tiozzo [16]). *Let $T : \Omega \to \Omega$ a measure-preserving, ergodic transformation of the probability measure space (Ω, λ), and let $f : \Omega \to \mathbb{R}_{\geqslant 0}$ any measurable, non-negative function. If the function*

$$g(\omega) := f(T\omega) - f(\omega)$$

belongs to $L^1(\Omega, \lambda)$, then for λ-almost every $\omega \in \Omega$ one has

$$\lim_{n\to\infty} \frac{f(T^n\omega)}{n} = 0.$$

For a point $x \in \mathbb{H}^2$, let $\operatorname{proj}(x)$ be the set of closest lattice points to x.

Proposition 4.3. *For almost every sample path $(w_n)_{n\in\mathbb{N}}$, with corresponding geodesic ray ρ_w, there exists a sequence of times $t_n \to \infty$ such that*

$$\lim_{n\to\infty} \frac{d_G(w_n, h_n)}{n} = 0$$

for any $h_n \in \operatorname{proj}(\rho_w(t_n))$.

Proof. In the proof, we consider the set $(G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ of bi-infinite sequences of group elements. For each sequence $w = (g_n)_{n\in\mathbb{Z}}$, we construct the forward random walk $w_n := g_1 \dots g_n$ and the backward random walk $w_{-n} := g_0^{-1} g_{-1}^{-1} \dots g_{-n}^{-1}$. Since both random walks converge almost surely, the maps

$$F^+(w) := \lim_{n\to\infty} w_n x_0 \in \partial\mathbb{H}^2$$

$$F^-(w) := \lim_{n\to\infty} w_{-n} x_0 \in \partial\mathbb{H}^2$$

are defined for almost every $w \in G^{\mathbb{Z}}$, and $F^-(w) \neq F^+(w)$ almost surely since the hitting measures are non-atomic. Hence, this defines for almost every $w \in G^{\mathbb{Z}}$ a bi-infinite geodesic in $\mathbb{H}^2$ whose endpoints are $F^+(w)$ and $F^-(w)$, which we denote by γ_w.

Let now $P(w)$ be the union of all closest points over all points in the geodesic γ_w, i.e.

$$P(w) = \bigcup_{x\in\gamma_w} \operatorname{proj}(x).$$

Note that P is equivariant, in the sense that

$$P(\sigma^n w) = w_n^{-1} P(w).$$

Let us now define the function $\varphi : G^{\mathbb{Z}} \to \mathbb{R}$ on the space of bi-infinite sample paths as

$$\varphi(w) := d_G(1, P(w))$$

i.e. the minimal word-metric distance between the base point X_0 and the set of closest lattice points to γ_w. The shift map $\sigma : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ acts on the space of sequences, ergodically with respect to the product measure $\mu^{\mathbb{Z}}$. By the equivariance of P, we have for each n the equality

$$\varphi(\sigma^n w) = d_G(w_n, P(w)). \tag{6}$$

We shall now apply Lemma 4.2, setting $(\Omega, \lambda) = (G^{\mathbb{Z}}, \mu^{\mathbb{Z}})$, $T = \sigma$, and $f = \varphi$. The only condition to be checked is the L^1-condition on the function $g(\omega) = f(T\omega) - f(\omega)$, which in this case becomes

$$g(\omega) = \varphi(\sigma w) - \varphi(w) = d_G(1, P(\sigma w)) - d_G(1, P(w)).$$

Now, using (6) we have

$$|d_G(1, P(\sigma w)) - d_G(1, P(w))| = |d_G(w_1, P(w)) - d_G(1, P(w))| \leq d_G(1, w_1)$$

which has finite integral precisely by the finite first moment assumption. Thus, it follows from Lemma 4.2 that for almost all bi-infinite paths w one gets

$$\lim_{n\to\infty} \frac{d_G(w_n, P(w))}{n} = 0.$$

By definition of $P(w)$, there exists a sequence of times t_n, and group elements $p_n \in G$, such that $p_n \in \text{proj}(\gamma_w(t_n))$, and furthermore

$$\lim_{n\to\infty} \frac{d_G(w_n, p_n)}{n} = 0. \tag{7}$$

The geodesic ray ρ_w starting at x_0 with the same endpoint as γ_w is asymptotic to γ_w, and we may parameterize ρ_w so that $d(\rho_w(t), \gamma_w(t)) \to 0$. In fact, a calculation using the hyperbolic metric shows that for any two asymptotic rays with this parameterization, there is a number K, depending on the rays, such that $d(\rho_w(t), \gamma_w(t)) \leq Ke^{-t}$. Furthermore, if two points distance at most d inside a cusp are at most distance ϵ apart, then the distance between their nearest point projections to the projected path are distance at most Ke^d apart, for some number K depending on ϵ. As the geodesic ray ρ starts outside a cusp, the distance of $\rho_w(t)$ inside a cusp is at most t, so this implies that there is a constant K, depending on w, such that if h_n is a point in $\text{proj}(\rho_w(t_n))$ then $d_G(p_n, h_n) \leq K$ for all n sufficiently large. Therefore,

$$\lim_{n\to\infty} \frac{d_G(w_n, h_n)}{n} = 0 \tag{8}$$

as required. □

We now extend the result from the sequence (t_n) along the geodesic to all points t along the geodesic.

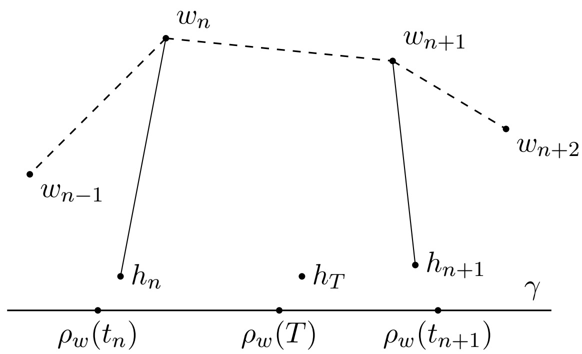

FIGURE 4. Intermediate times.

Proof of Theorem 1.3. Given a sample path w, let ρ_w be the geodesic ray joining the base point x_0 to the boundary point $F^+(w)$, and let (t_n) be the sequence of times given by Proposition 4.3, and $h_n \in \mathrm{proj}(\rho_w(t_n))$ a corresponding sequence of group elements. Let us now pick a time $T > 0$, and let $h_T X_0$ be a projection of $\rho_w(T)$ to the lattice. Since $t_n \to \infty$, there exists an index $n = n(T)$ such that $t_n \leqslant T \leqslant t_{n+1}$. By Proposition 2.2, there exist constants $C_1 > 0$, C_2 such that

$$d_G(h_n, h_T) \leqslant C_1 d_G(h_n, h_{n+1}) + C_2.$$

Moreover, by Proposition 4.3 and the triangle inequality,

$$\lim_{n\to\infty} \frac{d_G(h_n, h_{n+1})}{n} \leqslant \lim_{n\to\infty} \frac{d_G(h_n, w_n) + d_G(w_n, w_{n+1}) + d_G(w_{n+1}, h_{n+1})}{n} = 0$$

(where we used the finite first moment condition to ensure $d_G(w_n, w_{n+1})/n \to 0$). Thus, we also have

$$\lim_{n\to\infty} \frac{d_G(h_n, h_T)}{n} = 0$$

and again by Proposition 4.3,

$$\lim_{n\to\infty} \frac{d_G(w_n, h_T)}{n} \leqslant \lim_{n\to\infty} \frac{d_G(w_n, h_n) + d_G(h_n, h_T)}{n} = 0.$$

Similarly, since the relative metric is bounded above by the word metric,

$$\lim_{n\to\infty} \frac{d_{rel}(w_n, h_T)}{n} = 0.$$

Finally, by computing the ratio between the word and relative metric,

$$\lim_{T\to\infty} \frac{d_G(1, h_T)}{d_{rel}(1, h_T)} = \lim_{T\to\infty} \frac{\frac{d_G(1,h_T)}{n(T)}}{\frac{d_{rel}(1,h_T)}{n(T)}} = \lim_{n\to\infty} \frac{\frac{d_G(1,w_n)}{n}}{\frac{d_{rel}(1,w_n)}{n}} = \frac{c_1}{c_2} > 0.$$

This completes the proof of Theorem 1.3. $\square$

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