Linear-Time Algorithm for Morphic Imprimitivity Testing

Tomasz Kociumaka[1] Jakub Radoszewski[1]
Wojciech Rytter [1,2] **Tomasz Waleń**[3,1]

[1]Faculty of Mathematics, Informatics and Mechanics, University of Warsaw
`{kociumaka,jrad,rytter,walen}@mimuw.edu.pl`

[2]Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń

[3]International Institute of Molecular and Cell Biology in Warsaw

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Outline

1. Problem definition
2. Short introduction to existing solutions
3. Description of the new linear time solution

Problem definition

Morphic Imprimitivity Testing

For a input word $w \in \Sigma^n$, is there a *non-trivial* morphism h such that:

$$h(w) = w$$

Non-trivial means that h should not be an identity function.
The word w is *non-primitive* if such morphism exists,
otherwise it is *primitive*.

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Previous results

- it can be solved in $O((|\Sigma| + \log n) \cdot n)$ time (S. Holub 2009),
- slightly improved to $O(|\Sigma| \cdot n)$ time (S. Holub, V. Matocha, arXiv 2012).

Example

Simple case

Let

$$w = \texttt{abaacaca}$$

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More complicated case

Let

$$w = \texttt{aacabaaaacaacabaa}$$

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More complicated case

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$$w = \texttt{aac}\,\texttt{abaa}\,\texttt{aac}\,\texttt{aac}\,\texttt{abaa}$$

we can take:

$$h(a) = \epsilon \quad h(b) = \texttt{abaa} \quad h(c) = \texttt{aac}$$

Problem applications

Closely connected to several topics in formal language theory, and combinatorics on words:

- fixed points of morphisms,
- pattern languages,
- ambiguity of the morphisms.

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Reviewer's opinion

Although I cannot think of any actual applications, I find this question to be very natural

How to solve it? - Intuition

Theorem

For a word w, if there exists non-trivial morphism h, such that $h(w) = w$, then there exists non-trivial morphism h' such that:

- $h'(w) = w$
- *for all immortal letters* $x \in E$*:* $h'(x) = l_x\, x\, r_x$
  (i.e. $h'(\mathtt{b}) = \mathtt{abaa}$*)*
- *for all mortal letters* $x \notin E$*:* $h'(x) = \epsilon$

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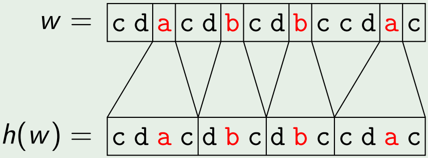

$w = $ c d a c d b c d b c c d a c

$h(w) = $ c d a c d b c d b c c d a c

$h(\mathtt{a}) = \mathtt{cdac}$
$h(\mathtt{b}) = \mathtt{dbc}$
$h(\mathtt{c}) = \epsilon$
$h(\mathtt{d}) = \epsilon$

Holub's algorithm

The algorithm maintains three sets:

- E – set of candidates for *immortal* letters,
- L and R – sets of interpositions.

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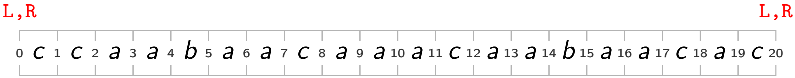
From triple (E, L, R) the actual morphism can be obtained:

- if the set $E \neq \Sigma$, then the morphism is non-trivial,
- from L, R we can deduce a way to divide input word to obtain morphism.

Holub's rule (a) – initialization of the algorithm

$L := L \cup \{0, n\},\ R := R \cup \{0, n\}$

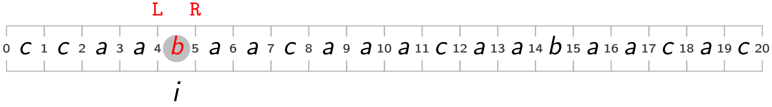
Example:



Holub's rule (b) – initialization of immortal letters

```
**if** $w[i] \in E$ **then**
    $L := L \cup \{i-1\}$ and $R := R \cup \{i\}$,
```

Example:

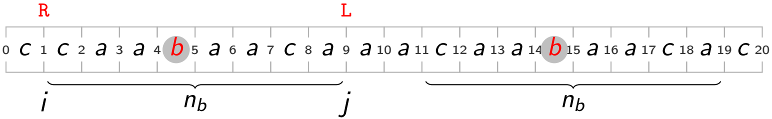


Holub's rule (c) – neighborhood marking

The neighborhood of letter x – n_x is the maximum factor that surrounds *each* occurrence of letter x in w.

```
**if** $w[i..j] = n_x$ for some $x \in E$ **then**
    $R := R \cup \{i-1\}$ and $L := L \cup \{j\}$,
```

Example:



Holub's rule (d) – copying rules

```
if $w[i..j] = w[i'..j'] = n_a$ for some $a \in E$ and $i - 1 \le k \le j$ then
    if $w[k] \in L$ then $L := L \cup \{i' + (k - i)\}$
    if $w[k] \in R$ then $R := R \cup \{i' + (k - i)\}$
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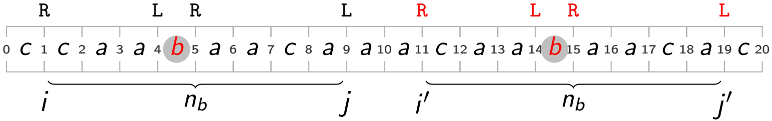
Example:



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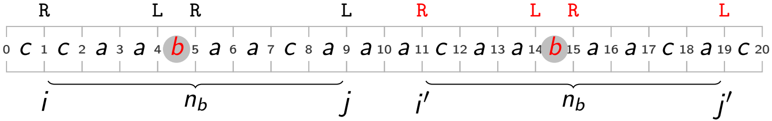

Problem

This rule is hard to implement efficiently!

Holub's rule (e) – new immortals letters

```
if $i < j$, $i \in L$, $j \in R$ then
    add $\alpha(w[(i+1)..j])$ to $E$ — letter $c \in w[(i+1)..j]$
        that has smallest number of occurrences in word $w$.
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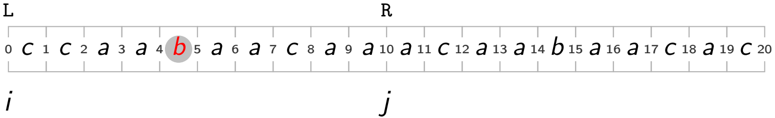
Example:



Holub's algorithm summary

Theorem

Extending a correct triple (E, L, R) *using any of the rules (a)-(e) leads to a correct triple. In particular, if any sequence of actions corresponding to (a)-(e) leads to* $E = \Sigma$ *then* w *is morphically primitive.*

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This is quite suprising that this set of simple rules, provides the solution for the problem.

Holub's algorithm summary

- simple implementation requires $O(n^2)$ time,
- this time complexity can be slightly improved using some preprocessing and data structures,
- unfortunately the obtaining linear time seems to be difficult task:
    - the non-determinism in rules choice is problematic,
    - rule (d) is the main bottleneck (it operates globally on the word).

What we have done? Outline

- modified set of rules (a),(b')–(e'), that are equivalent to Holub's rules but are easier to implement,
- strict ordering of rules application,
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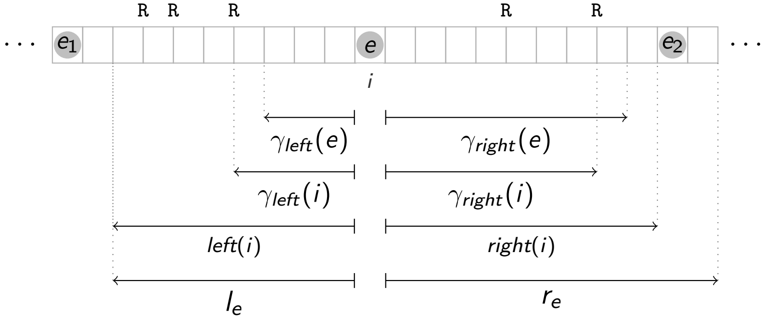
Result

As a consequence we obtained $O(n)$ running time algorithm.

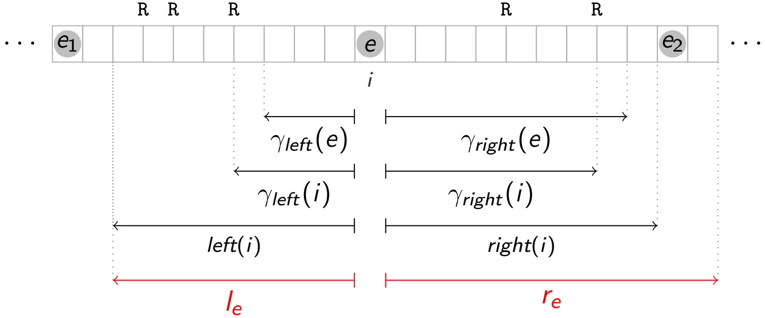
New neighborhood definitions

We introduced new definitions of neighborhood, to capture essential local neighborhood of the characters/word positions.



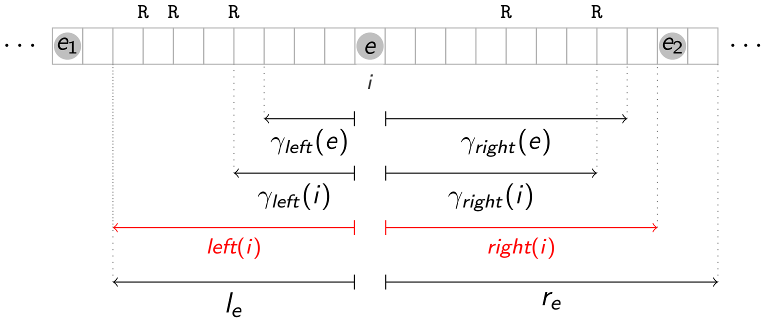
New neighborhood definitions

l_e – the length of the longest common suffix of all prefixes ending with e (minus 1) in word w.
r_e – the length of the longest common prefix of all suffixes starting with e (minus 1) in word w,



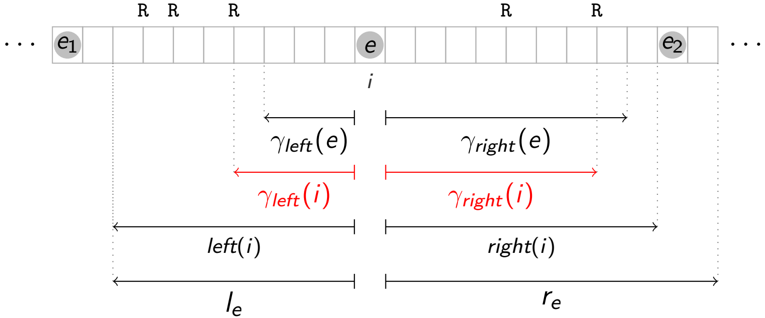
New neighborhood definitions

$$left(i) = \min(l_{w[i]},\ i - pred_E(i) - 1)$$
$$right(i) = \min(r_{w[i]},\ succ_E(i) - i - 1)$$



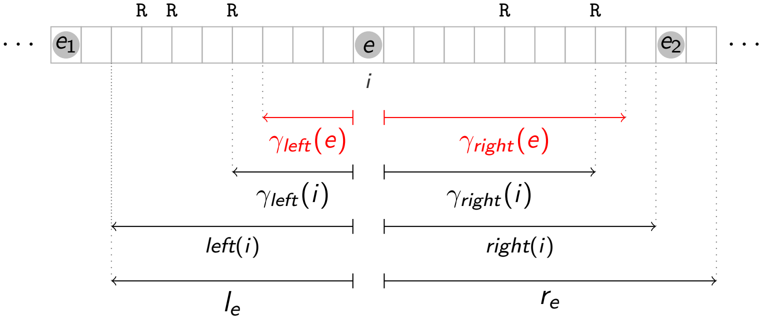
New neighborhood definitions

$$\gamma_{left}(i) = i - pred_R(i) - 1$$
$$\gamma_{right}(i) = pred_R(i + right(i) + 1) - i$$



New neighborhood definitions

$$\gamma_{left}(e) = \min\{\gamma_{left}(i) : i \in Occ(e)\}$$
$$\gamma_{right}(e) = \max\{\gamma_{right}(i) : i \in Occ(e)\}$$



New rule (b')

Old: **if** $w[i] \in E$ **then**
$L := L \cup \{i - 1\}$ and $R := R \cup \{i\}$,

New: **if** $w[i] \in E$ **then**
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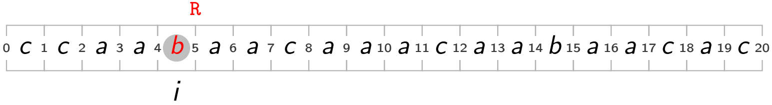
Example:



New rule (c')

Old: **if** $w[i..j] = n_x$ for some $x \in E$ **then**
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New: **if** $w[i] \in E$ **then**
$R := R \cup \{i - 1 - \mathit{left}(i)\}$ and $L := L \cup \{i + \mathit{right}(i)\}$,

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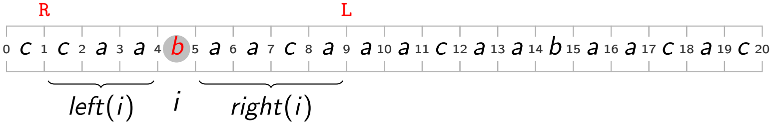

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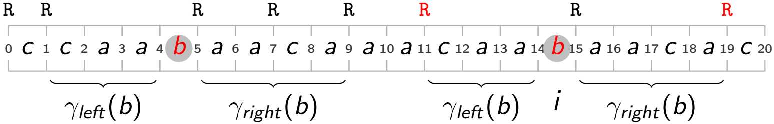
New: **if** $w[i] \in E$ **then**

$$R := R \cup \{i - 1 - \gamma_{left}(w[i]), i + \gamma_{right}(w[i])\}$$

Example:



New rule (e')

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New:
if $i < j$, $succ_R(i) = j$, $pred_L(j) = i$, $\{w[k] : i+1 \leq k \leq j\} \cap E = \emptyset$
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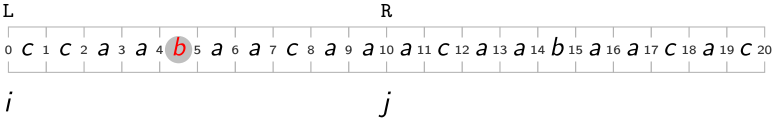

New rules correctness

Theorem

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Proof outline

We can show that using new rules we can simulate *essential* behavior of Holub's algorithm.

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Unfortunately that's not over, we have to deal with:

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Data structures:

- For answering $\alpha(i,j)$ queries in $O(1)$ time we use Range-Minimum-Queries (RMQ) data structure,
- For efficient computing the neighborhoods we use Suffix Arrays combined with Longest Common Prefix table.

Summary

- we presented a linear time algorithm for deciding if a word is morphically imprimitive,
- we started from the original quadratic algorithm by Holub, and transformed it by reducing the set of rules used by the algorithm,
- finally we proposed several efficient data structures that enabled linear-time implementation.

Thank you for your attention!