MATH/AMATH 516, Spring 2009 Homework 4 Due May 29 at 10:30 AM

You can discuss the problems with each other, but you must write up your answers on your own. Feel free to ask for a hint if you get stuck. [The page/equation numbers are from *Nonlinear Programming*, 2nd edition, 1999, 1st printing. They may differ for the 2nd printing.]

The answers to the two * problems are to be turned in jointly with your (randomly chosen) partner.

#1. [Manifold suboptimization: Exercise 2.5.2] Show by example that, in the manifold suboptimization (active-set) method, if more than one $i \in I^k = \{i \mid a_i^T x^k = b_i\}$ with negative multipliers μ_i are dropped from I^k, then the resulting direction d^k need not be a feasible direction.

#2. [Computing problem: Exercise 2.5.1] Use the manifold suboptimization method to solve the convex quadratic problem

$$\begin{aligned} \min \quad & f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \geq 1, \quad x \geq 0, \end{aligned}$$

starting from $x^0 = (0,0,1)^T$. Here you can choose $H^k = \nabla^2 f(x^k)$ since it is positive definite and choose α^k by minimization rule, i.e., α^k minimizes $\phi_k(\alpha) = f(x^k + \alpha d^k)$ over $\alpha \in [0, s^k]$, where s^k is the largest feasible stepsize given by $s^k = \min\{(b_i - a_i^T d^k)/a_i^T d^k \mid a_i^T d^k > 0\}$. [If you are coding in Matlab, you can use "A(index,:)" to access particular rows of a matrix A. For example, if index = [2 4], then A(index,:) would be the submatrix of A comprising its rows 2 and 4.]

#3.* [Polynomial-time interior-point method for convex QP.] Consider the convex quadratic program in standard form:

$$\begin{aligned} \min \quad & f(x) = \frac{1}{2} x^T Q x + c^T x \\ \text{s.t.} \quad & Ax = b, \quad x \geq 0, \end{aligned}$$

with $A \in \Re^{m \times n}$ having rank m ($n \geq m$) and $Q \in \Re^{n \times n}$ symmetric positive semidefinite. Define the wide neighborhood

$$\mathcal{N}(\gamma) = \{(x, y, \lambda, \epsilon) \mid Ax = b,\ y = Qx + c + A^T\lambda,\ x > 0,\ \epsilon > 0,\ \min_j x_j y_j \geq \gamma\epsilon,\ x^T y = n\epsilon\}.$$

For any $(x, y, \lambda, \epsilon) \in \mathcal{N}(\gamma)$, let (u, v, w) solve the Newton equation

$$Au = 0, \quad v = Qu + A^T w, \quad x_j y_j + u_j y_j + x_j v_j = \sigma\epsilon \qquad (0 < \sigma < 1).$$

Prove that $(x[\alpha], y[\alpha], \lambda[\alpha], \epsilon[\alpha]) \in \mathcal{N}(\gamma)$ and $\epsilon[\alpha] \leq (1 - \alpha C_1)\epsilon$ for all $0 < \alpha \leq C_2/n$, where C_1, C_2 are positive constants, $x[\alpha] = x + \alpha u$, $y[\alpha] = y + \alpha v$, $\lambda[\alpha] = \lambda + \alpha w$, $\epsilon[\alpha] = x[\alpha]^T y[\alpha]/n$. This yields a polynomial-time algorithm for solving convex quadratic programs. The proof in fact extends to other convex functions like $f(x) = -\sum_{i=1}^n \ln(x_i)$ and $f(x) = \sum_{i=1}^n x_i \ln(x_i)$!]

#4.* [Computing problem.] Below is a table showing the stock price P_{it} of company i in year t years (quoted from Nasdaq website):

Firm	1. Apr04	2. Apr05	3. Apr06	4. Apr07	5. Apr08
1. Amazon	40	34	36	40	75
2. Apple	9	43	60	90	160
3. Google	100	180	350	500	550
4. Microsoft	27	24	24	28	30

The return rate r_{it} at year t is $(P_{i,t+1} - P_{i,t})/P_{i,t}$. Define the sample expectation $\bar{r} = E_t[r_t]$ and covariance matrix $Q = E_t[(r_t - \bar{r})(r_t - \bar{r})^T]$, where $r_t = (r_{1t}, ..., r_{nt})^T$. Q is positive semidefinite and here $n = 4$. The portfolio selection problem is to choose a mix of the stocks to minimize risk while achieving a desired expected rate of return. This can be formulated as the optimization problem (ignoring transaction costs):

$$\min_x \quad x^T Q x \quad \text{s.t.} \quad \bar{r}^T x \geq \alpha \max_i \bar{r}_i, \quad \sum_{i=1}^n x_i = 1, \quad x_i \geq 0,$$

where $0 < \alpha < 1$, and x_i is the proportion of stock i in the portfolio. Take $\alpha = .9$. Using Matlab or your favorite computer language, implement an interior-point method to solve this problem. Note that $x_{\bar{i}} = \alpha$, and $x_i = (1-\alpha)/n$ for $i \neq \bar{i}$, where $\bar{r}_{\bar{i}} = \max_i \bar{r}_i$, is a feasible interior point. You need to introduce a variable for the inequality constraint. [Caution: This problem is a bit open-ended, so feel free to ask questions. Given x, λ can be chosen to achieve dual feasibility. Then ϵ and γ can be chosen so the starting point lies in the wide neighborhood $\mathcal{N}(\gamma)$.]

#5. [Extending the quadratic penalty method.] Consider the problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X, \; h_1(x) = 0, ..., h_m(x) = 0, \end{aligned}$$

where X is a nonempty closed set in $\Re^n$ and $f, h_1, ..., h_m$ are continuous functions from $\Re^n$ to $\Re$. Assume that this problem has a global minimum. Let ψ be any continuous function from $\Re$ to $[0, \infty)$ satisfying $\psi(\xi) = 0$ if and only if $\xi = 0$. For $k = 0, 1, 2...$, let $\{x^k\}$ be any global minimum of

$$\begin{aligned} \min \quad & f(x) + c^k \sum_{i=1}^m \psi(h_i(x)) \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{2.1}$$

where $c^k > 0$ (assuming a global minimum exists). Show that, if $\{c^k\} \to \infty$, then any cluster point of $\{x^k\}$ is a global minimum.