Master Thesis

Trans-Planckian Considerations in Inflationary Cosmology

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That I the force may recognise that binds creation's inmost energies.

Faust I

Contents

Abstract

This thesis investigates whether or not predictions stemming from inflationary theory are sensitive to changes of the high energy behaviour of the theory.

In the first part, established physics will be introduced. To begin with, standard cosmology will be reviewed. We will then introduce the formalism of quantum field theory in general backgrounds. In the third chapter we will consider cosmological perturbation theory and will eventually take the scalar perturbations to be quantum fluctuations.

In the second part, we will consider different approaches to modifying the high energy behaviour of inflationary theory. The consistency of the approaches will be subject of chapter 7 and in chapter 8 we will carefully consider the consequences of general modifications.

The question of sensibility to high energy changes essentially reduces to the question of which vacua are allowed in an inflating spacetime. We find strong evidence for the theory to be insensitive to changes of high energy behaviour, i.e. a certain class of vacua is found to be unphysical.

Diese Arbeit untersucht, ob die Vorhersagen der Theorie der kosmischen Inflation durch Veränderungen des Verhaltens der Theorie bei hohen Energien beeinflusst werden können.

Der erste Teil der Arbeit beschäftigt sich mit etablierter Physik. Am Anfang wird das kosmologische Standardmodell vorgestellt. Danach wird in die Quantenfeldtheorie auf gekrümmter Raumzeit eingeführt. Im dritten Kapitel wird die Theorie kosmologischer Perturbationen vorgestellt. Die Perturbationen werden schliesslich quantisiert und als Quantenfluktuationen behandelt.

Im zweiten Teil werden verschiedene Ansätze untersucht, die das Verhalten der inflationären Theorie bei hohen Energien verändern. Die Konsistenz der Ansätze wird in Kapitel 7 untersucht und in Kapitel 8 wird auf Konsequenzen der Ansätze eingegangen.

Die Frage, ob Vorhersagen der inflationären Kosmologie durch Veränderungen im Bereich der hohen Energien beeinflusst werden können, reduziert sich dann auf die Frage, welche Vakua für inflationäre Raumzeiten erlaubt sind. Wir finden starke Anhaltspunkte dafür, dass sich die Vorhersagen nicht beeinflussen lassen, bzw. dass eine gewisse Klasse von Vakuumszuständen unphysikalisch ist.

Introduction

Not too long ago, one could have said that whereas maths requires pen, paper and a paper bin, cosmology just requires pen and paper.
Fortunately, this statement no longer applies and recent years have seen cosmology turn into a quantitative science of highly precise and predictive nature.

Experiments [1, 2] have complemented decades of theoretical studies which eventually lead to the the standard model of big bang cosmology.
Not only is there undeniable evidence that the universe undergoes expansion (the light from distant galaxies is redshifted) but furthermore it is found that we live in an era in which the expansion is accelerating[1] due to the presence of a non-zero cosmological constant Λ. The physics within expanding space is governed by the principle of *freeze-out* and the concept of *symmetry breaking.*

When the interaction rate Γ of a particle species drops below the expansion rate H of the universe, a particle species is said to be frozen out or decoupled. At earlier times, i.e. higher energies, broken symmetries of physics are restored, e.g. different forces such as the electromagnetic and the weak force are unified. From an energy scale of about 1 TeV and below, i.e. after the first 10^{-10} seconds of the universe or at a temperature of about 10^{15}K, the physics is well understood. At first, the universe is dominated by highly relativistic particles, which are simply referred to as radiation within cosmology. With the expansion of space, different particle species decouple from the rest of the matter. At roughly 10^{10}K or 100 seconds after the initial singularity, nucleosynthesis occurs where light elements are formed. At an energy scale of 1 eV or about 10^{10} seconds after the big bang, matter and radiation density equal each other. Density fluctuations of the plasma propagate as acoustic oscillations. Roughly 380,000 years after the big bang or at an energy of 0.1 eV, protons and electrons combine[2] into neutral hydrogen atoms and photons cease to interact with the formerly free electrons. Therefore the photons now travel through space uninterruptedly.
It is these photons that we now refer to as cosmic microwave background radiation. Initially emitted at temperatures of the same order of magnitude as those at the sun's surface, the radiation has redshifted and cooled with the expansion of the universe. Its anisotropies resemble the acoustic oscillations of the plasma from which the radiation decoupled. It is furthermore the density perturbations of the plasma that will eventually lead to the gravitational instabilities required for structure formation.

[1] For an object at a given distance, the redshift is slightly smaller than expected [1]. Hence the rate of expansion was smaller in the past which implies that the expansion is accelerating.

[2] This process is called recombination. As they however are combined for the first time, the naming is misleading.

The above story of success in terms of explaining the universe as we observe it today requires some mechanism to induce the primordial density perturbations responsible for the anisotropies of the CMB and the formation of structure. There are additional shortcomings with big bang cosmology that will be considered in much detail later. For short, the universe requires finely tuned initial conditions. These initial conditions required by the theory in order to match observations are not included in standard big bang cosmology. Yet as we are not studying bullet trajectories but the universe, the initial conditions are of utmost interest.

The add-on to the standard model of cosmology providing the necessary initial conditions as well as a mechanism to produce density perturbations is called inflation [3, 4]. Assumed to have occurred at about 10^{-34} seconds after the big bang or at an energy scale of $\leq 10^{15}$ GeV, the universe expands acceleratingly with near exponential expansion.

The character of inflation is two-fold. For once, quantum fluctuations of the inflaton field cause inflation to last for different times in different regions thus inducing density perturbations. It is these quantum fluctuations that are the seed of the formation of structure.

On the other hand, inflation tears apart regions that have been in causal contact and thus creates a huge homogeneous patch that is much greater than the volume of causal contact without considering an accelerating phase.

Even more, inflation is predictive. The spectrum of primordial perturbations is calculated to be nearly scale invariant which is in accordance with observation [2].

Having described inflation as the current working hypothesis in order to explain primordial perturbations and the initial conditions of big bang cosmology, it is of crucial importance to investigate whether or not the inflationary paradigm stands on a conceptually solid footing.

As inflation is believed to have occurred shortly after the Planck and Grand unification era, it is of utmost interest to find out how unknown high energy physics might have influenced the physics of inflation. In other words, is inflation sensitive to any high energy changes of the theory? Fortunately, we obtain a physical observable, namely the power spectrum, from the inflationary paradigm. Thus investigating sensibility to high energy changes is not a mere theoretical consideration.

This thesis investigates the sensitivity of inflation to unknown high energy physics. We will undertake theoretical considerations such as how to parametrise effects of unknown physics and also investigate observable consequences of such effects. Furthermore, we will consider the consistency of different approaches and will eventually find the question to reduce to the issue of what vacua are allowed for inflating spacetimes. We will find that even without unknown high energy effects, the question of the correct vacuum does not remain unambiguous.

But first, let us start with a review of our current understanding of modern cosmology...

Conventions: Unless stated otherwise, we will work in units of $8\pi G = c = 1$. The metric signature will be $(+, -, -, -)$. Greek indices run from 0 to 3 whereas Roman ones run from 1 to 3. Furthermore, use of the Einstein summation convention is implied.

Part I

Modern Cosmology

Chapter 1

Spacetime Dynamics and Inflation

This chapter will introduce the concepts and the mathematical formalism representing our current understanding [5–12] of the universe.

1.1 Spacetime Geometry

In this section, we will review the most fundamental idea to cosmology, namely spacetime and its geometry. Starting with the cosmological principle we will then derive the equations believed to best describe our universe. Following a discussion of their causal consequences we will then consider the implied dynamics of the universe.

1.1.1 The Cosmological Principle

The driving assumptions behind all of modern cosmology are combined within the so called cosmological principle: *Our position in the universe - with respect to suitably large scales - is in no sense preferred, i.e. our universe is spatially* ***homogeneous*** *and* ***isotropic***. Homogeneity means that space is the same at all points. Isotropy means that space is the same in all directions about a point.

It is important to note that the phrasing of the cosmological principle as stated above requires isotropy about every point, not just one. A homogeneous space is not necessarily isotropic (considering a space with a homogeneous magnetic field, we find a preferred direction, i.e. the direction of the field lines) and likewise, a space isotropic about one point is not necessarily homogeneous (e.g. the centre of a spherically symmetric force field such as the gravitational field due to a point mass). However, a homogeneous universe isotropic about one point is isotropic about all points and isotropy about every point does imply homogeneity as we shall now prove [13].

Theoretical Motivation

Considering an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{1.1.1}$$

where therefore
$$\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho_\mu - \partial_\mu \xi^\rho(x), \tag{1.1.2}$$
we find a metric tensor to be invariant under such transformations, i.e. $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$, if the transformation may be written as $\xi^\mu(x) = \epsilon K^\mu(x)$ and the vector field $K^\mu(x)$ satisfies
$$D_\mu K_\nu(x) + D_\nu K_\mu(x) = 0, \tag{1.1.3}$$
which is the *Killing equation* with $D_\mu = \partial_\mu - \Gamma^\nu_{\rho\mu}$ being the covariant derivative. Every vector $K_\mu(x)$ represents an isometry of the metric.

Homogeneity implies that translations to all neighbouring points must leave the metric invariant. For a homogeneous, N-dimensional space there exist N linearly independent killing vectors which represent translational isometries of the metric. We thus may choose our killing vectors to be defined by
$$K^{(\lambda)}_\mu(x^*; x^*) = \delta^\lambda_\mu. \tag{1.1.4}$$
Isotropy implies that all rotations about a point must leave the metric invariant. Thus we require isometries to exist that leave a point x^* fixed and hence that
$$K^{(\lambda,\tau)}_\mu(x^*; x^*) = 0. \tag{1.1.5}$$
There are $N(N-1)/2$ linearly independent Killing vectors [14] that then may satisfy
$$K^{(\lambda,\tau)}_\mu(x; x^*) = -K^{(\lambda,\tau)}_\mu(x; x^*) \tag{1.1.6}$$
and whose derivatives are
$$D_\nu K^{(\lambda,\tau)}_\mu(x; x^*)\Big|_{x=x^*} = \frac{\partial}{\partial x^\nu} K^{(\lambda,\tau)}_\mu(x; x^*)\Big|_{x=x^*} = \delta^\lambda_\mu \delta^\tau_\nu - \delta^\lambda_\nu \delta^\tau_\nu. \tag{1.1.7}$$
Now, imposing isotropy about every point implies there are Killing vectors $K^{(\lambda,\tau)}_\mu(x, x^*)$ and $K^{(\lambda,\tau)}_\mu(x, x^* + dx^*)$ which both satisfy the above conditions at x^* and $x^* + dx^*$ respectively. Any linear combination of the two must also be a Killing vector and by Taylor's theorem, $\partial K^{(\lambda,\tau)}_\mu(x, x^*)/\partial x^{*\rho}$ also has to be a Killing vector. Now consider
$$\begin{aligned} 0 &= \frac{\partial}{\partial x^{*\rho}} \underbrace{K^{(\lambda,\tau)}_\mu(x^*, x^*)}_{\to 0} \\ &= \frac{\partial}{\partial x^\rho} K^{(\lambda,\tau)}_\mu(x, x^*)\Big|_{x=x^*} + \frac{\partial}{\partial x^{*\rho}} K^{(\lambda,\tau)}_\mu(x, x^*)\Big|_{x=x^*}. \end{aligned}$$
Combining the above with (1.1.7) yields
$$\frac{\partial}{\partial x^{*\rho}} K^{(\lambda,\tau)}_\mu(x, x^*)\Big|_{x=x^*} = -\delta^\lambda_\mu \delta^\tau_\rho + \delta^\tau_\mu \delta^\lambda_\rho. \tag{1.1.8}$$
We may therefore construct a Killing vector as
$$K^{(\tau)}_\mu(x; x^*) = \frac{\alpha}{N-1} \frac{\partial}{\partial x^{*\rho}} K^{(\rho,\tau)}_\mu(x, x^*), \tag{1.1.9}$$

where α is arbitrary. But this simply gives

$$K_{\mu}^{(\tau)}(x^*;x^*) = \underbrace{-\delta_{\mu}^{\rho}\delta_{\rho}^{\tau}}_{=0} + \delta_{\mu}^{\tau}\delta_{\rho}^{\rho} = \delta_{\mu}^{\tau}, \tag{1.1.10}$$

which is just (1.1.4). Thus isotropy about every point does imply homogeneity. This can be put to elegant use when arguing that there is nothing special about earth when observing space, hence observations in all directions from every point shall yield the same characteristics and therefore we have derived the cosmological principle from just one single assumption.

Note that the above result does not come as a surprise and may already be inferred from the start of the argument. When demanding isotropy *everywhere*, we have already constructed a homogeneous space, as in that it is isotropic everywhere, hence homogeneous.
It may however be shown that a 3-space isotropic about three non-colinear points must also be homogeneous in which case homogeneity must not be inferred from the construction of the gedankenexperiment.

Observational Status

Despite all the elegance of the cosmological principle and symmetric spaces it is important to note that even though the universe is found to be highly isotropic [2], deviations from homogeneity are being observed.
The recently discovered *Huge-LQG* [15], a large group of quasars, is estimated to be 1240 MPc or roughly 4 billion light years across in its longest dimension and thus challenges the assumption of large-scale homogeneity. Since this discovery is made at a redshift[1] of $z \approx 1.3$, we can argue that departure from large-scale homogeneity is a late time phenomenon and that the cosmological principle regains validity at much earlier times.
In this work, we consider the inflationary paradigm and its conceptual challenges. As inflation is assumed to have occurred at very early times, assuming validity of the cosmological principle is sufficiently motivated.

1.1.2 Friedmann-Robertson-Walker Spacetime

We now have the theoretical motivation and experimental indication to restrict the geometry of the spacetime under consideration significantly. Derivatives with respect to coordinate time shall henceforth be denoted with dots and derivatives with respect to conformal time, that will be introduced later on, shall be denoted with primes.

Slicing and Threading of Spacetime

In General Relativity there exist no global inertial frames [6]. Thus the question might arise according to which observer isotropy and homogeneity are found to

[1]Within cosmology, the redshift z of a distant object is defined as $1 + z = \frac{a_0}{a_e}$ [7], where a_0 is the scale factor of the universe today and a_e is the scale factor at the time of photon emission. Thus the larger the redshift the further away the object is and the further back in time one looks. The cosmic microwave background radiation decoupled roughly at a redshift of $z \approx 1000$, hence the scale factor today is about 1000 times bigger than at the time of decoupling. Thus photons emitted from an object at $z \approx 1.3$ have been emitted at late times.

be characteristics of spacetime. To shed light on this issue, consider slicing spacetime into time-ordered three-dimensional spacelike hypersurfaces denoted by Σ_t which are non-intersecting, homogeneous and isotropic. These hypersurfaces are labelled by a parameter t and thus this parameter can be considered a universal time coordinate in that a particular time refers to a given hypersurface. It is important to note that there is no preferred slicing, i.e. a spacelike hypersurface with $t = \text{const}$ may be constructed in any way. So slicing spacetime does not lead to preferred spatial or time coordinates. The time coordinate t denoting spacelike hypersurfaces is often called coordinate time. But it is important to keep in mind that there are no preferred coordinate systems as such hence the name does not imply some promoted status. Incorporating the cosmological principle requires the spacelike hypersurfaces to be maximally symmetric 3-spaces.

Likewise, we may thread spacetime into non-intersecting wordlines of constant coordinates. These worldlines can be associated with fundamental observers that are at rest with respect to the overall cosmological fluid. We will soon find it reasonable to define the slicing and threading in such a way that the worldlines of fundamental observers are orthogonal to the spacelike hypersurfaces.

Homogeneity and Isotropy of the Universe

Let us now connect the results of the previous two subsections and restrict the metric of a homogeneous and isotropic universe. Homogeneity demands equivalence of all points on some hypersurface Σ_t and isotropy requires all directions to be equivalent for fundamental observers.

Considering three non-colinear points each of which may be regarded as a fundamental observer, then isotropy demands that the formed triangle in 3-space must be similar to a triangle the non-colinear points form at some later time t. Homogeneity obviously makes the magnification factor of that triangle independent from position. Therefore, the spatial separation must only be affected by a common time-dependent factor. We set the speed of light c to unity and may write

$$ds^2 = dt^2 - a^2(t)dl^2, \tag{1.1.11}$$

where $a(t)$ is the scale factor, normalised to unity today, $dl^2 = g_{ij}dx^i dx^j$ is the line element on Σ_t and x^i are comoving coordinates with the indices i, j running from 1 to 3.

Now we may recall the concept of the fundamental observer and construct the observer to have constant comoving coordinates $x^i = \text{const}$ but parametrise $x^0 = \tau$. Along the wordline, $dx^i = 0$ and hence $ds = d\tau = dt$ (as c is set to unity). Thus the proper time of a fundamental observer equals the coordinate time. From the above, we can now immediately deduce

$$[u^\mu] \equiv \left[\frac{dx^\mu}{d\tau}\right] = (1, 0, 0, 0). \tag{1.1.12}$$

Any vector within Σ_t is of the form $[v^\mu] = (0, v^1, v^2, v^3)$ and thus

$$g_{\mu\nu}u^\mu v^\nu = 0. \tag{1.1.13}$$

Therefore we see that the fundamental observer's wordline is constructed such that it is orthogonal to the spacelike hypersurface as indicated before.

The Maximally Symmetric 3-Space

As stated before, the cosmological principle limits the hypersurfaces Σ_t to be maximally symmetric three-spaces. For a thorough discussion, see [6]. Here, we will outline the relevant arguments.

A general 3-space has a curvature tensor with six independent components, thus six functions need to be specified in order to describe the intrinsic geometry. Thanks to symmetries, we can reduce the number of independent components significantly. A maximally symmetric 3-space is just characterised by one number, its curvature K, which is - due to homogeneity - independent from position. The maximally symmetric 3-space is thus a space of constant curvature.

There exist three maximally symmetric three-spaces [11]. Flat Euclidean space satisfies

$$dl^2 = d\mathbf{x}^2 = \delta_{ij} dx^i dx^j , \tag{1.1.14}$$

and a sphere/hypersphere embedded in four-dimensional Euclidean space is described by

$$dl^2 = d\mathbf{x}^2 \pm dz^2, \; z^2 \pm \mathbf{x}^2 = a^2 \tag{1.1.15}$$

where a is the radius of the 3-sphere or some arbitrary constant for the case of the hypersphere. Rescaling to $z^2 \pm \mathbf{x}^2 = 1$ gives $dl^2 = a^2(d\mathbf{x}^2 \pm dz^2)$. Expressing dz in terms of $d\mathbf{x}$, we may write

$$dl^2 = a^2 \left[d\mathbf{x}^2 \pm \frac{\mathbf{x} \cdot d\mathbf{x}^2}{1 \mp \mathbf{x}^2} \right] . \tag{1.1.16}$$

We can recast the above in a general way and incorporate the Euclidean line element by writing

$$dl^2 = a^2 \left[d\mathbf{x}^2 \pm k \frac{\mathbf{x} \cdot d\mathbf{x}^2}{1 - k\mathbf{x}^2} \right] \equiv a^2 \tilde{g}_{ij} dx^i dx^j , \tag{1.1.17}$$

where k can take the values $-1, 0, +1$ for hyperbolic, Euclidean or spherical geometry respectively and $a^2 > 0$. We have now found the spatial line element for maximally symmetric 3-spaces.

To establish standard notation lets quickly review two coordinate transformations that, depending on the situation, may render the above more convenient to use. When going to spherical polars with $d\mathbf{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ and $\mathbf{x} \cdot d\mathbf{x} = rdr$, we can write

$$dl^2 = a^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega \right] , \; d\Omega \equiv d\theta^2 + \sin^2\theta d\phi^2 . \tag{1.1.18}$$

However, one often finds the redefinition $d\chi = dr/\sqrt{1 - kr^2}$, which then leads to

$$dl^2 = a^2 \left[d\chi^2 + S_k^2(\chi) d\Omega \right] , \tag{1.1.19}$$

where $S_k(\chi)$ may take the form $\sinh\chi$, χ or $\sin\chi$ for hyperbolic, Euclidean or spherical geometry respectively.

The Friedmann-Robertson-Walker Metric

Now we can simply substitute the spatial line element derived in the previous subsection into (1.1.11) in order to obtain a metric which satisfies the demands of the cosmological principle. We may write

$$ds^2 = dt^2 - a^2(t)\tilde{g}_{ij}dx^i dx^j,$$

which in polar coordinates is

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right]. \tag{1.1.20}$$

This is the Friedmann-Robertson-Walker metric. Note that we choose to set the scale factor to unity today, i.e. $a(t_0) = 1$. Also, the values of k for hyperbolic, flat and spherical geometry may now be coined as describing an open, flat or closed universe respectively. Furthermore, we may introduce so called *conformal time* $d\tau = dt/a(t)$, for which the FRW metric is found to be

$$\begin{aligned} ds^2 &= a^2(\tau)\left[d\tau^2 - \tilde{g}_{ij}dx^i dx^j\right] \\ &= a^2(\tau)\left[d\tau^2 - \left(d\chi^2 + S_k^2(\chi)d\Omega\right)\right] \end{aligned} \tag{1.1.21}$$

The cosmological principle has led us to find a spacetime in which physical distances relate to coordinate distances via a time-dependent scale factor, i.e. $d_{phys} = a(t)x$. The behaviour of the function $a(t)$ has yet to be determined. First, we will continue with a discussion of the causal structure of this spacetime.

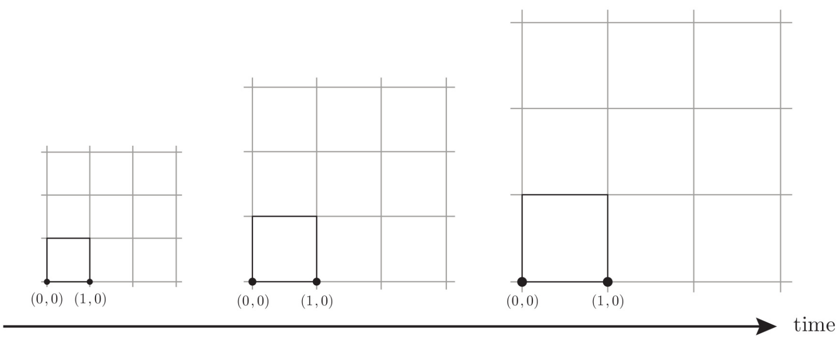

Figure 1.1: Objects at a certain comoving distance apart will always remain the same comoving distance apart. The physical separation increases with time as $d_{phys} = a(t)x$ [11].

1.1.3 Cosmological Horizons and Causal Structure

The causal structure of a spacetime depends on the propagation of light. Within cosmology, causal contact between two points is established when a photon has travelled from one point to the other. Strictly speaking, this definition of causality just requires causal contact in one direction and does not include a causal response for the causal interaction to be complete, i.e. sufficient causality for cosmology is one directional. One thus might argue that causality is seen in an oversimplified way in cosmological scenarios. However, defining causal interactions merely by photon propagation proves to be a sufficient approach for our purposes.

Before moving on let us quickly review one important feature of the spacetime we have just derived. The physical (or sometimes called proper) distances are obtained by multiplying the comoving coordinates with the scale factor. We therefore have to distinguish physical and comoving coordinates. Without peculiar motion, two galaxies fixed at comoving coordinates will always be the same comoving coordinate distance apart but their physical distance changes with the change of the scale factor. To avoid confusion we will always try to use comoving coordinates and only use physical coordinates if their use offers additional insight.

The propagation of light follows null geodesics, i.e. $ds^2 = 0$. When considering just radial propagation, i.e. $d\theta = d\phi = 0$, (1.1.21) simply reduces to

$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2\right]. \tag{1.1.22}$$

Thus, the trajectory of a light ray is found to be

$$\chi(\tau) = \tau + const., \tag{1.1.23}$$

and the comoving distance a light ray has travelled within an interval of coordinate time therefore is

$$\tau = \int_{t_1}^{t_2} \frac{dt}{a(t)}. \tag{1.1.24}$$

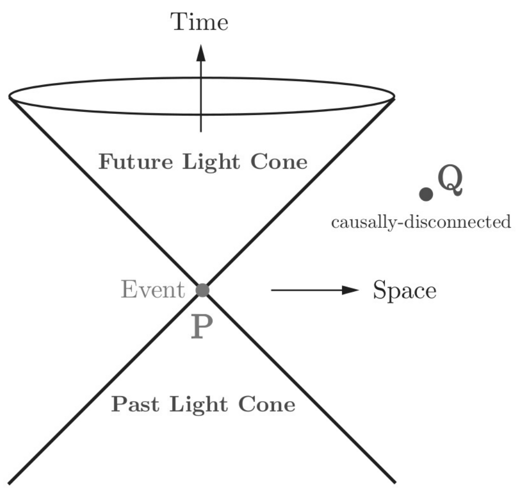

Figure 1.2: In a diagram of conformal time vs. comoving distance, light rays travel lines at a 45 degree angle, i.e. null geodesics for which $ds^2 = 0$. Interactions are only possible on timelike geodesics for which $ds^2 > 0$ whereas events connected by spacelike geodesics, i.e. $ds^2 < 0$, are out of causal contact [5].

The now following definitions of cosmological horizons simply differ in how one chooses the limits of integration.

If the integration is done from $t_1 = 0$ to some t_2, we may write

$$\tau_p = \int_0^{t_2} \frac{dt}{a(t)}. \tag{1.1.25}$$

This is the comoving distance a light ray has travelled from the beginning of time to some t_2, assuming there was an initial singularity and hence a beginning of time. As nothing can move faster through space than light, this integral sets a limit on the range of causal interactions that could have influenced an observer by the time t_2. We call τ_p the comoving *particle horizon* and find it to be the maximum width of an observer's past light cone.
Let us now keep in mind that causal interactions must not be undone, thus the particle horizon, i.e. the past light cone's maximum width must never decrease. An event within an observer's past light cone must never leave it.

If, however, we choose to integrate from some t_0 to infinity, we write

$$\tau_e = \int_{t_0}^{\infty} \frac{dt}{a(t)}. \tag{1.1.26}$$

This expression yields the range of all future possible interactions. Thus every point in space closer to an observer than τ_e at some t_0 can be causally influenced by this observer. We call τ_e the cosmological *event horizon* and recognise it as the maximum width of an observer's future light cone.

Now consider the following; the scale factor $a(t)$ may change its behaviour over time, thus integrating out to infinity not considering the changing behaviour seems to be a flawed concept. Intuitively, one would suggest

$$\tau_e = \int_{t_0}^{t_1} \frac{dt}{a_0(t)} + \int_{t_1}^{t_2} \frac{dt}{a_1(t)} + ... \tag{1.1.27}$$

until the final scale factor may be used and the integration could go out to infinity or some corresponding t_{max}. As an example, lets consider knowledge of the inflationary and the radiation dominated phases of the universe but no knowledge of any evolution beyond. An observer within the radiation dominated phase would, according to the above argument, calculate the event horizon of an observer during inflation to be

$$\tau_e = \underbrace{\int_{t_0}^{t_{reh}} \frac{dt}{e^{Ht}}}_{\text{finite}} + \underbrace{\int_{t_{reh}}^{\infty} \frac{dt}{t^{1/2}}}_{\to\infty} \to \infty, \tag{1.1.28}$$

where we have omitted normalisation constants as they do not influence the behaviour of the integrals necessary in this argument and t_{reh} is the time at the end of inflation. The event horizon does only seem to depend on the last known phase of evolution of the scale factor and the concept of an event horizon during an earlier stage does not seem to make any sense.

Yet we still seek a concept that yields the maximum range of future causal interaction. We may argue that as we might not have the knowledge about the future evolution of the scale factor we have no choice but to introduce this concept of an idealised maximum range of future causal interaction as in (1.1.26). The longer different phases of scale factor evolution persist, the better

this idealisation approximates a maximum distance, at least for some period of time during a certain phase. So equation (1.1.27) theoretically yields the true range of possible future interaction but is difficult to actually calculate as the future evolution might be unknown. Thus (1.1.26) gives a satisfying approximation. In the course of this work we will always refer to (1.1.26) as the event horizon and use it as the maximum distance of causal interactions for a certain phase of scale factor evolution.
Note that by using (1.1.26) we simply pretend a certain scale factor behaviour to go on forever and assign the so obtained event horizon to that phase of scale factor behaviour. So when we say that the event horizon is shrinking during the inflationary phase, what we mean is that the event horizon of an eternally inflating universe is shrinking and we thus simply apply this result to the inflationary phase of our universe. For if the inflationary phase is long enough the event horizon persists for at least some time before it changes its behaviour depending on the scale factor evolution.

In conclusion, we find that the particle horizon must never decrease and events may never leave it. The future range of causal interaction is, for our purposes, defined as in (1.1.26). Events may enter and leave this horizon and thus the range of future interactions may obviously vary. In fact, we will soon find the decreasing event horizon to be the definition of the inflationary phase. To introduce some convention; when describing the inflationary phase or cosmological perturbations, often the mere word horizon is used to refer to the event horizon. Once a scale has left the (event) horizon, no causal interaction is possible any more. This is what will be meant by horizon exit. As stated before, the (event) horizon grows after the inflationary phase thus events, i.e. objects emitting radiation may re-enter the (event) horizon and hence causal interaction is possible again. The particle horizon will always be referred to as such.

A word of warning: We will show that during the inflationary phase, event horizon and Hubble horizon coincide so that the word horizon may refer to either event or Hubble horizon which is physically the same during inflation. However, these horizons are not equivalent for any other phase. So the only two horizons that shall be used in the following are event and particle horizon explicitly denoted as such.

1.1.4 Superluminal Motion and Horizon Exit

The question often arises, though not important for the course of this work but closely related to the preceding section, whether we can observe objects that recede from us with a velocity faster than the speed of light c. Let us first recall that expansion of space may very well lead to galaxies receding with velocities greater than c and that this recession does not violate Special Relativity, as the recession is due to space itself expanding, not due to a galaxy's peculiar velocity within space.

Whether or not one can observe light emitted from a galaxy receding faster than the speed of light with respect to oneself simply depends on whether the observer lies within the galaxy's event horizon τ_e at the time of emission. If that is the case then the emitted light will be received and the galaxy hence be observed.

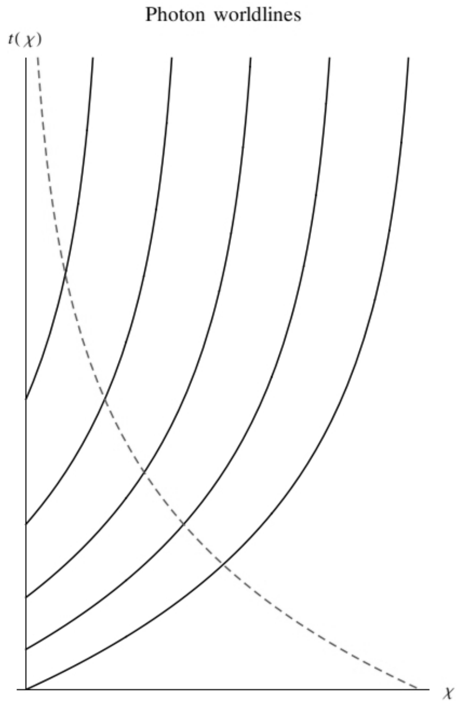

Figure 1.3: This is a coordinate time t vs. comoving distance χ diagram. The scale factor is $a = e^t$, thus the photon world lines are $t = -\ln(-\chi + t_i)$ where the units are $H = 1$. The bold lines are photon world lines emitted at different times t_i from an observer at $\chi = 0$ whereas the dashed line depicts the observers event horizon $t_e = -\ln(\chi)$. For this de Sitter universe we see that the event horizon is decreasing. As the photon world lines frame the future light cone of their event of emission we immediately may infer that the event horizon yields the maximum width of the future light cone.
If however the behaviour of the scale factor changes such that the worldlines decrease their slope at late times, the notion of the event horizon is invalidated.

For usual models of the scale factor (i.e. $a \propto t^\alpha, 0 < \alpha < 1$), we find

$$\tau_e \propto \int_{t_0}^{\infty} \frac{dt}{a(t)} \to \infty, \tag{1.1.29}$$

thus those universes do not have an event horizon and hence all events will eventually be observed.

When considering a dark energy or inflaton dominated universe (i.e. quasi de Sitter space, $a \propto e^{Ht}, H = const$), we obtain

$$\tau_e(t) \propto \frac{1}{H} e^{-Ht}, \tag{1.1.30}$$

and hence those universes actually have a shrinking event horizon in comoving coordinates. Therefore, events are leaving the range of possible future causal interactions and will cease to be observable eventually. In physical coordinates, (1.1.30) would yield $d_e = H^{-1}$, i.e. the event horizon is a constant distance away. Expansion of space then naturally carries objects outside of that horizon.

Summarising, we find only those receding galaxies to be observable in whose event horizon τ_e we are at the time of photon emission. We observe the galaxy as soon as the event of photon emission has entered our particle horizon. So it is important to keep in mind that observability and actual observation are two different concepts linked to the two different horizons.

Now lets ask what happens if a galaxy is being observed but then spacetime changes its behaviour to e.g. an eternal phase of accelerating expansion. Obviously, we can argue that the galaxy's τ_e decreases, hence at some time, the observer will be outside of the galaxy's τ_e and vice-versa - the galaxy has left the event horizon. But what does this process look like? The exit of the event

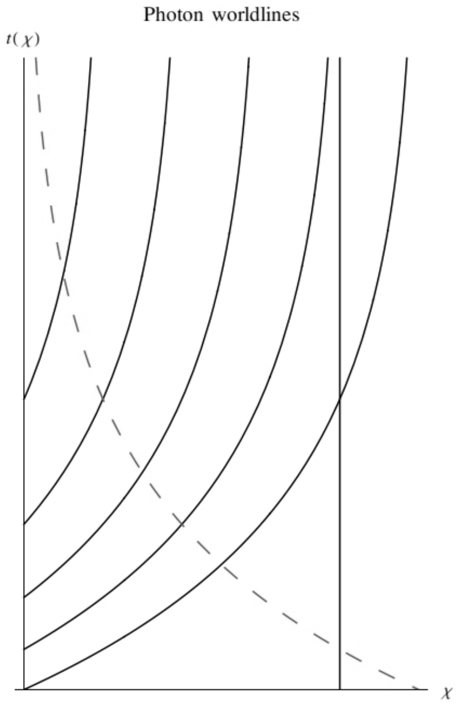

Figure 1.4: Consider another observer's wordline, i.e. the vertical line. This observer will leave the event horizon of the observer located at $\chi = 0$. Thus they will eventually be out of causal contact. This is exactly what is meant by horizon or Hubble horizon exit in inflationary scenarios.
However, due to the asymptotic behaviour of the photon world lines the observer located at $\chi \neq 0$ will never cease to receive signals from the observer located at $\chi = 0$ and vice versa. Yet the signals received in larger intervals represent more and more closely spaced emission events. Hence both observers will see each other as moving infinitely slowly through time and with infinite redshift. This is meant by the term *freeze-in.*

horizon cannot be observed as such. What can be observed though is exactly the same as what one would see when observing an object approaching the event horizon of a black hole. For the observer, the receding galaxy's time slows down infinitely. Thus once a galaxy has been observed, it will not escape our idealised capability of observation as such but it will leave our event horizon and will look like moving infinitely slowly through time.
The galaxy thus approaches infinite redshift and seems to be moving infinitely slowly in the time direction. This is what is meant by the term *freeze in.* We will later encounter exactly this behaviour when scales leave the event horizon during the inflationary phase and thus are *frozen in* at superhorizon scales.

1.1.5 Dynamics of FRW Spacetime

So far, we have considered the scale factor as some function of time but have not specified its exact behaviour, despite having anticipated its form in previous arguments about cosmological horizons. In order to do so, we need further restricting physics, namely the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1.31}$$

These equations link the behaviour of spacetime to its matter content. Having found forms of $T_{\mu\nu}$ allows us to calculate $G_{\mu\nu}$ and therefore to find an explicit form of $a(t)$ for a given dominating matter content. For notational ease we will work in units of $8\pi G = c = 1$.

Let us now quote the definition of the Einstein tensor, see e.g. [6, 7]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.1.32}$$

with

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}, \; R \equiv g^{\mu\nu}R_{\mu\nu}, \tag{1.1.33}$$

for

$$\Gamma^{\mu}_{\alpha\beta} \equiv \frac{g^{\mu\nu}}{2}\left[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right], \tag{1.1.34}$$

where $()_{,\beta} = \frac{\partial}{\partial x_\beta}$.

The Cosmological Fluid

We now seek a model for the energy-momentum tensor of the matter content of the universe. The energy-momentum tensor $T_{\mu\nu}$ may be decomposed into a 3-scalar, T_{00}, into the 3-vectors T_{i0} and T_{0j} as well as the 3-tensor T_{ij}. The requirement of isotropy requires the components of the 3-vectors to vanish, i.e. $T_{i0} = T_{0j} = 0$. Furthermore, isotropy about the origin implies that the 3-tensor T_{ij} is proportional to g_{ij}. Homogeneity then requires all constants of proportionality to be functions of time alone. Therefore, we find it instructive to approximate the matter content of the universe with a perfect fluid without shear- or bulk-viscosity and lacking any heat conducting properties. The only characterising elements are then pressure and density in the fluid's rest frame. We thus have

$$T^{\mu}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix},$$

or in covariant form

$$T^{\mu\nu} = (\rho + P)\, u^{\mu}u^{\nu} - Pg^{\mu\nu}, \tag{1.1.35}$$

with u^{μ} being an observers four-velocity. For $u^{\mu} = (1, 0, 0, 0)$, i.e. in the fluid's rest frame, we recover the above matrix.

Now, energy-momentum conservation requires

$$\nabla_{\mu}T^{\mu\nu} = 0. \tag{1.1.36}$$

Just taking the $\nu = 0$ equation, this is

$$\partial_{\mu}T^{\mu 0} + \Gamma^{\mu}_{\alpha\mu}\rho - \Gamma^{\alpha}_{0\mu}T^{\mu\alpha} = 0, \tag{1.1.37}$$

but isotropy means T_{i0} vanishes, hence the above further reduces to

$$\frac{d\rho}{dt} + \Gamma^{\mu}_{0\mu}\rho - \Gamma^{\alpha}_{0\mu}T^{\mu\alpha} = 0, \tag{1.1.38}$$

and evaluating the Christoffel symbols for the FRW metric then yields the continuity equation

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{1.1.39}$$

We have found an equation giving us the time-evolution of a matter content of the universe. Let us now explicitly calculate how certain matter components behave.

Although having used the term matter above to relate to anything within the universe, we will now restrict the word only to mean non-relativistic matter for which the pressure is much less than its energy density, i.e. $P \ll \rho$. Often this is also simply referred to as dust. We thus assume an equation of state $P = 0$ and (1.1.39) becomes

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho = 0. \tag{1.1.40}$$

This can either be solved directly or the solution might be read of by recasting the above as

$$\frac{1}{a^3}\frac{d}{dt}\left(\rho a^3\right) = 0. \tag{1.1.41}$$

Both methods lead to

$$\rho \propto \frac{1}{a^3}, \tag{1.1.42}$$

which is what one would have expected intuitively for matter in a space expanding in three dimensions.

The term radiation is used for anything whose pressure is about a third of its density, thus the equation of state reads $P = \frac{1}{3}\rho$, (1.1.39) becomes

$$\frac{1}{a^4}\frac{d}{dt}\left(\rho a^4\right) = 0, \tag{1.1.43}$$

and we find radiation density to evolve as

$$\rho \propto \frac{1}{a^4}, \tag{1.1.44}$$

which again yields an intuitive result as the additional factor of a^{-1} can be understood as being caused by the redshift due to expansion.

We will later find curvature density to fall off with $\rho \propto a^{-2}$ but now investigate another, rather special component of the universe first. An ideal cosmological constant[2] has an equation of state $P = -\rho$, hence we immediately find

$$\dot{\rho} = 0. \tag{1.1.45}$$

It is truly remarkable that this vacuum energy does not dilute with the expansion of space. Soon, we will find that it is this behaviour that gives rise to the inflationary phase.

It might be purposeful to already remark the following at this point. A true cosmological constant with the above equation of state would be, as the name suggests, constant in time, thus if such a phenomenon would be to govern a phase of the universe's expansion, e.g. inflation, then there would not be a way to ever stop that phase (disregarding models of quintessence for now), thus there exists no way for a graceful exit. We will thus consider an effective time-decaying cosmological constant that maintains $P = -\rho$ over a sufficient amount of time. How to realise this will be subject of section 1.2.

[2] A cosmological constant Λ may be added to the Einstein tensor (1.1.32) in the form of $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda$. This then has $T_{\mu\nu} = g_{\mu\nu}\Lambda$ and therefore $P = -\rho$.

The Cosmological Field Equations

Now we may finally relate the matter content described in the previous subsection to the evolution of the scale factor of the FRW spacetime. In order to do so, we compute the l.h.s. of (1.1.32),

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

Again, isotropy saves us some time as we only have to calculate the R_{00} and R_{ij} components. We find

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{1.1.46}$$

$$R_{ij} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right]. \tag{1.1.47}$$

Thus, the Ricci scalar is

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \tag{1.1.48}$$

Hence, the l.h.s. of (1.1.32) is found to be

$$G^0_0 = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right], \tag{1.1.49}$$

$$G^i_j = \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\delta^i_j. \tag{1.1.50}$$

We may then simply combine the above two results with (1.1.35),

$$T^{\mu\nu} = (\rho + P)\, u^\mu u^\nu - P g^{\mu\nu},$$

and obtain the Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2} \tag{1.1.51}$$

$$H^2 + \dot{H} = \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + P), \tag{1.1.52}$$

where we have chosen units of $8\pi G = c = 1$. From (1.1.51) we see that 3-curvature k is dominated by a^{-2} as mentioned earlier.

For an expanding universe, i.e. $\dot{a} > 0$, filled with non-relativistic matter or radiation, (1.1.52) immediately tells us that $\ddot{a} < 0$, thus there must be a singularity in the past for which $a(t = 0) = 0$. This assumes of course that General Relativity remains valid at all times. We will later find that an inflationary, i.e. de Sitter universe does not have a singularity in the past.

Now consider a flat universe at time $t = t_0$, i.e. today. The first Friedmann equation lets us then define the critical density as

$$\rho_{crit} = 3H_0^2. \tag{1.1.53}$$

We may then define dimensionless density parameters for the different matter components of the universe as

$$\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{crit,0}}. \tag{1.1.54}$$

Thus equation (1.1.51) can be recast as

$$\Omega - 1 = \frac{k}{\dot{a}^2}, \tag{1.1.55}$$

which is a form that will be of use later on.

Solutions of the Friedmann Equations for a Flat Universe

Now consier a flat universe ($k = 0$). For Radiation domination, $\rho \propto a^{-4}$ and (1.1.51) reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^4}, \tag{1.1.56}$$

which has the solution $a \propto t^{\frac{1}{2}}$.

Matter domination means $\rho \propto a^{-3}$, hence (1.1.51) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^3}, \tag{1.1.57}$$

which is solved by $a \propto t^{\frac{2}{3}}$.

Finally, we consider a cosmological constant for which $\rho = \Lambda =$ const., and hence

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \Lambda, \tag{1.1.58}$$

which has the solution $a \propto e^{\Lambda t}$.

To summarize, we have found:

	$P(\rho)$	$\rho(a)$	$a(t)$
RD	$\frac{1}{3}\rho$	a^{-4}	$t^{\frac{1}{2}}$
MD	0	a^{-3}	$t^{\frac{2}{3}}$
Λ	$-\rho$	a^0	$e^{\Lambda t}$

So if we assume that the universe must have been radiation dominated at very early times, we conclude that matter domination must take over at some point until the cosmological constant dominates for eternity.

We have now a full description of the dynamics of the universe, all of which was derived from the simple assumption that there is nothing preferred about earth when observing space. Our theory does not contradict observations and seems to describe the universe we inhabit rather accurately. As we will see in the next section, our theory does however require finely tuned initial conditions. Of course, it may be argued that a theory should yield the evolution of a system once initial conditions are given, but since we are not talking about bullet trajectories but the universe, the initial conditions are of great interest and a theory that does not encompass the universe's initial conditions will be highly unsatisfying.

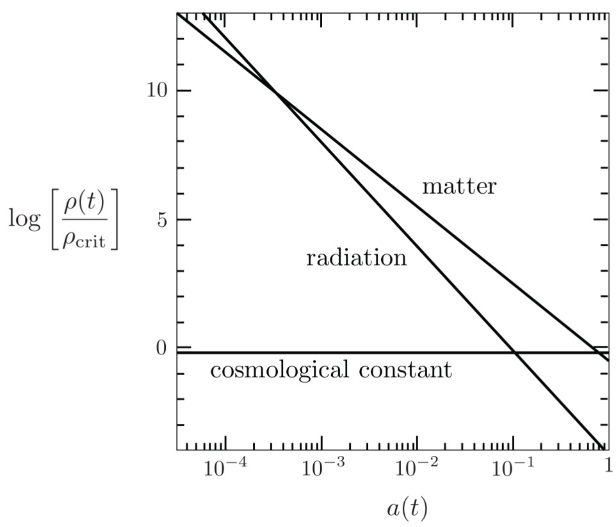

Figure 1.5: Radiation density falls off faster than the matter density. Hence when starting with a radiation dominated phase, matter domination will eventually take over. As the density of the cosmological constant does not dilute it will finally dominate over all other components [11].

1.2 The Inflationary Paradigm

The formalism introduced above describes the observations accurately and within an elegant mathematical framework. However, as pointed out already, the theory only yields the time evolution of the system and does not make any statement about its initial conditions. Furthermore, observational input requires the initial conditions to be fine-tuned to a very high degree. In this section, we will review the basic paradigm to approach this issue.

1.2.1 The Horizon Problem

In Big Bang Cosmology, we observe the cosmic microwave background radiation to be nearly isotropic, see e.g. [2]. However, considering the amount of time passed between the initial singularity and the time of decoupling of the CMB t_{dec} (in the Standard Model of Big Bang cosmology), we find that two events (i.e. decoupling of a CMB photon) on opposite sides of our visible universe cannot have overlapping past light cones and there is thus no reason for them to display isotropy to such a high degree[3]. In other words, the particle horizons of those two events share no single common event.

This is not a contradiction or flaw of Big Bang cosmology but simply an observation that Big Bang cosmology does not predict. One could therefore argue that the near isotropy of the CMB is a coincidence which does not need to

[3]Note that we are assuming one directional causal interactions to be sufficient for reaching thermal equilibrium, i.e. isotropy. This is obviously a questionable assumption and thus states the problem in its weakest form. Requiring at least a true interaction would make the fine-tuning problem even worse.

be explained. As this is highly unsatisfying to many people, the inflationary paradigm has been proposed. We may now solve this problem by requesting a certain behaviour from either of the horizons defined the first section.

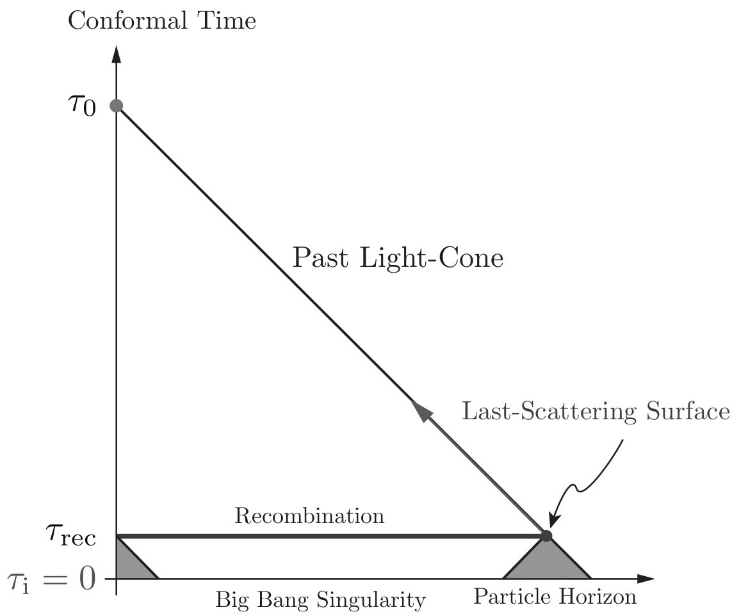

Figure 1.6: The past light cones of two events of decoupling share no common event in standard big bang cosmology [5]. Thus the isotropy of the CMB seems to be purely coincidental.

"Particle Horizon view point"

In order for the past light cones of two events of decoupling to actually overlap or equivalently, for their particle horizons to actually share events, we may simply require τ_p of those events to have a much bigger contribution from the past than from later times. Recall the definition

$$\tau_p = \int_0^{t_0} \frac{dt}{a(t)}.$$

Let us now recast the above as

$$\tau_p = \int_0^{t_{dec}} \frac{dt}{a(t)} = \int_{\ln a(0)}^{\ln a(t_{dec})} \frac{1}{aH} d\ln a. \tag{1.2.1}$$

So to fulfil our requirement of bigger contributions to the integral in earlier times, we may simply recognise

$$\frac{d}{dt}(\dot{a}^{-1}) < 0. \tag{1.2.2}$$

Now consider

$$\frac{d}{dt}(\dot{a}^{-1}) = -\frac{\ddot{a}}{\dot{a}^2} < 0 \rightarrow \ddot{a} > 0. \tag{1.2.3}$$

We have derived $\ddot{a} > 0$, i.e. accelerating expansion of space, from the simple argument that τ_p must have had bigger contributions in the past. Thus our considerations of overlapping past light cones have lead us directly to the condition defining the inflationary phase.

"Event Horizon view point"

We may also argue the following way. In order for the CMB to be nearly isotropic, we may simply request the entire visible universe to be within one event's τ_e before the beginning of inflation. In other words, τ_e was much bigger before inflation than it is after, or, τ_e has to be decreasing. Recall the definition (1.1.26)

$$\tau_e = \int_{t_i}^{\infty} \frac{dt}{a(t)}.$$

and recast this in a similar manner, i.e.

$$\tau_e = \int_{t_i}^{\infty} \frac{dt}{a(t)} = \int_{\ln a(t_i)}^{\ln a(\infty)} \frac{1}{aH} d\ln a, \tag{1.2.4}$$

we find the same constraint as above when requiring that τ_e has to decrease, namely

$$\frac{d}{dt}(\dot{a}^{-1}) = -\frac{\ddot{a}}{\dot{a}^2} < 0, \tag{1.2.5}$$

and thus arrive at the same $\ddot{a} > 0$, therefore having found the definition of the inflationary phase.

So we have deduced the condition for the inflationary phase from requiring that the particle horizon must have had bigger contributions in the past and that the event horizon is shrinking. This solves the horizon problem.

1.2.2 The Flatness Problem

From the position of the first peak of the CMB's angular power spectrum, it may be deduced that the universe is in a state very close to spatial flatness [16]. Now consider the Friedmann equation with the curvature term

$$H^2 = \frac{\rho}{3} - \frac{k}{a^2}. \tag{1.2.6}$$

Dividing both sides by the Hubble parameter yields

$$\Omega(a) - 1 = \frac{k}{(aH)^2}. \tag{1.2.7}$$

Now recall the behaviour of the scale factor for the phases of radiation and matter domination respectively, i.e.

$$a(t) \propto t^{1/2} \tag{1.2.8}$$

$$a(t) \propto t^{2/3}, \tag{1.2.9}$$

which have derivatives w.r.t. coordinate time of the form

$$\dot{a} \propto t^{-1/2} \tag{1.2.10}$$

$$\dot{a} \propto t^{-1/3}. \tag{1.2.11}$$

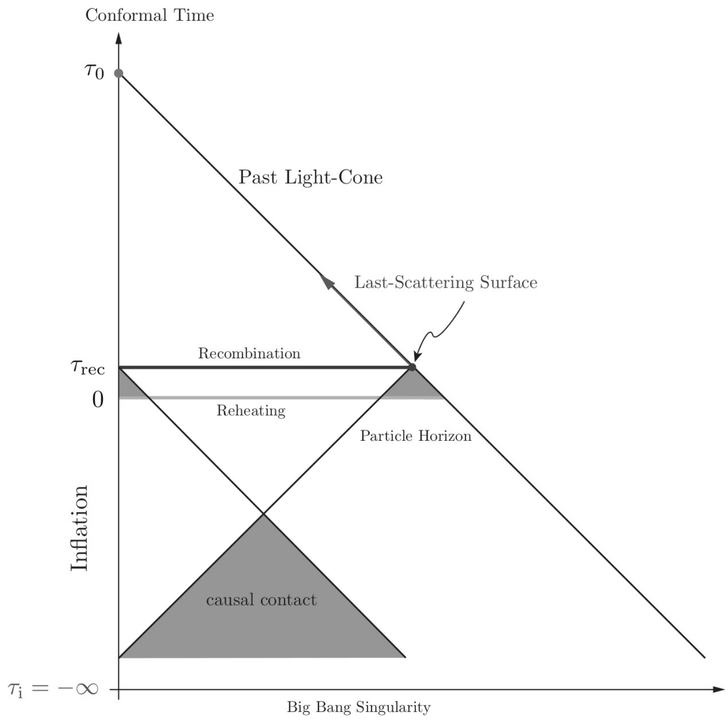

Figure 1.7: In principle, inflation provides enough conformal time before the surface of last scattering such that the entire visible universe was within one event's event horizon τ_e before inflation [5]. From the other point of view, the particle horizons now overlap thus the events of decoupling share a common past.

We see that $(aH)^2 = \dot{a}$ is always decreasing and hence that the r.h.s of (1.2.7) is diverging. Thus the state of spatial flatness is found to be an unstable state. As stated initially, the universe is close to spatial flatness nowadays and as it is shown to diverge from spatial flatness, we can deduce that the universe must have been extremely close to spatial flatness at early times. Of course the initial condition of flatness may be achieved by coincidence, but this is - similar to the horizon problem - highly unsatisfying. We thus seek a mechanism driving the universe to spatial flatness initially.

Again, consider the r.h.s. of (1.2.7). If we want the universe to be driven towards spatial flatness, we simply require the denominator, i.e. aH, to be growing. But note that this can be formally put as

$$\frac{d}{dt}(\dot{a}) > 0 \rightarrow \ddot{a} > 0, \tag{1.2.12}$$

and we have again derived the condition for the inflationary phase.

How much Inflation?

Now we can ask and answer the question of how much inflation is needed in order to solve our fine tuning problems. First recall the curvature term of the Friedmann equation and recast it as

$$\Omega_k = \frac{k^2}{a^2H^2}. \tag{1.2.13}$$

Let us now express the ratio of the curvature density at inflation to the curvature density today as

$$\frac{\Omega_{k,*}}{\Omega_{k,0}} = \left(\frac{H_0}{H_*}\right)^2 \left(\frac{a_0}{a_*}\right)^2 \approx \frac{t_*}{t_0}, \tag{1.2.14}$$

with $H_0/H_* \approx t_*/t_0$. Making the assumption that inflation takes place between the Planck era and the GUT phase transition, the above ratio is found to be between 10^{-60} and 10^{-54} [6].

As the ratio depends on a_*^{-2}, the scale factor has to grow by a factor of 27 to 30 which implies $60 - 70$ e-folds. Thus inflation needs to persist over about this amount of e-folds in order to solve the flatness problem.
Investigating the horizon problem yields a similar number of e-folds, hence both problems are overcome with the inflationary mechanism [7].

1.2.3 Physics of Inflation

We will now describe a toy model inflaton field where the inflaton is simply taken to be a scalar field and we do not further specify its potential $V(\phi)$. Note that for the remainder of this work, a specific inflationary model or potential is not required. We will always simply take the inflaton as being a scalar field without the need of further specifying its potential. This remains true for all chapters and hence all considerations are model-independent and thus of generall nature.

Conditions for Inflation

Let us now recall what we have found to be defining inflation [11],

$$\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0, \tag{1.2.15}$$

from which we deduce $\ddot{a} > 0$, i.e. a phase of accelerating expansion. Let us now introduce a new notation by writing

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + \dot{H}a}{(aH)^2} = -\frac{1}{a}(1-\epsilon) < 0, \tag{1.2.16}$$

where $\epsilon \equiv -\dot{H}/H^2$. The above equation (1.2.16) then implies $0 < \epsilon < 1$. Consider $H = \dot{a}/a$, from which we deduce $Hdt = d\ln a$. But Hdt measures the number of e-folds N of the duration of inflationary expansion, hence $dN = d\ln a$. Let us now recast the definition of ϵ as

$$\epsilon = -\frac{\dot{H}}{H^2} = -\frac{1}{H}\frac{dH}{Hdt} = -\frac{1}{H}\frac{dH}{dN} = -\frac{d\ln H}{dN} < 1. \tag{1.2.17}$$

From this we infer that the fractional change of H per e-fold is small but positive as $0 < \epsilon < 1$. To solve the horizon and flatness problems, we want inflation to occur over about 60-70 e-folds, thus we want ϵ to remain small over roughly this very number. Let us introduce

$$\eta \equiv \frac{d\ln\epsilon}{dN} = \frac{\dot{\epsilon}}{H\epsilon}, \tag{1.2.18}$$

which is therefore the fractional change of ϵ per e-fold. For inflation to persist we hence require $|\eta| < 1$.

What is the physical source for a universe with the above outlined behaviour? Recall the Friedmann equation

$$H' + H^2 = -\frac{1}{6}(\rho + 3P) = -\frac{H^2}{2}\left(1 + \frac{3P}{\rho}\right) \tag{1.2.19}$$

from which we deduce

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{P}{\rho}\right) < 1 \rightarrow \omega \equiv \frac{P}{\rho} < -\frac{1}{3}, \tag{1.2.20}$$

which tells us that the physical source of inflation must violate the strong energy condition. We may equally well infer that the source of inflation has to have negative pressure.

Field Theory

Let us know quickly recall some results from scalar field theory. Consider the following action for a scalar field

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}(\phi, \nabla_\mu\phi). \tag{1.2.21}$$

Varying the action with respect to the metric yields the stress-energy tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial(\nabla^\mu\phi)}\nabla_\nu\phi - g_{\mu\nu}\mathcal{L}. \tag{1.2.22}$$

Variation with respect to the field ϕ gives

$$\delta S_\phi = S_\phi[\phi + \delta\phi] - S_\phi[\phi] \tag{1.2.23}$$

$$= \int d^4x\sqrt{-g}\left[\mathcal{L}(\phi + \delta\phi, \nabla_\mu(\phi + \delta\phi)) - \mathcal{L}(\phi, \nabla_\mu\phi)\right], \tag{1.2.24}$$

which at first order is

$$\delta S_\phi = \int d^4x\sqrt{-g}\left[\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi)}\right)\right]\delta\phi. \tag{1.2.25}$$

This is stationary if the field satisfies the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi)}\right) = 0. \tag{1.2.26}$$

We will limit ourselves to the case of

$$S_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{1.2.27}$$

where the Euler-Lagrange equation may be used to obtain the Klein-Gordon equation

$$\Box\phi = -\frac{dV}{d\phi} \tag{1.2.28}$$

with $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. Considering $\sqrt{-g} = a^4$ this operator becomes

$$\Box = (\sqrt{-g})^{-1}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu). \tag{1.2.29}$$

The stress-energy tensor may then be shown to equal

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi - V(\phi)\right). \tag{1.2.30}$$

Let us now apply the above results and proceed to slow-roll inflation.

Slow-Roll Inflation

Consider a simple scalar field minimally coupled to gravity,

$$S = S_{EH} + S_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{1.2.31}$$

where we do not need to further specify the physical behaviour of this field but simply use it as our parameter of interest. Let us assume that this scalar field is the dominant ingredient of the inflationary universe and hence its stress-energy tensor dominates the dynamics of spacetime. As we want space to be homogeneous, the background value of the field ϕ can only depend on time. Evaluating (1.2.30) in this FRW setup of homogeneous and isotropic spacetime, it takes the form of a perfect fluid, i.e. $T_{\mu\nu} = diag(\rho, -P, -P, -P)$ and yields

$$\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi) \tag{1.2.32}$$

$$P_\phi = \frac{1}{2}\dot\phi^2 - V(\phi). \tag{1.2.33}$$

From the above, we may introduce the equation of state

$$\omega_\phi \equiv \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot\phi^2 - V(\phi)}{\frac{1}{2}\dot\phi^2 + V(\phi)}, \tag{1.2.34}$$

and already infer that $\omega < 0$ when the potential dominates over the kinetic term. The Friedmann equation may now be recast as

$$H^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + V\right] \tag{1.2.35}$$

after having substituted for ρ. Recalling

$$\Box = (\sqrt{-g})^{-1}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu),$$

(1.2.28) may be recast as

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{1.2.36}$$

Where $V' = dV/d\phi$. From the equation of motion and the rewritten Friedmann equation, we can derive

$$H' = -\frac{1}{2}\dot{\phi}^2. \tag{1.2.37}$$

Substituting the above into (1.2.17), we obtain

$$\epsilon = \frac{1}{2}\left(\frac{\dot{\phi}}{H}\right)^2. \tag{1.2.38}$$

From (1.2.35) and the condition $0 < \epsilon < 1$ we may deduce from the above the same condition as found earlier when considering the equation of state, namely that the potential has to dominate over the kinetic term. Note that this argument goes along the lines of the argument presented in the first subsection of 1.2.3, but now we are able to link our dynamical requirements to terms of a scalar field. Again, we would like $0 < \epsilon < 1$ to persist, or in other words, that the acceleration of the scalar field must have a small fractional change per e-fold. We introduce

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{1.2.39}$$

and take the time-derivative of (1.2.38) to find $\eta = 2(\epsilon - \delta)$. To summarise, both H and ϵ ought to have small fractional changes per e-fold during inflation, so we impose $\epsilon, \delta, \eta \ll 1$. Let us now impose these regimes onto the equations of motion. This is what is generally referred to as *slow-roll*, which is a sufficient but not necessary criterion for inflation.

The Friedmann equation (1.2.35) reduces to

$$H^2 \approx \frac{V}{3}, \tag{1.2.40}$$

and the Klein-Gordon equation (1.2.36) can be rewritten as

$$3H\dot{\phi} \approx -V'. \tag{1.2.41}$$

With the above approximations, we quickly find

$$\epsilon \approx \frac{1}{2}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_v, \tag{1.2.42}$$

and taking the time-derivative of our simplified Klein-Gordon equation, we arrive at

$$-\left(\frac{\ddot{\phi}}{H\dot{\phi}}+\frac{H'}{H^2}\right)\approx\frac{V''}{V}\equiv\eta_v. \tag{1.2.43}$$

These are called the slow-roll parameters. As long as $\eta_v, \epsilon_v < 1$, inflation persists. It might be noteworthy to stress that for slow-roll inflation to occur at all, $0 < \epsilon < 1$ is necessary. Yet this alone would not solve the problems inflation was set up to avoid. Thus in order for inflation to also solve the flatness and horizon problems, we need it to occur over many e-folds which is expressed in the requirement $\eta \ll 1$.

De Sitter Approximation

The above outlined slow-roll regime displays the background evolution (1.2.40), which is

$$H^2 \approx \frac{V}{3} \approx const. \tag{1.2.44}$$

Recalling $H = \dot{a}/a$, we find

$$H \approx const. \approx \frac{\dot{a}}{a} \rightarrow \frac{da}{a} \approx H dt, \tag{1.2.45}$$

from which we then deduce the well known

$$a(t) \propto e^{Ht}. \tag{1.2.46}$$

Therefore, during slow-roll inflation, spacetime may be well approximated by de Sitter space. However, as in perfect de Sitter space inflation would neither end nor would there be an initial singularity - strictly speaking the beginning of time, i.e. t_i, may now be chosen arbitrarily - we assume that spacetime does not resemble perfect de Sitter space. It is simply an approximation when considering physics well within the inflationary phase.

Let us now quickly lay out some characteristics of a de Sitter universe. Its comoving event horizon may be calculated to be[4]

$$\tau_e \propto \int_{t_i}^{\infty} \frac{1}{e^{Ht}} dt = \frac{1}{H} e^{-Ht_i}. \tag{1.2.47}$$

So during slow-roll inflation, the comoving event horizon is shrinking. However, the proper distance to the event horizon is

$$d_e \propto e^{Ht_i} \int_{t_i}^{\infty} \frac{1}{e^{Ht}} dt = \frac{1}{H}. \tag{1.2.48}$$

This is the already mentioned case that during a de Sitter phase, the physical distance to the event horizon remains constant. Note that what stays constant is the proper, i.e. physical distance to the event horizon whereas it is shrinking in comoving coordinates. The comoving particle horizon is

$$\tau_p \propto \int_{t_i}^{t_f} \frac{1}{e^{Ht}} dt = \frac{1}{H}\left(e^{-Ht_i} - e^{-Ht_f}\right). \tag{1.2.49}$$

[4]Technically, one could normalise the scale factor such that $a(t_0)$=1, i.e. comoving and proper distances coincide today. However the normalisation does not affect the behaviour that is emphasised in this subsection, hence the relevant horizons are given with proportionalities.

Here, the lower bound of integration is a matter of choice. Conventionally, we may set $t_i = 0$, thus obtaining

$$\tau_p \propto \int_0^{t_f} \frac{1}{e^{Ht}} dt = \frac{1}{H}\left(1 - e^{-Ht_f}\right). \tag{1.2.50}$$

This does not contradict our earlier considerations since, as predicted, the particle horizon is growing. Its asymptotic behaviour reflects the fact that for ever lasting de Sitter space, everything will be torn apart eventually thus no new causal interactions may contribute to the particle horizon. However, there is no reason why $t_i \to -\infty$ should not be allowed. This would lead us to

$$\tau_p \propto \int_{-\infty}^{t_f} \frac{1}{e^{Ht}} dt = \frac{1}{H}\left(e^{\infty} - e^{-Ht_f}\right) \to \infty. \tag{1.2.51}$$

But what is the interpretation of this? The above choice for the lower bound of integration simply reflects that in de Sitter space there might be in infinite amount of conformal time in the past. One could therefore also argue, that this choice of lower bound immediately solves the horizon problem of standard Big Bang cosmology as we then have an infinite amount of conformal time, i.e. an infinite particle horizon or causal past thus being able to explain the isotropy of the CMB. For completeness, lets quickly calculate the proper distance to the particle horizon for the conventional choice of the lower bound of integration. We find

$$d_p \propto e^{Ht_f} \int_0^{t_f} \frac{1}{e^{Ht}} dt = \frac{1}{H}\left(e^{Ht_f} - 1\right). \tag{1.2.52}$$

Again, we have the anticipated behaviour of a growing particle horizon.

Ending Inflation and Reheating

In order for the standard evolution of our universe to occur, inflation must end somehow. This is still poorly understood but in a simplified view, inflation ends once the slow roll conditions are no longer satisfied. It is assumed that once the inflaton is oscillating about its minimum value it decays into other fields and thus reheats the - due to inflation - cooled universe. In recent models however, quantum fluctuations of the inflaton field are considered and it is argued that these fluctuations constantly move the field further away from its minimum in some places and therefore inflation globally never ends. So there will be infinitely many inflating bubble universes where the number of inflating universes is much larger than the number of universes with standard evolution.
Note however that the notion of reheating only applies to the universe we inhabit. As the inflaton undergoes quantum fluctuations, it can occur that inflation violates slow-roll in one region but not in another. So globally, there will be an infinite number of bubble universes in an eternally inflating scenario.

Perhaps less understood than ending inflation is how to start it. In this first chapter, we have only effectively described the inflationary paradigm without outlining some underlying particle physics model. Thus our best understanding of the inflationary paradigm resembles that of a cosmological constant varying with time without knowing the detailed physics involved. Yet as this effective understanding provides an excellent fit to the observations we may keep our faith in this approach [18].

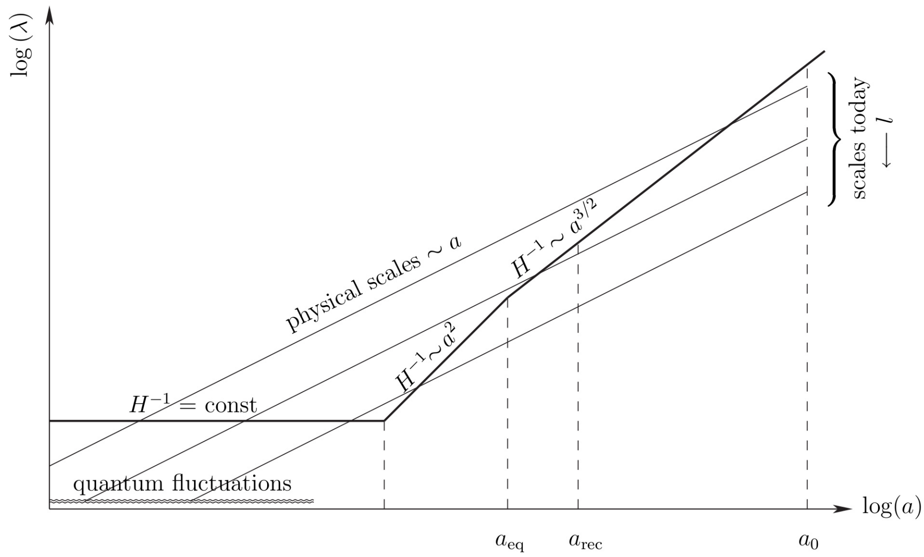

Figure 1.8: Physical scales leave the event horizon during inflation and re-enter the Hubble horizon afterwards. Note that H^{-1} is not the exact event horizon for a post-inflationary universe but is taken here as an approximation [17]. Keep in mind that in order for a scale to be observable, it has to enter the observer's event horizon but observation takes place when the event of photon emission has entered the observer's particle horizon.

Chapter 2

Quantum Field Theory in Curved Spacetime

The first chapter has described a homogeneous and isotropic universe within the framework of classical General Relativity. However, we seek a physical mechanism causing inhomogeneities that will eventually lead to the formation of structure. As we will later describe perturbations of the inflaton field that we take to be quantum fluctuations we need to familiarise ourselves with the theory of quantum fields on general backgrounds.
The purpose of this chapter is to establish fundamentals of quantum field theory in a curved background as they are of utmost importance later on. The treatment summarises and complements known results [19–23] and seeks to present them in a consistent and coherent manner and in a suitable notation that we will use throughout this work.

2.1 General Discussion

The symmetry group of Minkowski space is the Poincaré group. We may decompose a quantised scalar field as [19]

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right] \tag{2.1.1}$$

where the hats for denoting operators have been omitted in this section for notational ease, the vacuum is then defined as

$$a_i|0\rangle = 0, \tag{2.1.2}$$

and the mode function u_k obeys the equation of motion of the field ϕ. This vacuum is invariant under the action of the Poincaré group.

Now by the principle of general covariance, coordinate systems are physically irrelevant. Furthermore, a general spacetime may not have Poincaré symmetry. Thus in a curved, i.e. non-Minkowskian spacetime, we may decompose a scalar field with another arbitrary orthonormal set $\{\bar{u}_k\}$ as

$$\phi(x) = \sum_j \left[\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x) \right] \tag{2.1.3}$$

where we then also find a new vacuum state $\bar{a}_j|\bar{0}\rangle = 0$. This vacuum state is no longer invariant under the action of a certain group. Considering both sets of the above expansions to be complete, we may write the mode functions of one set in terms of the other set's mode functions, i.e.

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*) \tag{2.1.4}$$

$$u_i = \sum_i \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right). \tag{2.1.5}$$

The above are called *Bogolubov Transformations* with α, β being the Bogolubov coefficients. Considering the mode functions to be orthonormal, i.e.

$$(u_i, u_j) = \delta_{ij},\ (u_i^*, u_j^*) = -\delta_{ij},\ (u_i, u_j^*) = 0, \tag{2.1.6}$$

where $(u_i, u_j) = -i \int_\Sigma (u_i \partial_\mu u^* - u_i^* \partial_\mu u_j)_j \sqrt{-g_\Sigma} d\Sigma^\mu$, we may write

$$\begin{aligned}
(\bar{u}_j, u_k) &= \sum_i (\alpha_{ji} \underbrace{(u_i, u_k)}_{\delta ik} + \beta_{ji} \underbrace{(u_i^*, u_k)}_{0}) \\
&= \sum_i \alpha_{ji} \delta_{ik} \\
&= \alpha_{ji}
\end{aligned} \tag{2.1.7}$$

and likewise find

$$(\bar{u}_i, u_j^*) = -\beta_{ij}. \tag{2.1.8}$$

As shown explicitly in appendix A, the creation and annihilation operators are related via

$$a_i = \sum_j [\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger] \tag{2.1.9}$$

$$\bar{a}_j = \sum_i [\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger]. \tag{2.1.10}$$

This describes the mixing of the creation and annihilation operators for different expansions.

Finally we are ready to deduce a fundamental statement considering quantised fields on a general background which foreshadows later considerations. Recall the definition of one vacuum $\bar{a}_i|\bar{0}\rangle = 0$. But what happens when acting on the same state with the annihilation operator of a different orthonormal set? We have

$$\begin{aligned}
a_i|\bar{0}\rangle &= \sum_j [\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger]|\bar{0}\rangle \\
&= \sum_j \alpha_{ji} \underbrace{\bar{a}_i|\bar{0}\rangle}_{0} + \sum_j \beta_{ji}^* \underbrace{\bar{a}_j^\dagger|\bar{0}\rangle}_{|\bar{1}\rangle} \\
&= \sum_j \beta_{ji}^* |\bar{1}\rangle \neq 0.
\end{aligned}$$

So as long as $\beta_{ij} \neq 0$, the expansion's associated Fock spaces are different. To further illustrate this consider $N_i = a_i^\dagger a_i$ to yield

$$\langle \bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2. \tag{2.1.11}$$

To put it in words, the vacuum of the $\bar{u}_j$ modes contains $\sum_j |\beta_{ji}|^2$ particles in the u_i mode. Thus the particle concept becomes observer dependent. This is a most crucial result. When considering the wave-particle duality and it's mathematical form $E = hf$ as the first step in debunking the particle concept, the combination of quantum field theory and general relativity to yield observer dependent particles may be seen as the final one.

But with the particle concept out of our hands, what physical object could be a reasonable candidate to replace it? We might recall that tensor equations do hold in all coordinate systems. Thus rather than thinking about particles it would seem useful to employ e.g. the idea of the energy-momentum tensor, as e.g. if the expectation value of a tensor equation is zero in one frame it will be zero in all other reference frames. This is exactly the approach of chapter 7.

2.2 Conjugate Momentum

So far, we have only regarded a field and its mode equations yet not the field's conjugate momentum. When promoting some introduced field variable to field operator status, the variable's conjugate momentum is required in order to impose relevant commutation relations. In order to obtain an expression for the conjugate momentum, recall the action of a scalar field (1.2.27) but consider it to be massless

$$S = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{2.2.1}$$

where $g = det(g_{\mu\nu})$. Since $S = \int d^4x\mathcal{L}$, we may infer the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{2.2.2}$$

To obtain the field's conjugate momentum, consider

$$\pi_\phi = \frac{\partial\mathcal{L}}{\partial\phi'}. \tag{2.2.3}$$

Now let us investigate the case of a rescaled field $f = a\phi$ where $a = a(\tau)$ is the scale factor. If we want to quantise the rescaled field f we require its conjugate momentum which we then calculate as

$$\begin{aligned}
\pi_f &= \frac{\partial\mathcal{L}}{\partial f'} \\
&= \frac{\partial}{\partial f'}\left[\frac{\sqrt{-g}}{2a^2}\left\{(\partial_o\phi)^2 - \partial_i\phi\partial_j\phi\delta^i_j\right\}\right] \\
&= \frac{\partial}{\partial f'}\left[\frac{\sqrt{-g}}{2a^2}\left\{\left(\frac{1}{a}f' - \frac{a'}{a}f\right)^2 - \partial_i\phi\partial_j\phi\delta^i_j\right\}\right] \\
&= f' - \frac{a'}{a}f,
\end{aligned} \tag{2.2.4}$$

as there is no f'-dependence in the spatial derivatives and $\sqrt{-g} = a^4$ for flat FRW spacetime, i.e. $g_{\mu\nu} = a(\tau)^2 \cdot diag(1, -1, -1, -1)$. Thus we have obtained a relation between the field's and the rescaled field's conjugate momentum as

$$\pi_f = a(\tau)\pi_\phi. \tag{2.2.5}$$

We are now well equipped to establish our prescription of quantising a field in a flat de Sitter background.

2.3 Quantised Fields in a de Sitter Background

Let us now consider the case of a massless rescaled scalar field f in flat de Sitter space [20–23], i.e. a time-dependent background. In order to quantise the field, we impose the following commutation relations

$$[\hat{f}(\tau, \mathbf{x}), \hat{\pi}_f(\tau, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{2.3.1}$$

$$[\hat{f}(\tau, \mathbf{x}), \hat{f}(\tau, \mathbf{y})] = [\hat{\pi}_f(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = 0, \tag{2.3.2}$$

which implies

$$[\hat{f}(\tau, \mathbf{k}), \hat{\pi}^\dagger_f(\tau, \mathbf{k'})] = i\delta^{(3)}(\mathbf{k} - \mathbf{k'}). \tag{2.3.3}$$

The creation and annihilation operators satisfy

$$[\hat{a}_\mathbf{k}, \hat{a}^\dagger_{\mathbf{k'}}] = \delta^{(3)}(\mathbf{k} - \mathbf{k'}), \; [\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k'}}] = [\hat{a}^\dagger_\mathbf{k}, \hat{a}^\dagger_{\mathbf{k'}}] = 0. \tag{2.3.4}$$

Now, consider a component of the field operator $\hat{f}$ by taking the Fourier transform of an i-th component of equation (2.1.1)

$$\hat{f}_\mathbf{k}(\tau) = f_k(\tau)\hat{a}_\mathbf{k} + f^*_k(\tau)\hat{a}^\dagger_{-\mathbf{k}},$$

where the mode functions hence depend on conformal time τ and we may impose the normalisation condition on the mode functions $f_k(\tau)$

$$\langle f_k, f^*_k \rangle \equiv i(f^*_k f'_k - f'^*_k f_k) = 1. \tag{2.3.5}$$

Now, choosing different expansions such as in subsection 2.1 is essentially the same as choosing certain times τ, i.e. $a_\mathbf{k} \to \hat{a}_\mathbf{k}(\tau_0), \bar{a}_\mathbf{k} \to \hat{a}_\mathbf{k}(\tau_1)$.
To further illustrate this point and to establish a new notation, recast the above with the operators absorbing the time dependence and write

$$\hat{f}_\mathbf{k}(\tau) = \hat{a}_\mathbf{k}(\tau) + \hat{a}^\dagger_{-\mathbf{k}}(\tau), \tag{2.3.6}$$

Likewise, we may write the conjugate momentum (2.2.4) as

$$\hat{\pi}_\mathbf{k}(\tau) = -ik\left(\hat{a}_\mathbf{k}(\tau) - \hat{a}^\dagger_{-\mathbf{k}}(\tau)\right). \tag{2.3.7}$$

Similar to equation (2.1.9), we may now recast the creation and annihilation operators in terms of their values at some fixed time τ_0 as

$$\hat{a}_\mathbf{k}(\tau) = u_k(\tau)\hat{a}_\mathbf{k}(\tau_0) + v_k(\tau)\hat{a}^\dagger_{-\mathbf{k}}(\tau_0) \tag{2.3.8}$$

$$\hat{a}^\dagger_{-\mathbf{k}}(\tau) = u^*_k(\tau)\hat{a}^\dagger_{-\mathbf{k}}(\tau_0) + v^*_k(\tau)\hat{a}_\mathbf{k}(\tau_0), \tag{2.3.9}$$

which are Bogolubov transformations and yield the mixing of creation and annihilation operators as time proceeds. As the commutation relations have to be preserved under unitary time evolution, i.e. they have to obey the condition (2.3.5), we have the constraint

$$|u_k(\tau)|^2 - |v_k(\tau)|^2 = 1. \tag{2.3.10}$$

Substituting the above into equation (2.3.6), we obtain

$$\hat{f}_{\mathbf{k}}(\tau) = f_k(\tau)\hat{a}_{\mathbf{k}}(\tau_0) + f_k^*(\tau)\hat{a}^\dagger_{-\mathbf{k}}(\tau_0) \tag{2.3.11}$$

$$\hat{\pi}_{\mathbf{k}}(\tau) = -i\left(g_k(\tau)\hat{a}_{\mathbf{k}}(\tau_0) - g_k^*(\tau)\hat{a}^\dagger_{-\mathbf{k}}(\tau_0)\right), \tag{2.3.12}$$

with

$$f_k(\tau) = \frac{1}{\sqrt{2k}}\left(u_k(\tau) + v_k^*(\tau)\right) \tag{2.3.13}$$

$$g_k(\tau) = \sqrt{\frac{k}{2}}\left(u_k(\tau) - v_k^*(\tau)\right), \tag{2.3.14}$$

where equation (2.3.13) is a solution to the field equation of the field variable f and equation (2.3.14) may be obtained from the expression for the conjugate momentum. A vacuum may now be defined at any chosen time τ_0 to be

$$\hat{a}_{\mathbf{k}}(\tau_0)|0,\tau_0\rangle = 0. \tag{2.3.15}$$

Recalling equation (2.3.8), we can rewrite the above as

$$u_k(\tau_0)\hat{a}_{\mathbf{k}}(\tau_0) + v_k(\tau_0)\underbrace{\hat{a}^\dagger_{-\mathbf{k}}(\tau_0)|0,\tau_0\rangle}_{=|1,\tau_0\rangle} = 0, \tag{2.3.16}$$

which immediately sets $v_k(\tau_0) = 0$. Thus in order to define a vacuum state we need mode functions such that this condition is satisfied at the time the vacuum is imposed. Obviously, one may define a vacuum at any other time as

$$\hat{a}_{\mathbf{k}}(\tau_1)|0,\tau_1\rangle = 0. \tag{2.3.17}$$

However, at τ_1 we find

$$\hat{a}_{\mathbf{k}}(\tau_1)|0,\tau_0\rangle = \tag{2.3.18}$$

$$u_k(\tau_1)\underbrace{\hat{a}_{\mathbf{k}}(\tau_0)|0,\tau_0\rangle}_{=0} + v_k(\tau_1)\underbrace{\hat{a}^\dagger_{\mathbf{k}}(\tau_0)|0,\tau_0\rangle}_{=|1\rangle} \tag{2.3.19}$$

and thus the vacuum of τ_0 is not the vacuum at τ_1 any more. This phenomenon is called cosmological particle production.

At last, expanding the field variable $\hat{f}$ in terms of the Fourier modes (2.3.11), yields

$$\hat{f}(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\left(f_k(\tau)\hat{a}_{\mathbf{k}}(\tau_0)e^{i\mathbf{k}\cdot\mathbf{x}} + f_k^*(\tau)\hat{a}^\dagger_{\mathbf{k}}(\tau_0)e^{-i\mathbf{k}\cdot\mathbf{x}}\right), \tag{2.3.20}$$

which will be a form of use later on. We will make extensive use of the above formalism when quantising the variable describing the inflaton perturbations.

Chapter 3

The Perturbed Universe

The description of the first chapter applies to a perfectly homogeneous and isotropic universe. In this chapter, we want to enhance our understanding in order to be able to successfully describe deviations from the cosmological principle on small scales. Eventually, we will make use of the formalism introduced above in order to describe the quantum nature of the perturbations in the inflaton field.

3.1 Cosmological Perturbation Theory

The mere purpose of this section is to introduce all the relevant physics and notation in order to justify our proceedings in subsection 3.2.2.

So far, we have only considered a perfectly homogeneous and isotropic universe. Yet obviously, our universe is highly inhomogeneous on small scales. A perfectly homogeneous and isotropic universe would not allow for the formation of structure. Hence we need a method to describe small deviations from homogeneity, i.e. perturbations. For the course of this work we are only interested in perturbations of the dominating content of the universe during inflation, i.e. the inflaton field. With the tools of this section we can then successfully relate inflaton perturbations to the metric fluctuations in the next section.

Our strategy for section 3.2.2 will be perturbing the inflaton field, i.e.

$$\phi(\tau, \mathbf{x}) = \bar{\phi}(\tau) + \delta\phi(\tau, \mathbf{x}), \tag{3.1.1}$$

where the bar denotes the homogeneous background value. This will then be related to the perturbed metric which we write as

$$g_{\mu\nu}(\tau, \mathbf{x}) = \bar{g}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau, \mathbf{x}). \tag{3.1.2}$$

In the following, we will study how equation (3.1.1) leads to a perturbed energy-momentum tensor

$$T_{\mu\nu} = \bar{T}_{\mu\nu}(\tau) + \delta T_{\mu\nu}(\tau, \mathbf{x}). \tag{3.1.3}$$

As we always assume our perturbations to be small compared to the background values, we may expand the Einstein equations to linear order and work with

$$\delta G_{\mu\nu} = \delta T_{\mu\nu}, \tag{3.1.4}$$

where we have set $8\pi G = 1$ and we are working in conformal time τ. With this expansion we can then link the perturbations of the inflaton field to the perturbations of spacetime, i.e. the metric. This shall be our strategy for this section and provide us with all the necessary tools [5–7, 11, 24] for the physics to come.

3.1.1 Metric Fluctuations

The most general perturbed flat FRW metric is [5]

$$ds^2 = a^2(\tau)\left[(1+2\Psi)d\tau^2 - 2B_i dx^i d\tau - \left[(1-2\Phi)\delta_{ij} + 2E_{ij}\right]dx^i dx^j\right], \quad (3.1.5)$$

with conformal time being the time variable. $\Psi(\tau, \mathbf{x})$ is a 3-scalar named lapse, $B_i(\tau, \mathbf{x})$ is a 3-vector called shift, $\Phi(\tau, \mathbf{x})$ is a 3-scalar and called the spatial curvature perturbation and $E_{ij}(\tau, \mathbf{x})$ is a spatial shear 3-tensor being symmetric and traceless, i.e. $E^i_i = 0$. Equation (3.1.4) will relate these functions of the perturbed metric to the fluctuations in the inflaton field.

Fourier Space

To ease our lives considerably, we will often work in Fourier rather than real space. Recall

$$f(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} f_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (3.1.6)$$

Hence any partial differential equation for f turns into an ordinary differential equation for f_k. Individual Fourier modes decouple, i.e. f_k is independent [5] from $f_{k'}$ for $\mathbf{k} \neq \mathbf{k'}$ and $k = |\mathbf{k}|$.
Most importantly, we will make use of the following replacements

$$\partial_i \to ik_i \quad (3.1.7)$$

$$\nabla^2 \to -k^2. \quad (3.1.8)$$

3.1.2 Issues of Gauge

When analysing the homogeneous background universe, we are completely free to choose our preferred coordinate system as the physics involved is clearly coordinate independent.
However, the situation changes once dealing with small perturbations [24]. Technically speaking, when defining a perturbation $\delta Q(p)$ of Q at a point $p \in M$, where M is a spacetime manifold, as

$$\delta Q(p) = \bar{Q}(p) - Q(x(p)), \quad (3.1.9)$$

one might also choose an equally valid, second decomposition, such as

$$\delta\tilde{Q}(p) = \bar{Q}(p) - Q(\tilde{x}(p)). \quad (3.1.10)$$

For the above, we are dealing with an unperturbed background manifold N with fixed coordinates and a perturbed spacetime manifold M. Every coordinate choice for our perturbed spacetime corresponds to a different map from N to M and hence to a different definition of the perturbation δQ. In General Relativity, changing from one coordinate system infinitesimally to another is called a gauge transformation.

To give a more instructive picture of the problems related to different choices of gauge consider the following unperturbed quantity $\epsilon(t, \mathbf{x}) = \epsilon(t)$. We then introduce a new, equally allowed, time coordinate $\tilde{t} = t + \delta t(t, \mathbf{x})$. Now, we are interested in the form of $\tilde{\epsilon}(\tilde{t}, \mathbf{x}) \equiv \epsilon(t(\tilde{t}, \mathbf{x}))$. Assuming the usual $\delta t \ll t$, we find

$$\epsilon(t) = \epsilon(\tilde{t} - \delta t(t, \mathbf{x})) \approx \epsilon(\tilde{t}) - \frac{\partial \epsilon}{\partial t}\delta t \equiv \epsilon(\tilde{t}) + \delta\epsilon(\tilde{t}, \mathbf{x}). \tag{3.1.11}$$

The first term on the r.h.s. of the above may obviously be interpreted as the background value of the quantity in the new coordinates. More importantly however, we find the quantity to be perturbed. This perturbation is fictitious, i.e. non-physical and it only arises due to the change to new coordinates. Likewise, it may be possible to remove physical perturbations by choosing suitable coordinates.
To overcome these ambiguities, we will consider the complete set of perturbations, i.e. matter and metric fluctuations and will consider gauge invariant variables before fixing a certain gauge.

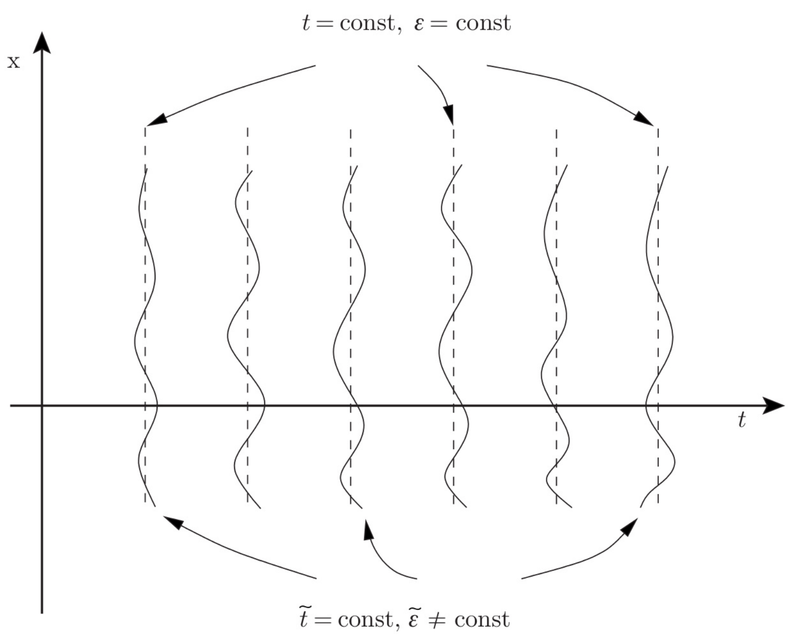

Figure 3.1: The dashed lines show the quantity ϵ in dependence of the time coordinate t while the solid lines show the quantity $\tilde{\epsilon}$ for the new time coordinate $\tilde{t} = t + \delta t(t, \mathbf{x})$. Whereas the quantity is unperturbed in one frame of reference, it is perturbed in the other [25].

Gauge Transformations

We will now derive the behaviour of scalar quantities, the metric tensor and the energy-momentum tensor under gauge transformations [5, 7, 11]. Considering, similar to the beginning of section 1.1.1, an infinitesimal coordinate, i.e. gauge transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x^\alpha), \tag{3.1.12}$$

we have

$$\frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\mu - \partial_\mu \xi^\rho(x^\alpha). \tag{3.1.13}$$

For notational ease, let us define

$$\xi^\mu \equiv (T, L^i). \tag{3.1.14}$$

Now consider a 4-scalar, e.g. the energy density ρ. We will split it into an unperturbed background value and a perturbation. The defining quality of a scalar is its invariance upon transformations and we will make use of this in order to derive the gauge transformation properties of scalar perturbations. Thus we have

$$\rho = \bar{\rho} + \delta\rho, \;\; \tilde{\rho}(\tilde{x}^\mu) = \rho(x^\mu). \tag{3.1.15}$$

Explicitly, the second equality of the above reads

$$\begin{aligned} \tilde{\rho}(\tilde{x}^\mu) &= \bar{\rho}(\tilde{\tau}) + \delta\tilde{\rho}(\tilde{x}^\mu) = \bar{\rho}(\tau) + T\bar{\rho}'(\tau) + \delta\tilde{\rho}(x^\mu) \\ &= \bar{\rho}(\tau) + \delta\rho(x^\mu), \end{aligned} \tag{3.1.16}$$

where $\delta\tilde{\rho}(\tilde{x}^\mu) = \delta\tilde{\rho}(x^\mu)$, as their difference is second order. We thus read off the transformation law

$$\delta\tilde{\rho} = \delta\rho - T\bar{\rho}'. \tag{3.1.17}$$

The behaviour of the functions of the metric (3.1.5) is easily found when considering the invariance of the line element, i.e.

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta = \tilde{g}_{\mu\nu}(\tilde{x})d\tilde{x}^\mu d\tilde{x}^\nu, \tag{3.1.18}$$

(note, that the scale factor must also be transformed), or, by simply applying the general transformation law for a tensor, i.e.

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x). \tag{3.1.19}$$

With the above tools, we quickly find the following behaviour for the functions of the metric 3.1.5 under gauge transformations to be

$$\tilde{\Psi} = \Psi - T' - \mathcal{H}T, \tag{3.1.20}$$

$$\tilde{\Phi} = \Phi + \mathcal{H}T + \frac{1}{3}\partial_i L^i, \tag{3.1.21}$$

$$\tilde{B}_i = B_i + \partial_i T - L_i', \tag{3.1.22}$$

$$\tilde{E}_{ij} = E_{ij} - \partial_{<i} L_{j>}, \tag{3.1.23}$$

where $\mathcal{H} = a'/a$ is the comoving Hubble parameter and $' = \partial_\tau$ denotes derivatives with respect to conformal time. We may deduce from the above that any two of the four functions Ψ, Φ, B and E can be made to vanish by choosing T and L_i appropriately.

In order to avoid earlier considered ambiguities related to gauge choices, the above may motivate us to construct the gauge invariant Bardeen potentials [24]

$$\Psi_B \equiv \Psi + \mathcal{H}(B - E') + B' - E'', \tag{3.1.24}$$

$$\Phi_B \equiv \Phi - \mathcal{H}(B - E'), \tag{3.1.25}$$

where now $\Psi_B = \tilde{\Psi}_B$ and $\Phi_B = \tilde{\Phi}_B$.

Gauge Choices

However, despite having gauge invariant variables to hand now, some calculations will be considerably simplified when working in a specific gauge.

When choosing T and L in such a way that $B = E = 0$, we speak of Newtonian or longitudinal gauge. The striking advantage of this gauge is that the remaining non-zero functions Ψ and Φ may simply be replaced by the Bardeen potentials to obtain gauge invariant expressions.
We will make use of this when outlining the derivation of the Einstein equation in Newtonian gauge. Once derived, the replacing yields gauge invariant expressions from which then the form in any other gauge may simply read off by setting two of the metric functions to chosen values.

Another useful gauge choice is spatially flat gauge where $\Phi = E = 0$. There, all fluctuations are parametrised in terms of the inflaton fluctuation $\delta\phi$. This will be our preferred choice of gauge when deriving an equation of motion for these fluctuations.

3.1.3 Energy-Momentum Perturbations

As 4-scalar quantities, we may define perturbations of pressure and density as

$$\rho(\tau, \mathbf{x}) \equiv \bar{\rho}(\tau) + \delta\rho(\tau, \mathbf{x}), \tag{3.1.26}$$

$$P(\tau, \mathbf{x}) \equiv \bar{P}(\tau) + \delta P(\tau, \mathbf{x}). \tag{3.1.27}$$

Thus density and pressure are now found to be functions of position. The 4-velocity u^μ may also be a function of position in a perturbed universe. Strictly speaking, a perturbed energy-momentum tensor may include anisotropic stress but this will always be negligible, hence we can write the energy-momentum tensor of the now perturbed fluid as

$$T^\mu_\nu = (\rho + P)u^\mu u_\nu - P\delta^\mu_\nu, \tag{3.1.28}$$

where we have omitted the Π^i_j term for anisotropic stress. In order to now find the expressions for the components of the perturbed energy-momentum tensor, we now have to obtain an expression for the perturbed 4-velocity u^μ. The constraint $g_{\mu\nu}u^\mu u^\nu = 1$ and the definition $u^\mu = dx^\mu/ds$ let us write

$$1 = g_{\mu\nu}\left(\frac{d\tau}{ds}\right)^2 \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.1.29}$$

$$= \left(\frac{d\tau}{ds}\right)^2 (g_{00} + \underbrace{2g_{0i}\frac{dx^i}{d\tau} + g_{ij}\frac{dx^i}{d\tau}\frac{x^j}{d\tau}}_{2^{nd} order}), \tag{3.1.30}$$

where we may call $dx^i/d\tau = v_i$ the coordinate velocity. When dropping second order terms, we quickly find

$$\frac{d\tau}{ds} = \frac{1}{a}\left(1 - \Psi\right). \tag{3.1.31}$$

Thus, we have

$$u^\mu = \frac{1}{a}\left(1 - \Psi, v^i\right) \tag{3.1.32}$$

as the perturbed 4-velocity. Lowering the index with the metric tensor yields the covariant expression

$$u_\mu = a\left(1+\Psi, -(v^i + B_i)\right) \tag{3.1.33}$$

With these results, we find the components of (3.1.28) to be

$$T^0_0 = \bar{\rho} + \delta\rho, \tag{3.1.34}$$
$$T^i_0 = (\bar{\rho} + \bar{P})v^i, \tag{3.1.35}$$
$$T^0_j = -(\bar{\rho} + \bar{P})(v_j + B_j), \tag{3.1.36}$$
$$T^i_j = -(\bar{P} + \delta P)\delta^i_j. \tag{3.1.37}$$

3.1.4 Einstein Equations

We will now consider an outline of the derivation of the perturbed Einstein equations $\delta G_{\mu\nu} = \delta T_{\mu\nu}$. Our ingredients will be the perturbed metric and the perturbed energy-momentum tensor from the previous subsections. As already hinted earlier, we will work in Newtonian gauge and substitute the Bardeen potentials in the end in order to arrive at the gauge invariant form.

Prerequisites

First, consider the metric and its inverse in Newtonian gauge, i.e.

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1+2\Psi & 0 \\ 0 & -(1-2\Phi)\delta_{ij} \end{pmatrix} \tag{3.1.38}$$

and

$$g_{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} 1-2\Psi & 0 \\ 0 & -(1+2\Phi)\delta^{ij} \end{pmatrix} \tag{3.1.39}$$

With the above definitions, we can derive the perturbed connection coefficients

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\tau}(g_{\tau\rho,\nu} + g_{\tau\nu,\rho} - g_{\nu\rho,\tau}). \tag{3.1.40}$$

Perturbed Einstein Equations

In order to obtain the perturbation of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, we start by calculating the perturbed Ricci tensor

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{3.1.41}$$

where we simply substitute the perturbed connection coefficients. Having dropped any notion of anisotropic stress from the beginning lets us now make a nice simplification. It can be deduced from the G_{ij} component of the Einstein tensor, namely $\Psi - \Phi = 8\pi G a^2 \delta\Pi$, that in the absence of anisotropic stress, ie. $\delta\Pi = 0$, we have $\Psi = \Phi$. We thus are left only with one degree of freedom in terms of which we will then write the perturbed Einstein equations.
Now the physics is all done and we are left with collecting our previous results and substituting them where appropriate.

After a significant amount of straightforward yet very tedious algebra, we eventually arrive at

$$-3\mathcal{H}(\Phi' + \mathcal{H}\Phi) + k^2\Phi = 4\pi G a^2 \delta\rho, \tag{3.1.42}$$
$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho} + \bar{P})v, \tag{3.1.43}$$
$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P. \tag{3.1.44}$$

Now remember the earlier statement that in Newtonian gauge the metric functions and Bardeen potentials are equal. Despite working with the assumption of negligible stress, we will now explicitly denote the functions Ψ_B and Φ_B as such for later convenience. We find the gauge invariant form of the perturbed Einstein equations, i.e. the equations put in terms of gauge invariant quantities, to be

$$-3\mathcal{H}(\Phi'_B + \mathcal{H}\Psi_B) + k^2\Phi_B = 4\pi G a^2 \delta\rho, \tag{3.1.45}$$
$$\Phi'_B + \mathcal{H}\Psi_B = -4\pi G a^2(\bar{\rho} + \bar{P})(v + B), \tag{3.1.46}$$
$$\Phi''_B + 2\mathcal{H}\Psi'_B + \mathcal{H}\Phi'_B + (2\mathcal{H}' + \mathcal{H}^2)\Psi_B = 4\pi G a^2 \delta P. \tag{3.1.47}$$

where Ψ_B and Φ_B are the Bardeen potentials

$$\Psi_B \equiv \Psi + \mathcal{H}(B - E') + B' - E'',$$
$$\Phi_B \equiv \Phi - \mathcal{H}(B - E').$$

3.1.5 Case Study: Spatially Flat Gauge

As mentioned earlier, we will now consider spatially flat gauge, i.e. $\Phi = E = 0$, as it will be the gauge in which we will calculate the equation of motion for the inflaton perturbations in the next section.

In this gauge, the perturbed Einstein equations may be directly inferred from the above results and read

$$-3\mathcal{H}^2\Psi - k^2\mathcal{H}B = 4\pi G a^2 \delta\rho, \tag{3.1.48}$$
$$\mathcal{H}\Psi = -4\pi G a^2(\bar{\rho} + \bar{P})(v + B), \tag{3.1.49}$$
$$2\mathcal{H}\Psi' + (2\mathcal{H}' + \mathcal{H}^2)\Psi = 4\pi G a^2 \delta P. \tag{3.1.50}$$

Now recall

$$T^0_j = -(\bar{\rho} + \bar{P})\partial_j(v + B)$$

and

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - V).$$

We recall that pressure and density for a scalar field only differ in the sign of the potential and that hence $\bar{\rho} + \bar{P} = a^{-2}\bar{\phi}'^2$ where the a^{-2} factor is due to differentiation with respect to conformal time. To linearised order, we can then recast the above for a scalar field as

$$T^0_j = \frac{\bar{\phi}'^2}{a^2}\partial_j\delta\phi. \tag{3.1.51}$$

In order to lay out the way for the calculations of the next subsection, consider taking the ∂_j derivative of (3.1.49). This is

$$\mathcal{H}\partial_j\Psi = -4\pi G a^2 \underbrace{(\bar{\rho}+\bar{P})\partial_j(v+B)}_{=T^0_j}, \tag{3.1.52}$$

and can simply be integrated to give

$$\Psi = 4\pi G\frac{\bar{\phi}'}{\mathcal{H}}\delta\phi. \tag{3.1.53}$$

Remembering $\epsilon = 4\pi G\bar{\phi}'^2/\mathcal{H}$, we arrive at

$$\Psi = \epsilon\frac{\mathcal{H}}{\bar{\phi}'}. \tag{3.1.54}$$

Now let us consider (3.1.48). From the perturbed energy-momentum tensor of a scalar field, we obtain

$$\delta\rho = \frac{1}{a^2}\left(\bar{\phi}'\delta\phi' - \bar{\phi}'^2\Psi\right) + V'\delta\phi. \tag{3.1.55}$$

From our treatment of inflation, we remember $a^2\partial_\phi V = -\bar{\phi}'' - 2\mathcal{H}\bar{\phi}'$ and thus, using (3.1.54), arrive at

$$\nabla^2 B = \epsilon\frac{\mathcal{H}}{\bar{\phi}'}\left(\delta\phi' + (\delta-\epsilon)\mathcal{H}\delta\phi\right), \tag{3.1.56}$$

with $\epsilon = 1 - \mathcal{H}'/\mathcal{H}^2$ and $\delta = 1 - \bar{\phi}''/\mathcal{H}\bar{\phi}'$. We will soon use these equations for the derivation of the Mukhanov-Sasaki equation.

3.2 Quantum Fluctuations and Initial Conditions

We have now accustomed ourselves to the mathematical description of a perturbed universe. In the cosmological community, it is generally understood that perturbations of the inflaton field $\delta\phi(\tau, \mathbf{x})$ cause the inflationary phase to last different amounts of time at different spatial coordinates thus inducing perturbations in the fabric of spacetime, i.e. metric perturbations and thus gravitational potentials then lead to perturbations in the initially homogeneous matter density.
However, this classical description does not say anything as to why the inflaton field is actually perturbed. The answer to that is that the inflaton field is considered to be a quantum object. To be more exact, the inflaton field may be split into a classical homogeneous background part $\bar{\phi}(\tau)$ and a quantised perturbation $\delta\hat{\phi}(\tau, \mathbf{x})$. The following subsections will describe the proceedings and eventually derive an equation of motion for the inflaton fluctuations. At last, we will "go quantum" and promote the inflaton perturbation to field operator status.

3.2.1 Prerequisites

Having discussed cosmological perturbation theory we will now turn to one of its most stunning applications. We will link the scalar perturbations of the inflaton field to the metric perturbations and will obtain an equation of motion of the form of a harmonic oscillator with time-dependent frequency [5, 6, 11].

We will consider a single scalar field minimally coupled to gravity which has an action of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{3.2.1}$$

Recall that inflationary spacetime may very well be approximated by de Sitter space for which we have[1]

$$ds^2 = a^2(\tau)\left(d\tau^2 - d\mathbf{x}^2\right),\ a(\tau) = -\frac{1}{H\tau}. \tag{3.2.2}$$

In general, we are confronted with five modes; the metric perturbations Ψ, Φ, B, E and the perturbation of the inflaton field $\delta\phi$. Invariance of (3.2.1) under gauge, i.e. coordinate transformations removes two modes of the metric fluctuations. Via the Einstein equation we will be able to link the other two to the inflaton perturbation which hence will be the only variable left. It is for this mode that we will find the equation of motion.

3.2.2 The Mukhanov-Sasaki Equation

We are now ready to derive the fundamental equation governing the coupling between inflaton and metric perturbations. It will be the most noteworthy result of the chapter and will bridge known physics with first trans-Planckian considerations to follow. The derivation is conveniently done in the ADM formalism and comoving gauge [26] or, more instructively, in spatially flat gauge. We will follow the latter approach.

In spatially flat gauge ($\Phi = E = 0$) the metric takes the form

$$ds^2 = a^2(\tau)\{(1+\Psi)d\tau^2 - 2\partial_i B dx^i d\tau - \delta_{ij}dx^i dx^j\} \tag{3.2.3}$$

We want to obtain an equation of motion for the inflaton perturbation $\delta\phi$ coupled to metric perturbations $\delta g_{\mu\nu}$. Consider the Klein-Gordon equation for a scalar field (i.e. the inflaton) in a general spacetime (i.e. $\eta_{\mu\nu} \to g_{\mu\nu}$) where $V' = \partial_\phi V$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right) + V'(\phi) = 0. \tag{3.2.4}$$

Substituting the metric (3.2.3) into (3.2.4), perturbing the inflaton field $\phi \to \bar{\phi} + \delta\phi$ and considering $\sqrt{-g} = a^4(1+\Psi)$ (to first order), we find

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \left(\nabla^2 - a^2V''\right)\delta\phi = \left(\Psi' - \nabla^2 B\right)\bar{\phi}' - 2a^2V'\Psi. \tag{3.2.5}$$

[1] The form of the scale factor follows when considering $a(t) = e^{Ht}$ and $d\tau = a^{-1}dt$. Solving for $d\tau$ then yields the relevant expression with $H = \dot{a}/a$.

In order to link the scalar potentials B and Ψ with the inflaton perturbation $\delta\phi$, we recall (3.1.54) and (3.1.56)

$$\Psi = \epsilon\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi \tag{3.2.6}$$

$$\nabla^2 B = \epsilon\frac{\mathcal{H}}{\bar{\phi}'}\left(\delta\phi' + (\delta - \epsilon)\,\mathcal{H}\delta\phi\right), \tag{3.2.7}$$

where $\epsilon = 1 - \mathcal{H}'/\mathcal{H}^2$ and $\delta = 1 - \bar{\phi}''/\mathcal{H}\bar{\phi}'$. Now considering the change of variables $\delta\phi = a^{-1}f$, we obtain

$$\delta\phi'' + 2\mathcal{H}\delta\phi' = \frac{1}{a}\left(f'' - \frac{a''}{a}f\right) \tag{3.2.8}$$

for the terms in the l.h.s. of (3.2.5). To get rid of V' and V'', we use the background equation of motion (3.2.4) and its derivative with respect to conformal time where $V' = \partial_\phi V$ and $V'' = \partial_\phi^2 V$

$$a^2 V' = -\bar{\phi}'' - 2\mathcal{H}\bar{\phi}' \tag{3.2.9}$$

$$-a^2 V'' = \frac{\bar{\phi}'''}{\bar{\phi}'} + 2\mathcal{H}' - 4\mathcal{H}^2. \tag{3.2.10}$$

Now we are all set to derive the equation of motion for the perturbation $\delta\phi$ of the inflaton field.

The strategy will be substituting (3.2.6), its derivative with respect to conformal time, (3.2.7), (3.2.8), (3.2.9) and (3.2.10) into (3.2.5). The derivative of (3.2.6) is found to be

$$\Psi' = \frac{\epsilon\mathcal{H}}{\bar{\phi}'}\delta\phi' - \frac{\epsilon\mathcal{H}\bar{\phi}''}{\bar{\phi}'^2}\delta\phi + \frac{1}{\bar{\phi}'}\left(\mathcal{H}' - \frac{\mathcal{H}''}{\mathcal{H}} + \frac{\mathcal{H}'^2}{\mathcal{H}^2}\right)\delta\phi. \tag{3.2.11}$$

Identifying $(\delta - \epsilon)\mathcal{H} = \mathcal{H}'/\mathcal{H} - \bar{\phi}''/\bar{\phi}'$, one can recast (3.2.7) as

$$\nabla^2 B = \frac{\epsilon\mathcal{H}}{\bar{\phi}'}\left(\delta\phi' + \frac{\mathcal{H}'}{\mathcal{H}}\delta\phi - \frac{\bar{\phi}''}{\bar{\phi}'}\delta\phi\right). \tag{3.2.12}$$

Thus we can rewrite the r.h.s. of (3.2.5) by substituting (3.2.6), (3.2.9), (3.2.11) and (3.2.12), while identifying $\epsilon\mathcal{H}' = \mathcal{H}' - \mathcal{H}^{2\prime}/\mathcal{H}^2$, $\epsilon\mathcal{H} = \mathcal{H} - \mathcal{H}'/\mathcal{H}$ and $\epsilon\mathcal{H}^2 = \mathcal{H}^2 - \mathcal{H}'$, as

$$r.h.s. = \left[-\frac{\mathcal{H}''}{\mathcal{H}} + 2\frac{\mathcal{H}'^2}{\mathcal{H}^2} + 2\left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right)\frac{\bar{\phi}''}{\bar{\phi}'} + 4\left(\mathcal{H}^2 - \mathcal{H}'\right)\right]\delta\phi. \tag{3.2.13}$$

Substituting (3.2.8), (3.2.10) and (3.2.13) into (3.2.5) and collecting terms, we finally arrive at

$$f'' - \nabla^2 f - \left(\frac{\bar{\phi}'''}{\bar{\phi}'} + \frac{a''}{a} - 2\mathcal{H}' - \frac{\mathcal{H}''}{\mathcal{H}} + 2\frac{\mathcal{H}'^2}{\mathcal{H}^2} + 2\mathcal{H}\frac{\bar{\phi}''}{\bar{\phi}'} - 2\frac{\mathcal{H}'}{\mathcal{H}}\frac{\bar{\phi}''}{\bar{\phi}'}\right) f = 0. \tag{3.2.14}$$

When defining $z = a\bar{\phi}'/\mathcal{H}$ and applying a Fourier transformation, (3.2.14) can nicely be recast as

$$f_k'' + \left(k^2 - \frac{z''}{z}\right) f_k = 0, \tag{3.2.15}$$

which is the so-called *Mukhanov-Sasaki* equation. This equation contains the coupling between inflaton and metric perturbations, does not require any slow-roll approximation, and is valid on sub- and superhorizon scales.

3.2.3 Limiting Behaviour and Slow-Roll

Before we continue and quantise the Mukhanov-Sasaki equation, let us first take a look at its behaviour in certain limiting cases which will be of crucial use later on.

Equation (3.2.15) is of the form of a simple harmonic oscillator with time dependent frequency

$$w_k{}^2(\tau) = k^2 - \frac{z''}{z}. \tag{3.2.16}$$

Recalling the slow-roll conditions $\mathcal{H} \approx const$, $d\bar{\phi}/dt \approx const$ and $\epsilon \ll 1$, we immediately see

$$\frac{z''}{z} \approx \frac{a''}{a} = \mathcal{H}^2(2-\epsilon) \approx 2\mathcal{H}^2. \tag{3.2.17}$$

Additionally to the slow-roll approximation, let us now consider the subhorizon limit, i.e. $k \gg \mathcal{H}$ or, equivalently during slow-roll, $k^2 \gg a''/a$. We find that (3.2.15) reduces to

$$f_k'' + k^2 f_k = 0, \tag{3.2.18}$$

which is the equation of a scalar field in Minkowski space, as one could have expected intuitively. When approaching the subhorizon limit during inflation, i.e. $k \gg \mathcal{H}$, spacetime looks similar to Minkowski space, thus one expects the fluctuations to show exactly this behaviour. The short distance physics should not be affected by the expansion of the universe.

3.2.4 Quantisation of the Mukhanov-Sasaki Equation

Recall that the derivation of the Mukhanov-Sasaki equation has been done entirely within the classical regime, i.e. the field variables have not been promoted to field operators. We have found an equation of motion for the perturbation of a scalar field within the framework of General Relativity, i.e. a classical theory. Let us now proceed and quantise equation (3.2.15).
Recall that the equation of motion obtained is for the rescaled variable $f = a\delta\phi$. We will now change our notation and treat the inflaton perturbation $\delta\phi$ as a quantised scalar field, i.e. $\delta\phi \to \hat{\phi}$. So in order to quantise $f \to \hat{f} = a\hat{\phi}$ we may simply quote the results from section 2.3.

We recall that the quantised field variable $\hat{f}$ may be expanded in terms of its Fourier modes (2.3.20)

$$\hat{f}(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(f_k(\tau)\hat{a}_{\mathbf{k}}(\tau_0)e^{i\mathbf{k}\cdot\mathbf{x}} + f_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger(\tau_0)e^{-i\mathbf{k}\cdot\mathbf{x}} \right),$$

where the mode functions $f_k(\tau)$ now are solutions to the equation of motion (3.2.15). Furthermore, we may recall relations (2.3.13) and (2.3.14)

$$f_k(\tau) = \frac{1}{\sqrt{2k}} u_k(\tau) + v_k^*(\tau)$$

$$g_k(\tau) = \sqrt{\frac{k}{2}} \left(u_k(\tau) - v_k^*(\tau) \right),$$

where from equation (2.2.4) we can infer

$$g_k = f_k' - \frac{a'}{a} f_k. \tag{3.2.19}$$

Now all we are left to do is to define a reasonable vacuum for our theory. We recall from section 2.3 that in order to impose a vacuum at some time τ_0 we need to have $v_k(\tau_0) = 0$. As we now have the equation of motion for the variable f at hands, we can obtain the functions $u_k(\tau)$ and $v_k(\tau)$ and motivate a choice of vacuum for our theory.

3.2.5 The Choice of Vacuum and Evolution

We have seen in section 2.3 that what may be the vacuum at some time τ_0 will not be the vacuum at some later time τ_1. However, we seek to define a reasonable vacuum state for the inflationary phase. In section 3.2.3 we already concluded that the physics of small scales should not be affected by the overall expansion of space and should resemble the behaviour of Minkowski space. So let us impose a vacuum by requiring that our choice of vacuum coincides with the Minkowski vacuum at small scales, i.e. large k and early times.
In 3.2.3, we have already considered the subhorizon limit and found that

$$w_k^2 = k^2 - \frac{z''}{z} \to k^2,$$

as the k^2 term dominates. So we have obtained the equation of motion for a simple harmonic oscillator and thus recall equation (3.2.18)

$$f_k'' + k^2 f_k = 0.$$

This has the familiar solutions $f_k \propto e^{\pm ik\tau}$. Only $f_k \propto e^{-ik\tau}$ resembles a state of minimal excitation, i.e. only the positive frequency solution $f_k \propto e^{-ik\tau}$ minimises the Hamiltonian of the system and thus may be regarded as the vacuum state. Therefore let us consider this to be our choice of vacuum during the inflationary phase while being at subhorizon scales. Formally, we specify

$$\lim_{\tau\to-\infty} f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \tag{3.2.20}$$

where the factor $1/\sqrt{2k}$ ensures that the normalisation condition (2.3.5) is satisfied. Note that this may be thought of as an initial condition on the evolution of perturbations whose equation of motion is (3.2.15).
Now, consider slow-roll inflation and take the spacetime to be resembled by de Sitter space. We have[2]

$$\frac{z''}{z} \approx \frac{a''}{a} = \frac{2}{\tau^2}. \tag{3.2.21}$$

Thus (3.2.15) in de Sitter space is found to be

$$f_k'' + \left(k^2 - \frac{2}{\tau^2}\right) f_k = 0. \tag{3.2.22}$$

The above has the general solution

$$f_k(\tau) = A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right), \tag{3.2.23}$$

[2]Recall that the scale factor is given by $a = -\frac{1}{H\tau}$. The equality then readily follows.

where in order to satisfy the normalisation condition (2.3.5) we have $|A_k|^2 - |B_k|^2 = 1$. Our definition of the vacuum may be used, as suggested, as an initial condition and fixes $B_k = 0,\ A_k = 1$ and thus lets the second term vanish. We have now found the unique mode function

$$f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{3.2.24}$$

that determines the future evolution of the perturbations.

Due to the requirement that our mode functions reduce to the Minkowski case at small scales we have obtained a solution to equation (3.2.15) that we will use as our mode function for the rescaled inflaton perturbations. We have however not yet fully argued for a vacuum state as in $\hat{a}_{\mathbf{k}}(\tau_0)|0, \tau_0\rangle = 0$. Let us therefore now seek to obtain the functions $u_k(\tau)$ and $v_k(\tau)$ of relations (2.3.13) and (2.3.14) in order to specify $v_k(\tau_0) = 0$ and thus explicitly set τ_0. From equation (3.2.24) and (3.2.19) we quickly find

$$u_k = \frac{1}{2}e^{-ik\tau}\left(2 - \frac{i}{k\tau}\right) \tag{3.2.25}$$

$$v_k = \frac{1}{2}e^{ik\tau}\frac{i}{k\tau}. \tag{3.2.26}$$

So in order to satisfy (2.3.16)

$$\begin{aligned} &\hat{a}_k(\tau_0)|0,\tau_0\rangle = \\ &u_k(\tau_0)\hat{a}_{\mathbf{k}}(\tau_0) + v_k(\tau_0)\underbrace{\hat{a}^\dagger_{-\mathbf{k}}(\tau_0)|0,\tau_0\rangle}_{\neq 0} = 0, \end{aligned} \tag{3.2.27}$$

we hence require $\tau_0 \to -\infty$ from which we immediately see that $v_k(\tau_0) = 0$. So for the fluctuations of the inflaton, we impose a Minkowski vacuum condition at the infinite past.

Let us now revisit the important physical points of our argument. The equation of motion obtained for the rescaled inflaton perturbations admits general solutions (3.2.23). Furthermore, we promote the rescaled field variable f to field operator status and thus seek to determine an initial vacuum. As the vacuum at one time will not be the vacuum at later times, we require a limiting case. First, we go to sufficiently small scales and motivate that spacetime looks Minkowskian as short scale physics should not be affected by the expansion of space. This already limits the general solution to the equation of motion. Second, we seek to satisfy $\hat{a}_k(\tau_0)|0, \tau_0\rangle = 0$ which we then only find to be satisfied for $\tau_0 \to -\infty$. Thus our condition that we want spacetime to look Minkowskian at small scales immediately yields a vacuum in the remote past. This is referred to as *Bunch-Davies vacuum* in the literature [5,11]. It might already be interesting to mention that it is not possible to impose the Bunch-Davies vacuum at a finite time in the past as obviously $v_k(\tau_0) = 0$ is only satisfied for $\tau_0 \to -\infty$. The case will however look different when not requiring the Minkowski behaviour for small scales. This will further be investigated in chapter 6.

3.2.6 Power Spectrum

The above considerations entail elegant physics however we have not yet identified an actual observable quantity. Let us recall what the physical implications of the above analysis are.

We have successfully split up the inflaton field into a homogeneous but time dependent background and a space and time dependent perturbed part. We then quantised the inflaton field by quantising the perturbations and thus recognising the inflaton perturbations as quantum fluctuations. Yet unlike the quantum fluctuations of the electromagnetic field that are visible through the Casimir effect [27], the fluctuations of the inflaton field are not directly measurable for us today.
First of all we don't know if the proposed inflaton field still exists today with an insignificant vacuum expectation value (or *vev* for short) or if it has fully decayed during the end of inflation and the phase of pre- or reheating. Secondly, we are still lacking a convincing particle physics model for the inflaton hence any sort of direct measurement seems to be a priori impossible. But luckily this is not the end of the story.
As mentioned before in the introduction to section 3.2 and also in [5–12], quantum fluctuations of the inflaton get amplified during inflation and lead to density perturbations that are considered the seeds of structure formation. Moreover, as caused by the inflaton fluctuations, the density perturbations share characteristics such as e.g. the power spectrum with the inflationary fluctuations. Furthermore, the density perturbations leave their imprint on the cosmic microwave background radiation. Hence measuring the CMB may be considered the experiment to test the predictions of the previous subsections.

Let us now introduce our actual physical observable, namely the power spectrum of the fluctuations. The power spectrum of the field $\hat{\phi} = a^{-1}\hat{f}$ may be calculated as

$$\begin{aligned}
\langle 0|\hat{\phi}_k^\dagger \hat{\phi}_{k'}|0\rangle &= \delta(k-k')a^{-2}|f_{k'}|^2 \\
&= \delta(k-k')\left|\frac{e^{-ik'\tau}}{\sqrt{2k'}}\left(1-\frac{i}{k'\tau}\right)\right|^2 \\
&= \frac{1}{2k}\left(1+\frac{1}{(k\tau)^2}\right) \\
&= \frac{H^2}{2k^3}(1+k^2\tau^2) \\
&\to \frac{H^2}{2k^3},
\end{aligned}$$

or

$$P_\phi = \frac{k^3}{2\pi^2}\langle 0|\hat{\phi}_k^\dagger \hat{\phi}_{k'}|0\rangle = \left(\frac{H}{2\pi}\right)^2, \qquad (3.2.28)$$

where for the last step we have taken the limit of late times, i.e. $k\tau \to 0$ as we are interested in the spectrum of the modes at (event) horizon crossing, i.e. freeze-in. The striking result is that our quantum theory of cosmological perturbations has predicted a scale invariant spectrum (3.2.28), i.e. independent of k. This result is in nearly perfect agreement with observations [2].
However, the universe does not resemble perfect de Sitter space during inflation as inflation must have started and must end at some point, hence we know that we must have $H \to H(\tau)$. The time-dependence also induces a dependence[3] on

[3] We will further investigate and quantify this effect in section 6.3.

k and thus we actually expect small deviations from scale invariance [18] which is in agreement with observations [28]. The exact deviation of scale invariance depends on the inflationary model assumed.

Part II

Trans-Planckian Considerations

Chapter 4

Reconsidering Inflation

The first part of this work describes a story of success. We have considered shortcomings of the standard cosmological model and have seen how the introduction of a phase of accelerating expansion of space in the very early universe sets the initial conditions we need in order to explain what is observed today. And not only does inflation successfully yield the required initial conditions it furthermore predicts the near scale invariance of the fluctuation spectrum of the cosmic microwave background.
Yet the success comes with a price. Rather than worrying about initial conditions of standard big bang cosmology we now have to worry about the actual particle physics model of the inflaton. So in a way one might argue that our ignorance has simply been shifted or reparametrised. Nevertheless, the multiple problems of initial conditions and scale invariance of the CMB spectrum have at least been reduced to the mere problem of finding the inflaton field.
In this chapter we want to quickly summarise remaining conceptual issues regarding inflation and introduce the so called *trans-Planckian problem* of the inflationary paradigm.

4.1 Do we indeed need Inflation?

In the light of the previous arguments, this section's heading might seem confusing, but as stressed in the above discussions, the horizon as well as the flatness problem are not contradicting the predictions of standard Big Bang cosmology. They simply highlight where standard cosmology looses its predictive power. It is therefore reasonable to ask whether or not inflation is sufficiently motivated. Furthermore, in our argument of section 1.2.1, we chose the lower bound of integration to be basically at the initial singularity yet there is no reason why we should hold faith in classical geometry at those very early times. We implicitly assumed that including regions beyond the quantum gravity wall (i.e. when GR and QFT break down) does not lead to a behaviour making our argument unnecessary.
Consider the comoving particle horizon τ_p taken from the beginning of the uni-

verse up to the end of inflation t_{reh}

$$\tau_p = \tau_{QG} + \underbrace{\int_{\delta t}^{t_{reh}} \frac{dt}{a(t)}}_{inflation}, \tag{4.1.1}$$

where the first term on the r.h.s. is supposed to be a contribution stemming from a full theory of quantum gravity. As long as a full theory of quantum gravity is not discovered, we cannot exclude the possibility that the contribution to the particle horizon from quantum gravity is such that inflation is rendered unnecessary afterwards. In other words, quantum gravity might solve the initial conditions problem of standard cosmology without invoking a phase of accelerating expansion. Obviously though, this is simply a conceptual criticism as a theory of quantum gravity is not yet discovered. Also, the remarkable success of the inflationary paradigm sheds very convincing light on the theory.
Thus in principle, a full theory of quantum gravity might render inflation unnecessary but as of now this is mere speculation.

4.2 The Trans-Planckian Problem

Let us now consider another conceptual challenge to the inflationary paradigm, namely the trans-Planckian problem.
In the literature, see e.g. [29–31], the trans-Planckian problem is often referred to as the exponential expansion of the universe during inflation increasing the size of scales that are smaller than the Planck length to sizes that are of cosmological interest today. It is often quoted that certain inflationary models provide a number of e-folds large enough so that e.g. the diameter of the entire visible universe of today is of the order of the Planck length before inflation and that hence one has to consider trans-Planckian effects when studying inflationary theory.
This phrasing however is misleading as it suggests that one does not have to consider any trans-Planckian effects for models with fewer e-folds, or in other words, that the issue of trans-Planckian effects is simply e-fold dependent. Furthermore, we will actually see that all subsequent considerations in this work neither rely on a certain inflationary model nor require a certain number of e-folds. Thus whereas the phrasing might pretend to yield an easy to understand explanation of the trans-Planckian problem, it is by no means an accurate description.

Let us now consider the true essence of the trans-Planckian problem of the inflationary paradigm. When deriving the equation governing the fluctuations of the inflaton field (3.2.15) and hence calculating an actual observable, i.e. the power spectrum, we implicitly assumed the approach taken to be valid up to arbitrarily high energies. There is no inherent cut-off in the range of the wavenumber k and thus the high energy behaviour of the theory is assumed to be of the same form as the low energy one.

We know that the physical description we have at hands looses its validity at energies approaching the Planck scale, i.e. when gravity is expected to become as strong as the other forces and some unified theory needs to be invoked. A theory of quantum gravity is not yet found and as the Planck scale

is of $\mathcal{O}(10^{19})$ GeV, probing this energy scale lies well outside the possibilities of current collider-based experiments.
Considering these limits to our current physical understanding it seems important to investigate whether or not the inflationary paradigm is sensitive to changes of the high energy behaviour of the theory, as we - for the reasons outlined above - cannot expect the theory to behave at high energies as it does at low energies.
Fortunately, this investigation does not remain a mere theoretical undertaking. As we obtain an observable physical quantity from inflationary theory, we may try to find out whether or not unknown high energy behaviour can influence that very quantity, while at the same time maintaining consistency within the paradigm of course[1].
Thus the meaning of the term *trans-Planckian problem* of inflationary cosmology can simply be understood as investigating whether or not the inflationary paradigm is sensitive to high energy changes of the theory.

In the following chapters we will consider different approaches to this investigation.
First, we will introduce ad-hoc changes to the theory in order to mimic any high energy behaviour. As this eventually can be understood as a proof of concept rather than an approach to yield significant quantitative results we will then consider to introduce a cut-off scale at which all the trans-Planckian, i.e. high energy physics, is parametrised in terms of initial conditions of our theory. We will then investigate whether or not some initial conditions spoil the perturbative approach and if they are consistent with understood low energy physics in order to formulate an answer as to whether or not inflation is sensitive to changes of high energy physics.

[1] When considering any influence of high energy changes to the theory on the power spectrum, one of course has to ensure that the high energy changes do not spoil inflationary theory. In technical terms, any change of the behaviour must not lead to back-reaction such that the perturbative approach underlying all of this work is destroyed. We will consider this in most detail in chapter 7.

Chapter 5

From Black Holes to Inflation

In this chapter we review the first attempt to make quantitative statements about the effects of trans-Planckian, i.e. unknown physics on physics which is well understood and believed to hold up to some high physical scale which we will denote by $\mathcal{K}_c$ in this chapter.

5.1 The Trans-Planckian Problem in Black Hole Physics

The trans-Planckian problem was first formulated in the context of Hawking radiation [32–34] and recently again in [35].
The standard derivation of black hole evaporation, i.e. Hawking radiation, involves radiation modes of arbitrarily high frequencies. Yet obviously, at energy scales of the order of the Planck mass, one looses faith in known physics such as QFT or GR. So having a derivation that includes modes with frequencies that exceed the range of applicability of known physics is a conceptual problem.

Thus the question is raised whether or not the physics of Hawking radiation is sensitive to UV corrections.
To obtain an answer, modified dispersion relations for radiation modes at high frequencies are introduced. It is found that the thermal spectrum is not influenced by any change of the high frequency behaviour of the theory. Despite having found the robustness of the derivation of Hawking radiation via numerical methods, it is assumed that the insensitivity to high frequency corrections follows from the behaviour of the modes being governed by longer timescales near the black hole horizon. Thus the modes may always adjust adiabatically to any change of dispersion relation and hence the standard predictions are not altered. It is therefore concluded that Hawking radiation is robust to any trans-Planckian physics.
Due to these considerations it is natural to first address the trans-Planckian problem of inflationary cosmology with the same approach, the details of which will be subject of the following sections.

5.2 The WKB Approximation and Non-Adiabatic Transitions

This section is crucial to this chapter as it presents in detail the mechanism by which the departure from adiabatic behaviour may lead to the mixing of positive and negative frequency solutions of an harmonic oscillator type equation such as e.g. the Mukhanov-Sasaki equation (3.2.15). It follows the treatment of [19, 36] and establishes our notation.

Consider a harmonic oscillator where the frequency is time dependent

$$f'' + \omega^2(\tau) f = 0. \tag{5.2.1}$$

In case of equation (3.2.15), $\omega(\tau) = k^2 - z''/z$ or $\omega(\tau) \to k^2$ for early times with z being dependent on the background cosmology. Now consider the WKB solution

$$f^{WKB}(\tau) = \frac{1}{\sqrt{2\omega(\tau)}} e^{-i\int_{-\infty}^{\tau} \omega' d\tau'}. \tag{5.2.2}$$

The above is an approximate solution to equation (5.2.1). The exact solution may be expanded in terms of f^{WKB} and $(f^{WKB})^*$

$$f(\tau) = A(\tau) f^{WKB}(\tau) + B(\tau) \left(f^{WKB}(\tau)\right)^*, \tag{5.2.3}$$

where A and B are time dependent in general. The WKB or adiabatic limit means that the change in frequency $\partial_t \omega$ is much smaller than the scale of the frequency ω itself, i.e. $\partial_t \omega / \omega^2 \to 0$ as $\partial_t \omega$ has units of ω^2.We seek constant coefficients A and B for the WKB limit and thus impose

$$i\partial_\tau f(\tau) = \omega(\tau) \left[A(\tau) f^{WKB} - B(\tau) \left(f^{WKB}\right)^*\right] \tag{5.2.4}$$

and the Wronskian $i(f^* \partial_\tau f - f \partial_\tau f^*) = |A(\tau)|^2 - |B(\tau)|^2 = const$. Inspecting equation (5.2.4) and equation (5.3.2) yields $A(\tau) \to A, B(\tau) \to B$. We now want to obtain an expression giving the time evolution of the coefficients A and B when departing the WKB regime. Therefore, we differentiate equation (5.3.2) and equate the result with equation (5.2.4). For the WKB regime, the two expressions of course equal each other yet for a non-adiabatic regime, i.e. $\partial_t A, \partial_t B, \partial_t \omega / \omega^2 \neq 0$, we obtain

$$f^{WKB} \partial_\tau A + \left(f^{WKB}\right)^* \partial_\tau B - \frac{1}{\sqrt{2}} \frac{\partial_\tau \omega}{\omega} \left[A f^{WKB} + B \left(f^{WKB}\right)^*\right] = 0. \tag{5.2.5}$$

Then, differentiating equation (5.2.4) again and inserting the result into equation (5.2.1), we finally arrive at

$$\partial_\tau B = \frac{1}{\sqrt{2}} \frac{\partial_\tau \omega}{\omega} e^{2i\int_{-\infty}^{\tau} \omega' d\tau'} A, \tag{5.2.6}$$

$$\partial_\tau A = \frac{1}{\sqrt{2}} \frac{\partial_\tau \omega}{\omega} e^{-2i\int_{-\infty}^{\tau} \omega' d\tau'} B. \tag{5.2.7}$$

Considering a case where $A = 1, B = 0$ at early times, we find that for later times

$$B(\tau) \approx \int_{-\infty}^{\tau} d\tau \frac{1}{\sqrt{2}} \frac{\partial_\tau \omega}{\omega} e^{2i\int_{-\infty}^{\tau} \omega' d\tau'}. \tag{5.2.8}$$

What does the obtained result tell us? Leaving the adiabatic regime leads to a non-zero B in equation (5.3.2), given the initial values were $A = 1, B = 0$. We can thus identify the departure of adiabaticity as the mechanism that leads to a mixing of positive and negative frequency solutions and thus non-adiabatic transitions or particle creation in our cosmological context.

It may furthermore be shown that the norm of expression 5.2.8 is given as [36]

$$|B|^2 \propto e^{-4\Im \int_{-\infty}^{\tau} \omega' d\tau'}. \tag{5.2.9}$$

Hence we conclude that the norm $|B^2|$ is non-zero when the frequency has a complex part. Following [19], let us at last introduce the parameter Q for which

$$\omega(\tau) = \Omega(\tau) - \underbrace{\left(\frac{3}{4}\frac{\dot{\omega}^2}{\omega^2} + \frac{3}{2}\frac{\ddot{\omega}}{\omega}\right)}_{=Q}, \tag{5.2.10}$$

where $\Omega^2 = m^2 + a^{-2}k^2$ is the energy of a particle at momentum k. So in the adiabatic regime, the parameter Q has to fulfil $|Q/\omega^2| \ll 1$.

We have shown that departure of adiabaticity and a complex frequency, i.e. dispersion relation, lead to a non-zero negative energy contribution, i.e. particle creation and thus a deviation from a given standard vacuum.

5.3 Modified Dispersion Relations

As indicated in the first section of this chapter, we will now turn to modified dispersion relations in order to encapsulate possible effects of trans-Planckian physics on the known regime. Certainly, this approach is partly guided by guesswork. As trans-Planckian physics is unknown by definition, we seek to model something we don't know by introducing changes to the established formalism and seek our modifications to reduce to the standard treatment at lower energies. In the following, the dispersion relation of the equation of motion (3.2.15) of the inflaton perturbations will be modified to mimic the effect of unknown physics. Before considering specific changes let us consider the general implications of this approach [37]. This may conveniently be done with the results of the previous section.

Generally, approaches of this kind [37–39] share common features. First, recall

$$f_k'' + \left(k^2 - \frac{z''}{z}\right) f_k = 0.$$

Now, the k^2-term in the above expression is replaced with some function $aF(k,\tau)$, where a is the scale factor, i.e. $k \to aF(k,\tau)$. The function $F(k,\tau)$ encodes all the trans-Planckian physics.

As pointed out in the first paragraph of this section, we require our modification to approach the standard behaviour for lower energies. Therefore, we need to introduce a *physical* scale below which the behaviour of the function F reproduces the known result. Hence, we require $aF(k,\tau) \to k$ for $F \ll \mathcal{K}_c$, where $\mathcal{K}_c$ denotes the physical cutoff scale beyond which we believe trans-Planckian effects to be of importance. One may specify $\mathcal{K}_c$ to be the GUT or the Planck scale or simply some high energy scale Λ. This is not important as of now.

However, implementing a physical cut-off breaks the Lorentz invariance of our treatment as a physical scale is singled out. Thus, trying to describe unknown physics in this way immediately introduces a departure from known principles. As we want to lay out the general framework, the behaviour of F beyond $\mathcal{K}_c$ need not be specified. We can just recast the above as

$$f_k'' + \underbrace{\left(a^2F^2 - \frac{z''}{z}\right)}_{=\omega^2} f_k = 0. \tag{5.3.1}$$

This is just of the form of equation (5.2.1). Thus it is instructive to apply the formalism of the previous section. If we assume the inflaton fluctuations to be in their WKB vacuum state at early times, we can write the vacuum solution as

$$f_k^{WKB}(\tau) = \frac{1}{\sqrt{2\omega(\tau)}} e^{-i\int_{-\infty}^{\tau_f}\omega' d\tau'}.$$

The exact evolution is given as in the previous section and reads

$$f_k(\tau) = A(\tau) f^{WKB}(\tau) + B(\tau)\left(f^{WKB}(\tau)\right)^*.$$

We may recall from subsection 3.2.5 that for early times, the term z''/z in equation (5.3.1) is negligible. This means that the behaviour of the $aF(k,\tau)$-term alone determines whether or not the adiabatic regime is maintained. Thus when $aF(k,\tau)$ departs adiabaticity, the second term of equation (5.3.2) becomes non-zero according to equation (5.2.8)

$$B(\tau) \approx \int_{\tau_i}^{\tau_f} d\tau \frac{1}{2}\frac{\partial_\tau \omega}{\omega} e^{2i\int_{-\infty}^{\tau_f}\omega' d\tau'}.$$

Are there visible consequences of this? Recall the definition of the k-th component of the power spectrum of the inflaton fluctuations to be

$$\begin{aligned} P_k &\propto k^3 |f_k|^2 \\ &= \underbrace{|f_k^{(0)}|^2}_{B=0} \left(|A|^2 + |B|^2 + A^*B + B^*A\right), \end{aligned} \tag{5.3.2}$$

where the $|f_k^{(0)}|$-term is the unmodulated result for the standard vacuum and the term in square brackets approaches 1 for $B \to 0$ considering $|A|^2 - |B|^2 = 1$. For $B \to 0$ we hence recover the standard result.

To summarise, we have introduced a modified dispersion relation $k \to aF(k,\tau)$ where $z''/z \approx a''/a \ll aF(k,\tau)$. We have given the behaviour of the function F by requiring $aF(k,\tau) \approx k$ for $F \ll \mathcal{K}_c$ and have left the behaviour of F for $F \gg \mathcal{K}_c$ unspecified, where $\mathcal{K}_c$ is some high physical scale up to which we hold faith in known physics. Then the mere departure of adiabaticity of F in the regime $F \gg \mathcal{K}_c$ leads to a non-standard evolution of the inflaton fluctuations and hence a modified spectrum.

5.4 Scale Separation

So far we have concerned ourselves only with a general framework to describe deviations from the standard treatment of inflaton fluctuations. Let us now get a little bit more specific by introducing a method in order to be able to quickly examine whether or not a given dispersion relation will lead to a non-adiabatic transition. Recall section 5.2 and consider the adiabatic parameter ω'/ω^2. In the adiabatic regime, this parameter is close or equal to zero, meaning that the change of frequency is negligible compared to the frequency's scale. Now, consider from before the scenario in which the frequency is given by the dominating part of equation (5.3.1), i.e. $\omega \approx aF(k,\tau)$. We may therefore recast the adiabatic parameter as

$$\begin{aligned} \frac{\omega'}{\omega^2} &\approx \frac{aF' + Fa'}{(aF)^2} \\ &= \frac{1}{aF^2}\frac{dF}{dk_{phys}}\left(-a^2 k_{phys}\frac{da}{d\tau}\right) + \frac{H}{aF} \\ &= \frac{H}{aF} - \frac{Hk_{phys}}{a^2F^2}\frac{dF}{dk_{phys}}. \end{aligned} \tag{5.4.1}$$

Now let us impose that the scales of the theory are well separated, i.e. $H \ll k_{phys} \ll \mathcal{K}_c$. This is a perfectly motivated assumption as observational evidence suggests [2] that for most common models $H/M_{pl} \approx \mathcal{O}\left(10^{-5}\right)$, where M_{pl} is the Planck mass and thus a good candidate for some high scale $\mathcal{K}_c$ beyond one has to invoke new physics.

Let us now consider two generic classes of dispersion relations and use them to evaluate the above expression. First, take a dispersion relation implementing a cut-off by $F \to \mathcal{K}_c$ for $k \to \infty$. Immediately, the second term of (5.4.1) vanishes as the derivative becomes zero and, due to scale separation, the first term is negligible. Similarly, for a dispersion relation of the form $F \propto k^n$, the second term behaves as $n/k^{n+1} \to 0$ where the first term is again negligible due to scale separation. Thus adiabaticity for these two classes of dispersion relations may be obtained by having the scales well separated and remains maintained throughout the evolution up to horizon exit and thus freeze-in.

In conclusion, we may now add scale separation to the features responsible for adiabatic evolution. Note that this statement is not true in general but applies only to certain models.

5.5 Examples

Now consider two concrete examples [38]. Introducing a power-law inflationary model

$$a(\tau) = l_0|\tau|^{1+\beta}, \tag{5.5.1}$$

with $\beta \leq -2$ (so $\beta = -2$ corresponds to de Sitter space), the mode evolution may be subdivided into a sub-Planck regime $\tau_i < \tau < \tau_c$ and a regime of standard evolution $\tau_c < \tau < \tau_H$. The times when the mode crosses the cut-off scale l_c and the Hubble horizon H^{-1} are[1]

$$\tau_c = \left(\frac{kl_c}{2\pi l_0}\right)^{\frac{1}{1+\beta}} \tag{5.5.2}$$

$$\tau_H = \frac{2\pi}{k}(1+\beta), \tag{5.5.3}$$

and

$$a^{-2} \propto k^{2+2\beta}. \tag{5.5.4}$$

Now consider the dispersion relation

$$F(k) = k_c \tanh^{1/p}\left[\left(\frac{k}{k_c}\right)^p\right], \tag{5.5.5}$$

for which it can be shown that the dominating contribution of the mode function is $f \propto \tau^{1/2}$. Therefore one finds $f(\tau_H) \propto k^{-1/2}$. A final calculation for the de Sitter case then yields

$$P_\phi \propto k^3 k^{-1} k^{-2} = k^0. \tag{5.5.6}$$

For a different dispersion relation

$$F^2(k) = k^2 + k^2 b_m \left(\frac{k}{k_c}\right)^{2m}, \tag{5.5.7}$$

where $b_m < 0$ and $m \in \mathbb{Z}$, a similar calculation yields

$$P_\phi \propto k^{2\beta+4}. \tag{5.5.8}$$

So for the de Sitter case $\beta = -2$, the exponent is zero and hence a modulation of the spectrum does not occur whereas the non-de Sitter case yields a non-scale invariant spectrum.

5.6 Conclusions

With the motivation coming from similar considerations in black hole physics it is investigated whether or not changing the behaviour of the dispersion relation beyond some cut-off scale has a significant effect. As seen in section 5.2, non-adiabatic behaviour may lead to transitions and thus a mixture of positive and negative frequency solutions. Thus one may infer that the inflationary paradigm is not insensitive to high energy modifications opposite to what was found in the black hole case. However, this approach should be regarded as a proof of concept rather than an attempt to make quantitative statements about trans-Planckian physics. The form of trans-Planckian modifications has to be guessed as it is hard to motivate some specific behaviour of the dispersion relation beyond $\mathcal{K}_c$. Therefore, this approach demonstrates that the violation of the adiabatic regime can in principle have an effect, yet this is already everything one can learn from this ansatz.

[1] The first relation may be readily obtained when equating the physical wavelength $2\pi a/k$ with the cut-off scale l_c. Substituting the scale factor and solving for τ_c yields (5.5.2). The second relation may be obtained when substituting the scale factor into the definition of the Hubble horizon $H^{-1} = a/\mathcal{H} = a^2/a'$. Equating the physical wavelength with the Hubble horizon and solving for τ gives the second relation.

Chapter 6

Paraphrasing the Problem

Before, we have tried to mimic the effects of unknown trans-Planckian physics by introducing specific changes to our physical description, i.e. modifying the dispersion relation of equation (3.2.15) to yield equation (5.3.1). One may however rightfully question that approach as it requires certain assumptions about the behaviour of unknown, trans-Planckian physics. As we have seen when laying out the general framework in section 5.2, the crucial effect of any non-adiabatic evolution is to generate a non-zero B_k, i.e. a mixing between positive and negative frequency solutions and hence particle creation.

Now, rather than describing a specific non-adiabatic behaviour above some high scale $\mathcal{K}_c$, let us disregard the physics above that scale completely and consider its effects on the known regime merely through the initial conditions imposed on the inflaton fluctuations, i.e. requiring the trans-Planckian physics to set the modes of the inflaton fluctuations to a general inital state with non-zero B_k at a fixed physical scale, which we shall call Λ from now onwards. Therefore, the modes start to evolve at that physical scale and finite time in the past and we assume no mode evolution beforehand.

We have thus parametrised the unknown physics simply by starting the inflaton fluctuations in a general initial state at some finite time in the past. In the following sections we will concern ourselves with the details of the outlined approach.

We will furthermore carefully investigate possible observational features and will finally consider whether or not an introduced cut-off restricts the choice of allowed vacua.

6.1 Introducing a Physical Cut-Off

This and the following section puts to use the formalism developed in section 2.3 which was based on [20]. Recall from section 3.2.5 that the general solution to equation (3.2.15) in de Sitter space is equation (3.2.23)

$$f_k(\tau) = A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right).$$

In the conventional treatment, we have required the above to coincide with the Minkowski vacuum in the infinite past, therefore we have used the Bunch-Davies

vacuum as the initial condition for the above equation which hence set $A_k = 1$ and $B_k = 0$ and thus had us find the future mode evolution. Now the whole point about unknown trans-Planckian physics is that the modes might not be placed in the Bunch-Davies vacuum state initially.
In the introductory paragraph we stressed that the physical scale at which we start the mode evolution corresponds to a finite time in the past. Recall that the comoving momentum is related to the physical momentum via $k = ap$, where a is the scale factor. Our physical momentum cut-off is Λ, hence we assume a fluctuation mode to start with comoving momentum $k = a_i\Lambda$, where a_i is the scale factor at the time when the mode starts its evolution. In the de Sitter case we recall that $a = -(H\tau)^{-1}$. We identify τ_i as the initial time of mode evolution and may hence write

$$k = -\frac{\Lambda}{H\tau_i}.$$

Thus we find the k-dependent initial time at which we start to evolve the modes in a general, i.e. $B_k \neq 0$ state to be

$$\tau_i = -\frac{\Lambda}{Hk}. \tag{6.1.1}$$

To summarise, as we have introduced a physical cut-off scale Λ beyond which unknown physics is believed to be necessary and at which the modes are placed in their most general initial state, we have lost the notion of the infinite past but start the mode evolution at a fixed, k-dependent initial time. The implications of this are subject of the next section.

6.2 Imposing Initial Conditions

Let us now study in detail the proceedings when having trans-Planckian physics put the inflaton fluctuation in a general initial state at a finite time in the past. We recall equation (3.2.23)

$$f_k(\tau) = A_k \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + B_k \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right),$$

and find it's conjugate momentum (2.2.4) to be

$$g_k(\tau) = A_k\sqrt{\frac{k}{2}}e^{-ik\tau} - B_k\sqrt{\frac{k}{2}}e^{ik\tau}. \tag{6.2.1}$$

Now recalling equations (2.3.13) and (2.3.14) we may compare them with the above initial states and hence find

$$u_k = \frac{1}{2}\left(A_k e^{-ik\tau}\left(2 - \frac{i}{k\tau}\right) + B_k e^{ik\tau}\frac{i}{k\tau}\right), \tag{6.2.2}$$

$$v_k^* = \frac{1}{2}\left(B_k e^{ik\tau}\left(2 + \frac{i}{k\tau}\right) - A_k e^{-ik\tau}\frac{i}{k\tau}\right). \tag{6.2.3}$$

We then seek to fix a vacuum at the time (6.1.1) with the initial state from above as

$$a_k(\tau_i)|0, \tau_i\rangle = 0. \tag{6.2.4}$$

Substituting equation (2.3.8) into the above expression yields

$$\begin{aligned}&\left(u_k(\tau_i)\hat{a}_k(\tau_i)+v_k(\tau_i)\hat{a}^\dagger_{-k}(\tau_i)\right)|0,\tau_i\rangle\\&=\underbrace{u_k(\tau_i)\hat{a}_k(\tau_i)|0,\tau_i\rangle}_{=0}+\underbrace{v_k(\tau_i)\hat{a}^\dagger_{-k}(\tau_i)|0,\tau_i\rangle}_{\neq 0},\end{aligned}$$

which by condition (6.2.4) sets

$$v_k(\tau_i)=0, \tag{6.2.5}$$

where hence the same applies to the complex conjugate. This result applied to equation (6.2.3) presents a relation between the two coefficients A_k and B_k additionally to the constraint $|A_k|^2-|B_k|^2=1$. We find

$$B_k=\frac{ie^{-2ik\tau_i}}{2k\tau_i+i}A_k. \tag{6.2.6}$$

Let us now compute the power spectrum (3.2.28)

$$\begin{aligned}P_\phi&=\frac{k^3}{2\pi^2a^2}\langle 0|f_k^\dagger f_{k'}|0\rangle\\&=\frac{k^3}{2\pi^2a^2}\left|A_k\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)+B_k\frac{e^{ik\tau}}{\sqrt{2k}}\left(1+\frac{i}{k\tau}\right)\right|^2\\&=\frac{k^3}{2\pi^2a^2}\left(1+\frac{1}{(k\tau)^2}\right)\frac{1}{2k}\left(|A_k|^2+|B_k|^2+A_k^*B_k+A_kB_k^*\right)\\&\to\underbrace{\left(\frac{H}{2\pi}\right)^2}_{B_k=0}\left(|A_k|^2+|B_k|^2+A_k^*B_k+A_kB_k^*\right),\end{aligned} \tag{6.2.7}$$

where the first term resembles the spectrum for the $B_k=0$ case and we have again considered the late time limit. Note that the above expression is similar to (5.3.2), we may hence infer an equivalence between the two approaches in terms of modifying the power spectrum.
From constraint (6.2.6) and $|A_k|^2-|B_k|^2=1$, we deduce

$$|A_k|^2=\left(1-\left|\frac{i}{2k\tau_i+i}\right|^2\right)^{-1}. \tag{6.2.8}$$

Furthermore, we may find[1]

$$\begin{aligned}|B_k|^2&=\frac{1}{(2k\tau_i)^2}\\&=\frac{H^2}{4\Lambda^2},\end{aligned} \tag{6.2.9}$$

[1] In chapter 7 we will examine whether or not the value of B_k is is bounded by further considerations. We will see that to be the case and hence find an indication as to the order of magnitude of the scale of new physics.

where the last line results from expression (6.1.1). We can now calculate the actual correction to the power spectrum of the Bunch-Davies vacuum as[2]

$$\begin{aligned} P_\phi &= \left(\frac{H}{2\pi}\right)^2 \left[1 + |B_k|^2 \left(2 + \frac{2k\tau_i + i}{i} e^{2ik\tau_i} + \frac{2k\tau_i - i}{-i} e^{-2ik\tau_i}\right)\right] \\ &= \left(\frac{H}{2\pi}\right)^2 \left[1 + \frac{H}{\Lambda} \sin\left(2\frac{\Lambda}{H}\right) + \left(\frac{H}{\Lambda}\right)^2 \cos^2\left(\frac{\Lambda}{H}\right)\right], \end{aligned} \tag{6.2.10}$$

where in contrast to [20] we have obtained an expression up to second order. However, the main result is that the leading order correction to the power spectrum is

$$P_\phi = \left(\frac{H}{2\pi}\right)^2 \left[1 + \mathcal{O}\left(\frac{H}{\Lambda}\right)\right]. \tag{6.2.11}$$

So when introducing a physical scale Λ at which the mode evolution starts, i.e. a time-like hypersurface, and when placing the modes in a general state initially, we obtain a correction to the power spectrum linear in H/Λ.
Note that this analysis has been conducted in de Sitter space. A similar treatment of non-de Sitter power law inflation yields the same result [40].

6.3 Observable Consequences

In the previous section we found that trans-Planckian physics may place the inflaton fluctuation in a general vacuum state which then leads to a modulation of the power spectrum linear in the parameter H/Λ. Based on [41] we will now investigate what this further implies for observations.

Recall the Hubble slow-roll parameter

$$\epsilon = -\frac{\dot{H}}{H^2}.$$

In our standard treatment so far we have always considered near de Sitter space in which the Hubble parameter is constant. A non-varying Hubble parameter would leave us with the obtained result which would simply be an overall modulation of the amplitude of the power spectrum and hence a signature difficult to single out.
Yet as inflation ends at some point, this approximation breaks down and we can assume a time dependence $H \rightarrow H(t)$ which we may translate into a dependence on k. From the above, after having applied the chain rule twice[3], we then find

$$\frac{dH}{dk} = -\frac{\dot{a}}{a^2}\epsilon\frac{da}{dk},$$

which, when considering H at horizon crossing, i.e. $k = aH$, yields

$$\frac{dH}{dk} = -\epsilon\frac{H}{k}. \tag{6.3.1}$$

[2]We make use of the fact that for odd functions we have $f(x) = -f(-x)$ and furthermore consider the trigonometric identity $\cos(2x) = 2\cos^2(x) - 1$.

[3]First, we write $dH/dt = dH/dk \cdot dk/dt$. Then we make use of the fact that $da/dt \cdot dt/dk = da/dk$. Later we will use that from $k = aH$ we may infer the relation $da/dk = H^{-1}$.

The solution for the above differential equation then is

$$H \propto k^{-\epsilon}. \tag{6.3.2}$$

Considering the result of the previous section (6.2.11), we see that the argument of the first order correction is $2\Lambda/H$ or $2\Lambda k^{\epsilon}$ when making use of this section's results. We thus find that the k-dependence of H leads to a modulation of $P_{\phi}(k)$ with a periodicity given as

$$\frac{\Delta k}{k} \approx \frac{\pi H}{\epsilon \Lambda}. \tag{6.3.3}$$

Note that in perfect de Sitter, i.e. $\epsilon \to 0$, the above expression diverges which resembles the fact that without the k-dependence of H, we would not find any periodicity, i.e. the periodicity would approach infinity, and we would simply be left with the modulation of the amplitude from the previous section.
Thus considering a k-dependence translates into a more encouraging observational feature. We know from recent results of the Planck satellite [2] that $H/M_{pl} \leq 10^{-5}$. Taking $H = 10^{13}$ GeV, $\epsilon = 0.01$ [41] and considering the cut-off scale to be the Planck scale, we find

$$\frac{\Delta k}{k} = \Delta \ln k \approx \mathcal{O}(10^{-3}), \tag{6.3.4}$$

which means that there are $\mathcal{O}(10^{-3})$ oscillations per logarithmic interval which is a feature challenging to detect. If however the cut-off scale is taken to be below the Planck scale, e.g. the string scale $\Lambda \to M_{string} = \mathcal{O}(10^{16})$, one finds a periodicity of order unity which would be easier to single out.
As such a signal has not been detected yet [2] we may either conclude that the scale of new physics is indeed of the same order than the Planck scale or, obviously, that the considerations of this chapter do not relate to the physics of our universe.

6.4 Bunch-Davies Mode Functions in de Sitter Space

In the previous two sections we examined the effects of starting the inflaton fluctuations in a general state at a finite time in the past. In contrast to the standard treatment of inflaton fluctuations we introduced a cut-off and a non-Bunch-Davies vacuum initial state.

One may however argue that taking a non-Bunch-Davies state as the initial one is a questionable assumption, as we will discuss in section 7.3 and continue to evaluate in chapter 8. Yet one may certainly agree that we need a theory of quantum gravity in order to make statements about physics beyond the Planck scale. Thus whether or not one considers the concept of trans-Planckian physics causing a non-vacuum initial state to be plausible, it is certainly justified to start the mode evolution at a cut-off scale. We will now investigate whether or not the mere introduction of a cut-off will cause observable corrections to the power spectrum when taking the modes to be in their vacuum state. We will furthermore concern ourselves with the consistency of such an approach within the here applied formalism which will eventually shed interesting light on the choice of vacuum in de Sitter space.

Recall that the general solution to equation (3.2.15) is equation (3.2.23) which has been quoted above already. In section 3.2.5 we then found the evolution of the rescaled variable f_k by requiring our solution to coincide with the Minkowski vacuum in the infinite past, thus imposing an initial condition in which the positive energy solution $\propto e^{-ik\tau}$ is singled out.
When introducing a cut-off as in equation (6.1.1) we now want our solution to mimic the Minkowski case, i.e. $\propto e^{-ik\tau}$, but at a finite time $\tau_i = -H/(k\Lambda)$. We thus obtain a new initial condition (3.2.20)

$$\lim_{\tau \to \tau_i} f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \left(1 + i\frac{H}{\Lambda}\right), \tag{6.4.1}$$

from which we find the full mode evolution to be

$$f(\tau) = A_k \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right), \tag{6.4.2}$$

where

$$A_k = \left(1 + i\frac{H}{\Lambda}\right) \tag{6.4.3}$$

Let us now investigate the correction to the power spectrum (3.2.28)

$$\begin{aligned} P_\phi &= \frac{k^3}{2\pi^2 a^2} \langle 0 | f_k^\dagger f_{k'} | 0 \rangle \\ &= \frac{k^3}{2\pi^2 a^2} \left| \left(1 + i\frac{H}{\Lambda}\right) \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{kt}\right) \right|^2 \\ &= \frac{k^3}{2\pi^2 a^2} \left| \left(1 + \frac{H}{\Lambda k\tau}\right) + i\left(\frac{H}{\Lambda} - \frac{1}{k\tau}\right) \right|^2 \\ &\underbrace{\to}_{\tau \to -\infty} \left(\frac{H}{2\pi}\right)^2 \left[1 + \left(\frac{H}{\Lambda}\right)^2\right]. \end{aligned} \tag{6.4.4}$$

Unfortunately, the above argument has several flaws which invalidate its conclusion. It is intriguing that equation (6.4.2) is indeed a valid solution to the Mukhanov-Sasaki equation (3.2.15). However, (6.4.2) does not satisfy the normalisation condition $|A_k|^2 - |B_k|^2 = 1$. In (6.4.2), we have $A_k = 1 + iH/\Lambda$ for which $|A_k|^2 > 1$ and which then implies that we need a $B_k \neq 0$ in order for the normalisation condition to be satisfied. So despite having found a valid solution to (3.2.15), it cannot be used as a mode function for inflationary fluctuations as the normalisation condition is violated.
Furthermore, recall from section 2.3 that in order to impose a vacuum, one has to find mode functions such that it is possible to have $v_k(\tau_0) = 0$ for τ_0 being the time at which the vacuum is fixed. Now recall that for the Bunch-Davies vacuum state, we had (3.2.26)

$$v_k = \frac{1}{2} e^{ik\tau} \frac{i}{k\tau}.$$

So in order to satisfy (2.3.16)

$$\hat{a}_k(\tau_0)|0,\tau_0\rangle = \\ u_k(\tau_0)\hat{a}_{\mathbf{k}}(\tau_0) + v_k(\tau_0)\underbrace{\hat{a}^\dagger_{-\mathbf{k}}(\tau_0)|0,\tau_0\rangle}_{\neq 0} = 0,$$

we hence required $\tau_0 \to -\infty$. From the above we immediately see that we cannot impose the Bunch-Davies vacuum at any other time than the infinite past [42]. As in the above consideration we seek to impose the Bunch-Davies state at a finite time in the past, we will not be able to do so.

Thus as soon as we want impose the initial state of the inflaton fluctuation at a finite time in the past, we have to consider a general vacuum state with $A_k, B_k \neq 0$. We therefore have to consider the functions $u_k(\tau), v_k(\tau)$ from section 6.2 for which we then find relation (6.2.6)

$$B_k = \frac{ie^{-2ik\tau_i}}{2k\tau_i + i} A_k,$$

and which then lead to corrections of linear order in H/Λ.

6.5 Conclusions

In this chapter we parametrised any trans-Planckian effects with the initial conditions of the inflaton fluctuation's evolution. We introduced a physical cut-off scale Λ beyond which we assumed no mode evolution. Placing the inflaton modes in a non-Bunch-Davies vacuum initial state then lead to corrections of order H/Λ to the power spectrum P_ϕ. We furthermore were able to extract an additional observational signature, which is more promising the lower the cut-off scale is taken to be.

More interestingly, we then investigated whether or not the cut-off allows the Bunch-Davies state to be specified at a finite and k-dependent time in the past. It is found that the Bunch-Davies vacuum has to be fixed in the infinite past and that a non-Bunch-Davies vacuum state has to be invoked as soon as considering a physical cut-off scale.

So a byproduct of this chapter is the finding that it is only consistent to speak of a Bunch-Davies vacuum state when having access to the infinite past $\tau_0 \to -\infty$ in order to satisfy $\hat{a}_{\mathbf{k}}(\tau_0)|0,\tau_0\rangle = 0$. Hence when considering inflation in de Sitter space, does the inflationary phase require an eternal past in order for the notion of the vacuum to be meaningful? The authors of [43] simply coin this insight a theoretical uncertainty.

So let us now consider a quasi de Sitter inflationary phase with a finite duration, i.e. finite past to be equivalent to introducing a cut-off scale at which to start the mode evolution. One might then argue that any inflationary phase with a finite past may only allow for non-Bunch-Davies vacuum states with $A_k, B_k \neq 0$ as the notion of the infinite past to fix the vacuum is lost.

As we will see in chapter 7 and 8 however, the notion of a non-Bunch-Davies vacuum is not as uncomplicated as it might seem at first sight. So in the light of these findings, the Bunch-Davies vacuum appears to require an inflationary model with an eternal past as otherwise, a non-Bunch-Davies vacuum state has to be fixed at the beginning of inflation, i.e. the cut-off scale. Anticipating the results of chapter 7 and 8 we may already argue that this theoretical uncertainty of the Bunch-Davies state actually leads to vacua which do not behave well. Thus either inflation is quasi de Sitter with an eternal past or the formalism yields vacua with difficult to overcome challenges.

At last, let us consider the following; when imposing a physical cut-off scale and thus obtaining a fixed finite time for the mode creation, we did not consider uncertainties in either energy scale or time.
If we define the cut-off as the scale at which conventional physics holds, we should in principle also take the uncertainty of quantities into account. So if we do not neglect the uncertainty principle, we should not define an exact cut-off and hence obtain a value of $|B_k|^2$ subject to some uncertainty.
If every created mode has this uncertainty, then we have to consider many different vacua as there is not simply a single point in time of mode creation.

Chapter 7

Back-Reaction

So far we have either considered mechanisms that put the inflaton fluctuation in a non-vacuum state or have deliberately placed the fluctuations in such a state initially at a time-like hypersurface corresponding to some fixed physical scale. However, having non-vacuum modes means that the inflaton fluctuations are in an excited state. One could hence think of inflaton particles being present. Since the particle concept in non-Minkowskian space is somewhat ambiguous as seen in section 2.1, it is more instructive and fully sufficient to consider the excited field modes without a particle interpretation.

Due to these excitations of the inflaton fluctuations we have an energy density present additionally to that from the inflationary background. It is now important to note that the approach taken in order to describe inflaton fluctuations was a perturbative one, i.e. it has always been assumed that the perturbations were small compared to the unperturbed background values. However, having an additional energy density due to the fluctuation modes being excited might very reasonably spoil that perturbative approach and hence invalidate our entire analysis. This had first been noted by [44, 45] where the authors obtained upper limits on the coefficient B_k leading to a non-Bunch-Davies vacuum state. However, they did not account fully for the mode evolution of the inflaton fluctuations but took a limit of the infinite past. Thus we will present a more exact result below.

7.1 Prerequisites

In this section we will establish all the expressions needed to perform the above mentioned calculation

7.1.1 De Sitter Space and Conformal Time

Let us quickly recall the mathematical description of the de Sitter background. We will be working with conformal time $d\tau = a^{-1}dt$ so that for $a = e^{Ht}$ we find

$$a = -\frac{1}{H\tau}.$$

Furthermore we consider our space to be flat, hence the line element is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= \frac{1}{(H\tau)^2}\left(d\tau^2 - d\vec{x}^2\right) \end{aligned} \tag{7.1.1}$$

so that $g_{00} = (H\tau)^{-2}$, $g^{00} = (H\tau)^2$ and $g_{\mu\nu} = a^2 \cdot diag(1, -1, -1, -1)$.

7.1.2 Field Theory of Inflaton Fluctuations

So far we have been considering the rescaled field $\hat{f} = a\hat{\phi}$ to derive e.g. an equation of motion for the inflaton perturbation which we here denote by $\hat{\phi}$. Now we are interested in the energy density of the actual fluctuation $\hat{\phi}$, not the rescaled one. So when considering the actual fluctuations we have to multiply the mode function of the rescaled field $\hat{f}$ with the inverse of the scale factor in order to obtain the mode functions for the inflaton fluctuation $\hat{\phi}$. Thus, expression (2.3.20) rewritten in terms of the mode functions of the fluctuation is

$$\hat{\phi}(\tau, x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(h_k(\tau)\hat{a}_{\mathbf{k}}(\tau_0)e^{i\mathbf{k}\cdot\mathbf{x}} + h_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger(\tau_0)e^{-i\mathbf{k}\cdot\mathbf{x}} \right), \tag{7.1.2}$$

where

$$\begin{aligned} h_k(\tau) &= A_k c(\tau) + B_k c^*(\tau) \\ &= A_k \frac{-H\tau}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau} + B_k \frac{-H\tau}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right)e^{ik\tau}. \end{aligned} \tag{7.1.3}$$

As the particle concept is irritating in a time-dependent background, we will consider the energy-momentum tensor rather than counting inflaton particles. The energy-momentum tensor for a complex scalar field in the rest frame of the considered quantity is given by

$$T^\mu_\nu = g^{\mu\nu}\left(\partial_\lambda\phi\partial_\nu\phi^\dagger - \frac{1}{2}g_{\lambda\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi^\dagger\right). \tag{7.1.4}$$

As we seek the energy density we are interested in the 00-component T^0_0. We write

$$\begin{aligned} T^0_0 &= g^{00}\left(\partial_0\phi\partial\phi^\dagger_0 - \frac{1}{2}g_{00}\left(g^{00}\partial_0\phi\partial_0\phi^\dagger + g^{ij}\partial_i\phi\partial_j\phi^\dagger\right)\right) \\ &= \frac{1}{2}g^{00}\left(\partial_0\phi\partial_0\phi^\dagger + \partial_i\phi\partial_j\phi^\dagger\delta^i_j\right). \end{aligned}$$

Now, we need to recall the following properties of Fourier transforms

$$\mathcal{F}\{\partial_i\phi_k\} = ik\phi_k, \tag{7.1.5}$$

$$\mathcal{F}\{\phi^\dagger_k\} = \phi^\dagger_{-k}, \tag{7.1.6}$$

from which we see that $\mathcal{F}\{\partial_i\phi^\dagger_k\} = -ik\phi^\dagger_{-k}$. Fourier-transforming the obtained expression for T^0_0 finally yields

$$T^0_0 = \frac{1}{2}g^{00}\left(\partial_0\phi\partial_0\phi^\dagger + k^2\phi\phi^\dagger\right). \tag{7.1.7}$$

7.2 Energy-Momentum Tensor for Fluctuations

With the tools from the previous section, we may now calculate the energy-density of excited inflaton fluctuations.

7.2.1 Calculation and Renormalisation

We seek

$$\begin{aligned}\langle\rho\rangle &= \langle 0|T_0^0|0\rangle \\ &= \frac{1}{2}g^{00}\langle 0|\partial_0\phi\partial_0\phi^\dagger + k^2\phi\phi^\dagger|0\rangle. \end{aligned} \tag{7.2.1}$$

Recalling expression (7.1.2), we may recast the above as[1]

$$\begin{aligned}\langle\rho\rangle = \frac{1}{2}g^{00}\int\frac{d^3k}{(2\pi)^3}&\langle 0|\left(h'\hat{a}+h^{*\prime}\hat{a}^\dagger\right)\left(h^{*\prime}\hat{a}^\dagger+h'\hat{a}\right) \\ &+ k^2\left(h\hat{a}+h^*\hat{a}^\dagger\right)\left(h^*\hat{a}^\dagger+h\hat{a}\right)|0\rangle \\ &= \frac{1}{2}g^{00}\int\frac{d^3k}{(2\pi)^3}\left(h'h^{*\prime}\langle 0|\hat{a}\hat{a}^\dagger|0\rangle + k^2hh^*\langle 0|\hat{a}\hat{a}^\dagger|0\rangle\right) \\ &= \frac{1}{2}g^{00}\int\frac{d^3k}{(2\pi)^3}\left(h'h^{*\prime}+k^2hh^*\right). \end{aligned} \tag{7.2.2}$$

This is the expression for the energy density of the fluctuations of the inflaton field. In the following we will evaluate that expression but unlike [44,45] we will consider the time evolution of the fluctuations as well to obtain a more precise result. However, we will use a similar renormalisation technique as in [45]. In order to renormalise expression (7.2.2), we need to know the energy contribution of the vacuum, i.e. the $B_k = 0$-state, first. We thus calculate

$$\langle\rho\rangle_{vac} = \frac{1}{2}g^{00}\int\frac{d^3k}{(2\pi)^3}|A_k|^2\left(c'c^{*\prime}+k^2cc^*\right), \tag{7.2.3}$$

with

$$c(\tau) = \frac{-H\tau}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)e^{-ik\tau}, \tag{7.2.4}$$

$$c'(\tau) = \frac{H\tau}{\sqrt{2k}}\left(ike^{-ik\tau}\right). \tag{7.2.5}$$

Substituting the above into the expression for $\langle\rho\rangle_{vac}$, we find

$$\langle\rho\rangle_{vac} = \frac{1}{2a^2}\int\frac{d^3k}{(2\pi)^3}\cdot k|A_k|^2\left(1+\frac{1}{2(k\tau)^2}\right). \tag{7.2.6}$$

Now, consider the general mode equation and its derivative

$$\begin{aligned} h(\tau) &= A_k\frac{-H\tau}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)e^{-ik\tau} + B_k\frac{-H\tau}{\sqrt{2k}}\left(1+\frac{i}{k\tau}\right)e^{ik\tau}, \\ h'(\tau) &= \frac{-H\tau}{\sqrt{2k}}ik\left(B_ke^{ik\tau}-A_ke^{-ik\tau}\right). \end{aligned} \tag{7.2.7}$$

[1] In the step to the last line we make use of the fact that the states are ormalised to unity, i.e. $\langle 1||1\rangle = 1$.

Substituting the above into expression (7.2.2) gives

$$\langle\rho\rangle = \frac{(-1)^2}{4(2\pi^3)a^4}\int d^3k\cdot k(|A_k|^2+|B_k|^2-\{A_kB_k^*e^{-2ik\tau}+B_kA_k^*e^{2ik\tau}\})$$
$$+\left(|A_k|^2+|B_k|^2\right)\left(1+\frac{1}{(k\tau)^2}\right)+\{A_kB_k^*\left(1-\frac{i}{k\tau}\right)^2e^{-2ik\tau}$$
$$+B_kA_k^*\left(1+\frac{i}{k\tau}\right)^2e^{2ik\tau}\}.$$

To make any sense from this rather tedious expression, we will first employ a time average so that the oscillatory terms in curly brackets average out. We then are left with

$$\langle\rho\rangle_{avg} = \frac{1}{2(2\pi)^3a^4}\int d^3k\cdot k\left[\left(|A_k|^2+|B_k|^2\right)\left(1+\frac{1}{2(k\tau)^2}\right)\right]. \tag{7.2.8}$$

Now we simply subtract the contribution of the vacuum to arrive at the expression for the energy density due to excited inflaton perturbations. We find

$$\langle\rho(\tau)\rangle_{ren} = \frac{1}{2(2\pi)^3a^4}\int d^3k\cdot k|B_k|^2\left(1+\frac{1}{2(k\tau)^2}\right). \tag{7.2.9}$$

Let us at last recast the above expression in terms of physical momentum $p = a^{-1}k$ and recall $d^3k = 4\pi k^2dk$ to obtain[2]

$$\langle\rho(\tau)\rangle_{ren} = \frac{1}{4\pi^2}\int dp|B_k|^2\left(p^3+\frac{1}{2}H^2p\right). \tag{7.2.10}$$

The above is the expression for the energy density of the excited inflaton perturbations in terms of their physical momentum. Note that this is equivalent to the expression obtained by [44,45] and [46,47] whose authors also investigate this effect, but with an additional H^2-term. This term is due to considering the time evolution of the modes and not assuming the infinite past. Like the similar expression for the vacuum, this is divergent for any non-zero B_k. Yet unlike the vacuum state we may not simply cure the divergence by e.g. normal ordering, i.e. disregarding it. In chapter 6 we introduced the idea of a physical cut-off scale at which we start the mode evolution in a general initial state. Thus disregarding the physics beyond that scale lets us immediately set that scale as the upper limit of the integral in the above expression. The exact proceedings will be subject of the next subsection.

7.2.2 Choosing a Momentum Cut-Off

Now consider some cut-off C as the upper limit of the momentum integral (7.2.10), where we don't specify yet whether it is a physical or comoving one. Furthermore lets drop the subscript for notational ease and write

$$\langle\rho(\tau)\rangle = \frac{1}{4\pi^2}\int_H^C dp|B_k|^2\left(p^3+\frac{1}{2}H^2p\right). \tag{7.2.11}$$

[2]The substitution $d^3k = 4\pi k^2dk$ is allowed for spherical symmetries. As we are considering spacetime to be isotropic, we can make use of this trick.

The lower limit of the integral is taken to be the scale of inflation H as modes with lower momenta are stretched beyond the (event) horizon. Also, for perturbations with $k << aH$ one can't treat the inflaton fluctuations as a scalar field [48, 49].

Evaluating the above expression simply yields

$$\langle\rho(\tau)\rangle = \frac{|B_k|^2}{16\pi^2}\left(C^4 - 2H^4 + H^2C^2\right). \tag{7.2.12}$$

When, as suggested in chapter 6, we choose a physical cut-off scale Λ, we may immediately recast the above as

$$\begin{aligned}\langle\rho(\tau)\rangle &= \frac{|B_k|^2}{16\pi^2}\left(\Lambda^4 - 2H^4 + H^2\Lambda^2\right) \\ &= |B_k|^2\mathcal{O}(\Lambda^4),\end{aligned} \tag{7.2.13}$$

which is constant in time. Comparing the above with the energy density of the inflationary background $M_{pl}^2H^2$ [30, 45], we may deduce an upper limit for the coefficient B_k to be

$$|B_k|^2 < \frac{M_{pl}^2H^2}{\Lambda^4}. \tag{7.2.14}$$

If we consider the cut-off scale Λ to be the Planck scale M_{pl}, we can infer

$$|B_k|^2 < \left(\frac{H}{M_{pl}}\right)^2. \tag{7.2.15}$$

So the considerations regarding trans-Planckian physics require condition (7.2.15) to be fulfilled as otherwise the perturbative approach to cosmological perturbations is invalidated.
Now recall the expression for $|B_k|^2$ from chapter 6 as

$$|B_k|^2 = \frac{H^2}{4\Lambda^2}.$$

This is the expression for $|B_k|^2$ when the inflaton fluctuations are placed in a non-Bunch-Davies vacuum state at a cut-off scale. For any scale Λ for which we have $\Lambda < M_{pl}$, e.g. $\Lambda = M_{string}$, condition (7.2.14) is readily satisfied. If however one takes the cut-off scale to be the Planck scale, than the value of $|B_k|^2$ is of the same order as the upper limit (7.2.15). The inequality is still satisfied due to the factor of 1/4 yet it might be worrisome that the value is of the same order of magnitude as its limit. We have

$$|B_k|^2 < \frac{M_{pl}^2H^2}{\Lambda^4} = \frac{M_{pl}^2}{\Lambda^2}\cdot\frac{H^2}{\Lambda^2}, \tag{7.2.16}$$

where we see that the first factor on the r.h.s. is greater than or equal to unity depending on whether or not the cut-off scale is taken to be the Planck scale. So if $\Lambda < M_{pl}$ then condition (7.2.15) is easily satisfied whereas $|B_k|^2$ is of the same order as the upper limit for $\Lambda = M_{pl}$ where the first factor on the r.h.s. is unity.
Thus in the light of considerations of back-reaction, it seems that a lower cut-off scale is favoured as opposed to considering the cut-off to be at the Planck scale.

However, let us now investigate the behaviour of expression (7.2.10) for a comoving momentum cut-off. For a comoving cut-off Λ, we have $C = a^{-1}\Lambda$ and we find straight away that

$$\begin{aligned} \langle\rho(\tau)\rangle &= \frac{|B_k|^2}{16\pi^2}\left[\left(\frac{\Lambda}{a}\right)^4 - 2H^4 + H^2\left(\frac{\Lambda}{a}\right)^2\right] \\ &\propto \frac{1}{a^4}, \end{aligned} \tag{7.2.17}$$

which hence inflates away quickly. Thus the behaviour of the energy density of excited inflaton fluctuations depends on the nature of the cut-off.

7.3 Conclusions

The subject of this chapter was a consistency investigation. Starting off modes in a non-vacuum state introduces an additional energy density which has to be smaller than that of the background as otherwise the perturbative approach underlying all of this analysis is spoiled.

The follow-up considerations however are twofold. When choosing a physical momentum cut-off and thereby breaking local Lorentz invariance, we find an energy density constant in time. This provides a convenient upper limit on the coefficient B_k yet brings with it some conceptional challenges. First, taking the cut-off scale to be the Planck scale comes dangerously close to the upper limit imposed due to back-reaction. Hence it can be concluded that the cut-off scale should be lower than the Planck scale in order for the perturbative approach not to be endangered. Furthermore, due to the expansion of space, excited modes are being redshifted. Choosing a physical momentum cut-off means that there are modes starting to evolve from that scale for all times hence the excitation does not redshift away. As these modes start off being excited already and thus start with an energy density above that of the vacuum one could therefore ask where the reservoir[3] is from which the modes take their initial energy [50]. Furthermore it can be argued that a reservoir of energy providing excited modes to inflaton fluctuations for the duration of the inflationary phase should somehow backreact on its own already and hence spoil the inflationary background anyway. As the inflationary background is obviously not destroyed and our universe does exist, one may thus argue that the physics related to a constant physical cut-off are unphysical or at least do not resemble reality in our universe.

Technically, one has to recall that the fluctuation modes have constant wave number k, hence the cut-off scale can be understood as a boundary condition in k-space. Yet due to the time-dependence of the background, a physical cut-off implies mode creation, or in other words, a time-dependent, i.e. growing Hilbert space and it is not yet obvious how to realise that while preserving unitarity and local Lorentz invariance [51].

Applying modified dispersion relations to mimic this energy source does not better the situation. For once, it is nothing but a guess that unknown physics may be effectively described this way and also, to obtain a specific value for B_k and hence be able to make a quantitative statement about the correction to the

[3]See appendix B for an example of how such a reservoir might be realised.

power spectrum requires the choice of a specific dispersion relation for which a good motivation is hard to find.
Hence in the inflationary community, a cut-off is always understood as being comoving. Thus the expected behaviour of any excitation is exactly that of equation (7.2.17), i.e. any excitation inflates away within a few e-folds.
However, providing this conventionally unphysical energy reservoir might be exactly the point of the unknown trans-Planckian physics. We have reached a point at which one result, i.e. constant energy density for non-vacuum inflaton fluctuations emerging from a physical cut-off, only seems to make sense when we invoke trust in unknown physics.

Chapter 8

The Case for the Bunch-Davies State?

As seen in the previous chapter, it seems that a non-Bunch-Davies vacuum initial state in an expanding spacetime requires some conventionally unphysical behaviour in order not just to inflate away. In this chapter we will concentrate on the question of whether or not a non-vacuum initial state may be motivated. Because non-Bunch-Davies vacuum states arise in the context of modified dispersion relations as well as when introducing a cut-off scale to the theory, we will first examine their general features and then reconsider their behaviour when introducing a physical cut-off.

8.1 Alpha-Vacua

So far we have characterised the initial state of the inflaton fluctuations with the two parameters A_k and B_k for which $|A_k|^2 - |B_k|^2 = 1$. The constraint on these two parameters already points at a different form of representation, namely [52]

$$A_k = \cosh\alpha \tag{8.1.1}$$

$$B_k = e^{i\delta}\sinh\alpha, \tag{8.1.2}$$

where $\alpha \in [0, \infty)$ and $\delta \in (-\pi, \pi)$. The case $\alpha = 0$ is equivalent to the case $B_k = 0$ and hence resembles the Bunch-Davies Vacuum. Due to this reparametrisation, any state with a non-zero B_k and hence a non-zero α is called α-vacuum. However [50, 53, 54] use a different reparametrisation given by

$$A_k = \frac{1}{\sqrt{1 - e^{\alpha+\alpha^*}}} \tag{8.1.3}$$

$$B_k = \frac{e^{\alpha}}{\sqrt{1 - e^{\alpha+\alpha^*}}}, \tag{8.1.4}$$

where $\alpha \to -\infty$ corresponds to the Bunch-Davies vacuum and $\Re(\alpha) < 0$. We will employ the latter reparametrisation. Let us recall from section 2.1 that for a decomposed field operator ϕ

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x)\right]$$

we may introduce a Bologubov transformation in order to expand the field in a different set of modes for which the new set of creation and annihilation operators is

$$\begin{aligned} a_i &= \sum_j [\alpha_{ji}\bar{a}_j + \beta_{ji}^*\bar{a}_j^\dagger] \\ \bar{a}_j &= \sum_i [\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger]. \end{aligned}$$

which in the light of relation (8.1.3) and (8.1.4) and considering $B_k \to \beta_k = \beta_{ik}\delta_k^i$ can be recast as

$$a_k^\alpha = \underbrace{\frac{1}{\sqrt{1-e^{\alpha+\alpha^*}}}}_{=N_\alpha}\left(a_k - e^{\alpha^*}a_k^\dagger\right) \tag{8.1.5}$$

where the parameter α denotes the transformation and a similar expression for $(a_k^\alpha)^\dagger$ may easily be constructed. The alpha-vacuum state is then defined as

$$a_k^\alpha|\alpha\rangle = 0. \tag{8.1.6}$$

8.1.1 Particle Content

Let us now investigate a first difference between alpha-vacua and the Bunch-Davies vacuum state. In [19] it was suggested to introduce the concept of a particle detector in order to be able to make meaningful statements about particles in general spacetimes. Note however that the wording is misleading. This is not an attempt to reintroduce the problematic and observer-dependent particle concept. But as the energy-momentum tensor cannot be measured as such we need a prescription of how to obtain actual observables. This is done by the concept of coupling a detector to a field and investigating the detector's excitations, hence the measured excitations are interpreted as particles. To phrase it in more philosophic terms; the electron is nothing but the deflection of the ammeter's pointer.

So consider an idealised Unruh detector moving along a worldline $x^\mu(\tau)$ with the detector-field interaction Lagrangian $cm(\tau)\phi\left[x(\tau)\right]$ where c is a small coupling constant and $m(\tau)$ is some operator measuring the state of the detector. The detector may have a ground and excited states and the field is assumed to be in a Bunch-Davies vacuum state. Considering a general trajectory, the detector as well as the field will undergo a transition to some excited state where the amplitude of the transition may be calculated with first-order perturbation theory via

$$ic\langle E_1, \phi_1| \int m(\tau)\phi[x(\tau)]d\tau|0, E_0\rangle. \tag{8.1.7}$$

With this ansatz one then finds that the detector behaves as if immersed in a bath of thermal quanta with temperature

$$T = \frac{1}{2\pi}. \tag{8.1.8}$$

Thus it is shown that the Bunch-Davies vacuum is of thermal character.

The authors of [55] apply the same ansatz to the above introduced alpha-vacua. It is found that the detector does not show the behaviour as when excited by thermal quanta. The ratio of probabilities per unit time for the detector to make a transition from E_0 to $E_1 = E_0 + \delta E$ and vice versa is

$$\mathcal{R}_{ij} = \frac{\dot{P}_{i\to j}}{\dot{P}_{j\to i}} = e^{-2\pi\delta E}\left|\frac{1+e^{\alpha+\pi\delta E}}{1+e^{\alpha-\pi\delta E}}\right|^2 = f_\alpha(\delta E), \tag{8.1.9}$$

where $i \to j$ denotes a transition from E_0 to E_1. For the Bunch-Davies case $\alpha \to -\infty$ we recover the standard result

$$\mathcal{R}_{ij} \to \frac{\rho(E_j)}{\rho(E_i)} = e^{-2\pi\delta E}, \tag{8.1.10}$$

where $\rho(E)$ is the energy density of the state of energy E and the equilibrium temperature is (8.1.8). Now consider the principle of detailed balance, i.e.

$$\mathcal{R}_{ij}\mathcal{R}_{jk} = \mathcal{R}_{ik}. \tag{8.1.11}$$

Yet this implies $f_\alpha(\delta E)^2 = f_\alpha(2\delta E)$. Obviously, relation (8.1.9) cannot satisfy this for any other value of α other than $\alpha \to -\infty$. As pointed out in [50], while quoting the same non-thermal character, a system with a finite α may reach an equilibrium but this is violating at least one of detailed balance's properties of unitarity, time reversal or equipartition and is explicitly non-thermal. Furthermore, considering ratio (8.1.9) for $\delta E \gg \pi$ and finite α, we see that

$$\mathcal{R}_{ij} \to e^{2\alpha}, \tag{8.1.12}$$

which says nothing else but that the transition probability is constant regardless of the size of δE and hence hints at the alpha-vacuum having infinite energy.

To summarise, in contrast to the Bunch-Davies vacuum state, alpha-vacua do not share the thermal character. Furthermore, the ratio of transition amplitudes per unit time seems to indicate a divergence for the alpha-state.

8.1.2 Regularisation

Let us know consider the issue of regularising alpha-states. As alpha-vacua are excitations above the Bunch-Davies state that extend up to arbitrarily high energies, they do not just inflate away. A consequence is the divergent behaviour of $T_{\mu\nu}$ for an alpha-state.

From chapter 7 we recall the energy momentum tensor of a scalar field (7.1.4)

$$T^\mu_\nu = g^{\mu\nu}\left(\partial_\lambda\phi\partial_\nu\phi^\dagger - \frac{1}{2}g_{\lambda\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi^\dagger\right).$$

When investigating the theory for an alpha-state, i.e. $\langle\alpha|T^\nu_\mu|\alpha\rangle$, the above expression is clearly divergent without some cut-off. Certainly, the expression $\langle 0|T^\nu_\mu|0\rangle$ for the Bunch-Davies vacuum is also divergent. However, this diverges in the same way as the expression of the Minkowski vacuum. A consistent regulator of the Bunch-Davies vacuum is thus also a regulator of flat space. There exist reasonable regularisation procedures [56] for the Minkowski vacuum and therefore also for the Bunch-Davies vacuum that approaches the Minkowski vacuum on scales small compared to the curvature of spacetime.

To illustrate our argument, recall the expression for the energy density (7.2.11) of an alpha-state

$$\langle\rho(\tau)\rangle = \frac{1}{4\pi^2}\int dp \underbrace{\left|\frac{e^{\alpha}}{\sqrt{1-e^{\alpha+\alpha^*}}}\right|^2}_{=|B_k|^2}\left(p^3+\frac{1}{2}H^2 p\right). \tag{8.1.13}$$

Note that this has been obtained by already disregarding the contribution from the positive energy solution of the field's mode equation, i.e. the Bunch-Davies contribution. In order to formulate a field theory, we need to regulate the above alpha-vacuum contribution as well. Yet the above expression is obviously alpha-dependent. That means that the regulator that we seek in order to render $T_{\mu\nu}$ finite will be dependent on the parameter α. Due to the symmetry of de Sitter space we might simply expand the field ϕ in terms of yet another set of creation and annihilation operators as discussed in section 2.1. Consider this other set to be represented by some other value of the parameter α, e.g. β.

Now we immediately see that a theory which is regulated for a state $|\alpha\rangle$ is divergent for any state $|\beta \neq \alpha\rangle$. Thus it seems to be impossible to consistently formulate a field theory with an underlying alpha-vacuum as there exists no reasonable regularisation procedure and divergences persist for any other value of α than the one chosen to regulate. Furthermore, any alpha dependent regulator does not regulate flat space.

This result is underlined by the conclusions of [54], where the authors attempt to compute the one loop effective action of an alpha-vacuum. They find that non-local divergent counter-terms are needed in order to render interaction's Green's functions finite and that the renormalisation procedure thus fails. It is hence stressed that quantum field theory in a general alpha-vacuum does not seem to make sense unless further knowledge about high energy physics is available. The authors of [57] present the same findings.

At last it is important to stress that without a cut-off and a reasonable regularisation scheme, alpha-vacua backreact with the spacetime and render the perturbative approach to the theory of inflationary fluctuations invalid.

8.1.3 Cut-Off

In the previous subsections we have investigated some properties of alpha-vacua. We found that they are non-thermal and that the constant transition amplitude independent of δE already hints towards the alpha-states having infinite energy. Furthermore, we found that the regulator needed in order to be able to formulate a consistent field theory is dependent on the parameter α. Thus if the vacuum of one mode expansion is regulated, all others including the Bunch-Davies vacuum remain divergent.

As suggested in chapter 6 it might be reasonable to introduce a cut-off scale where any physics beyond the cut-off is disregarded. Let us know evaluate wheter or not this tackles the problems the alpha-vacua are suffering from.

Considering the non-thermal behaviour we immediately see that the constant transition amplitude is in no way altered by the introduction of the cut-off. However, we cannot have transitions to arbitrarily high energies any more as obviously the theory is now cut off at a certain scale. So in principle the cut-off does not cause the alpha-state to change its behaviour such that it would show the same characteristics as the Bunch-Davies state.

Considering the divergence of $T_{\mu\nu}$ we see that the theory clearly does not show a divergence any more yet even regulating a finite amount still requires an alpha dependent regulator as the field ϕ may still be arbitrarily expanded. Furthermore we arrive at the issue addressed in the last section of chapter 7. If we introduce a cut-off to the theory and consider the alpha-state to be a finite excitation above the Bunch-Davies vacuum then we have to decide whether the cut-off is meant to be a physical or comoving one. In the latter case, the finite excitation just inflates away as already shown by equation (7.2.17). If chosen to be a physical cut-off, then we require a mechanism that pumps in excited modes as space expands, i.e. we require a Hilbert space growing with time as remarked in the previous chapter. Not only could one question why this pumping mechanism does not already backreact on its own but it is also important to note that the mechanism needs to be stopped at the end of inflation in order not to overpopulate the post-inflationary universe. This however might obviously be related to the fate of the inflaton field itself.

In conclusion, a cut-off cures the divergent character but at the price of introducing some mode creation mechanism when regarded as a physical one. Furthermore, as calculated in chapter 7, fine-tuning is needed for the parameter B_k not to invalidate the perturbative approach. When however simply introducing a comoving cut-off the alpha-vacuum quickly relaxes to the Bunch-Davies state.

8.2 Effective Field Theory

Having seen that alpha-vacua either show divergent behaviour or require some unconventional mechanism in order to yield sensible physical behaviour, let us now review another approach to the influence of unknown physics on the observed power spectrum of the CMB.

The issue of possible trans-Planckian effects has also been studied in the context of effective field theory. It is important to highlight the physical features of the approach.

In the following we will consider the treatment of [58] and highlight the physical characteristics. Let us first investigate differences to our approach taken in chapter 6. We placed the modes in their general vacuum state once they emerged from a new physics hypersurface corresponding to the cut-off scale. However the authors of [58] place the modes in the vacuum state[1] and employ an effective field theory at the scale of horizon exit H and not at a fixed scale for which $k \gg aH$. Conceptually, it is not specified why some physics before horizon exit at $k = aH$ might not influence the mode evolution such that the EFT approach is invalidated. And it is this possibility of a mechanism becoming important a certain number of e-folds before horizon exit that motivated the authors of [59–61] to question the validity of the EFT approach as we will outline later.

It is assumed that the effect of any trans-Planckian physics may be represented by integrating it out and writing an effective field theory for the inflaton

[1] It is assumed that a Bunch-Davies vacuum may be fixed at the infinite past [50].

field. These assumptions require only low-energy locality as EFTs are local theories. Furthermore, the usual assumptions of the scale of inflation being much smaller than the Planck mass, i.e. $H \ll M_{pl}$, and the scale of new physics being much larger than the scale of inflation, i.e. $\Lambda \gg H$ are made. The authors of [58] then continue to formulate an effective field theory for the inflaton field ϕ at horizon exit.

8.2.1 Local Action and the Power-Spectrum

In order to obtain phenomenologically acceptable values of $\delta\rho/\rho$ at the end of inflation, inflaton self-interactions are required to be very weak. In this model, inflaton interactions will henceforth be ignored. Then considering that curvature of de Sitter space is proportional to H^2, the most general Euclidean local action is of the form

$$S_{eff}[\phi] = \int d^4p\phi(p)\phi(-p)[\frac{p^2}{2} + \frac{H^2}{2} + c_0 H^2 \left(\frac{H^2}{\Lambda^2}\right) + c_1 p^2 \left(\frac{H^2}{\Lambda^2}\right) + c_2 \left(\frac{p^4}{\Lambda^2}\right) + c_3 \left(\frac{p^4}{\Lambda^2}\right)\left(\frac{H^2}{\Lambda^2}\right) + c_4 \frac{p^6}{\Lambda^4} + ...]. \tag{8.2.1}$$

Information about new physics is thus contained in the coefficients c_i and the scale of new physics Λ. From the above it then follows that the two-point function and hence the form of the power-spectrum is given by

$$\langle\phi(p)\phi(-p)\rangle|_{p=H} = H^2(1 + c_0 \left(\frac{H^2}{\Lambda^2}\right) + c_1 \left(\frac{H^2}{\Lambda^2}\right) + c_2 \left(\frac{H^2}{\Lambda^2}\right) + c_3 \left(\frac{H^2}{\Lambda^2}\right)^2 + c_4 \left(\frac{H^2}{\Lambda^2}\right)^2 + ...). \tag{8.2.2}$$

Corrections thus are thus given as a power series in terms of H^2/Λ^2.

8.2.2 Possible Criticism

As mentioned in the introduction to this section, the authors of [59–61] question the validity of the above approach. It is argued that some non-adiabatic physics acts within a number of e-folds before the end of inflation such that a mixing of positive and negative frequency solutions to the mode equation and hence non-adiabatic transitions occur. Two things about this reasoning are important to note. First, it is along the same lines as the ideas of chapter 5. In both cases some non-adiabatic behaviour leads to the inflaton fluctuations being in an excited state. The details of this have been presented in section 5.2 and apply equally to the reasoning of [59–61]. Secondly, whereas the ideas of chapter 5 tried to mimic the behaviour of physics from an unknown trans-Planckian regime and aimed at affecting the inflaton fluctuation modes at the scale beyond which new physics needs to be invoked, the below reasoning applies to non-adiabatic behaviour before event horizon exit. So this is explicitly not a trans-Planckian consideration as it seeks to modify the modes at the scale H and not at the scale of new physics Λ. However it tries to cause a non-vacuum state and thus invalidate the treatment of section 8.2.

To demonstrate their argument, a standard hybrid inflationary model is constructed and its influence on the power-spectrum computed. The model is defined as

$$-\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + V(\phi,\chi)\right], \tag{8.2.3}$$

where

$$V(\phi,\chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda(\chi^2 - v^2)^2 + \frac{1}{2}g\chi^2\phi^2 + \frac{1}{12}\tilde{\lambda}\phi^4. \tag{8.2.4}$$

Conditions for inflation to occur are given in section two of [59]. It may then be shown that the field χ obeys

$$\ddot{\chi} + 3H\dot{\chi} + M^2(\phi)\chi \approx 0, \tag{8.2.5}$$

where $M^2 \approx g\phi^2$. This may then be solved by

$$\chi(t) = \chi_0 a(t)^{-3/2}\cos[g\phi^2(t - t_0)]. \tag{8.2.6}$$

Within the framework of this model, the equation of motion of an inflaton perturbation with wave number k is

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left[k^2 e^{-2Ht} + V''(\phi) - g\chi^2(t)\right]\delta\phi_k = 0. \tag{8.2.7}$$

The coupling of ϕ to χ through the time-dependent 'mass' term may lead to a mixture of positive and negative frequency components in the solutions for $\delta\phi$ which are purely positive initially, provided χ behaves such that the 'mass' term in equation (8.2.7) departs adiabaticity. This is demonstrated explicitly in section three of [59]. The authors therefore claim to identify non-adiabaticity to invalidate the effective field theory approach of [58].

There are however several short-comings with the above argument. Even though it shows that in principle a mechanism causing mode excitation before horizon exit can exist, the argument strongly relies on a certain class of inflationary models, namely hybrid ones, which seem to be slightly disfavoured in their standard form by recent observations [28]. Furthermore, as pointed out in section 8.1, a mixing of positive and negative frequency solutions i.e. an alpha-vacuum may show fundamentally sick behaviour. This was overcome in chapter 6 by implementing a physical cut-off to the theory but at the price of an unconventional mode creation mechanism. However, such a cut-off is not invoked in the hybrid model outlined above and hence the challenges regarding alpha-vacua remain. Thus the effective field theory approach is invalidated by explicitly producing non-local alpha-vacua. In context of trans-Planckian considerations this is challenging. When having considered alpha-vacua as initial states for the inflaton fluctuations it could always be argued that alpha-vacua need unknown trans-Planckian input in order to make physical sense. However as the alpha-vacua obtained by non-adiabatic physics before horizon exit are caused by well established hybrid inflationary models without any hidden elements to the theory, it is not possible to argue that some unknown trans-Planckian contribution might eventually render the states well behaved.

The criticism [59–61] to [58] hence seems to be unconvincing as it is eventually violating locality in order to invalidate the effective field theory approach and does not seem to be able to rely on unkown physics in order to avoid that price.

8.3 Conclusions

The peculiarities of non-Bunch-Davies vacuum states discovered in chapter 7, namely the consequences of the decision whether or not an introduced cut-off scale is a physical or comoving one, suggested to investigate the behaviour of the so called alpha-vacua further.
As described in section 2.1, these vacuum states are Bogolubov transformations or rotations of the standard Bunch-Davies vacuum. In principle, an arbitrary number of these de Sitter invariant vacua exist, i.e. one may always expand the field operator ϕ in terms of some other set of creation and annihilation operators. The question then arises if all states allowed are equally good contestants for an initial state of the inflaton fluctuations.

When first investigating the nature of alpha states we disregarded the cut-off in order to treat them generally and be able to decide whether or not the introduction of a cut-off then cures some of the issues discovered in the general treatment.

It is found that whereas the Bunch-Davies vacuum displays thermal character, the alpha-vacua do not do so. Specifically, the ratio of transition probabilities per unit time between two energy states of an idealised detector coupled to the field of inflaton fluctuations is independent of the size of the energy jump and thus constant. This means that all energy transitions are equally likely. As alpha-vacua extend to arbitrarily high energies, this indicates that jumps to basically infinite energy are just as likely as a transition with small δE. Hence this non-thermal behaviour hints at a divergence, or in other words, infinite energy of the alpha state.
Like the Bunch-Davies vacuum state, alpha vacua also show divergent behaviour when considering $\langle\alpha|T_{\mu\nu}|\alpha\rangle$. Thus it is required to find a reasonable regularisation method. It is then found that a regulator may always only regulate one specific alpha vacuum. Thus the regulator is alpha-dependent. As mentioned before, arbitrarily many decompositions are de Sitter invariant - in the notation of (8.1.3) we have $\Re(\alpha) \in (-\infty, 0)$ - so once one alpha state is regulated all the others are not. This regulator will also not regulate the divergent contribution from the Bunch-Davies state and hence not regulate flat space as well. Additionally, it was quoted that an attempt at renormalising the one loop effective action with an alpha-state leads to non-local and divergent counter terms.

A possible attempt to cure the issues arising above may be the introduction of a cut-off scale.
We immediately see that jumps to arbitrarily high energies are not possible any more as a courtesy of the cut-off. However, the non-thermal character is not changed regardless of any upper limit. Within the allowed range of energies, a thermal equilibrium will not be reached.
Considering regularisation procedures we have obviously cured the divergences as all vacua are now cut off at a certain scale. However, the coefficient B_k now has to undergo fine tuning in order not to invalidate the perturbative approach as shown in chapter 7.
Also, this finally is the point at which one has to decide whether or not the cut-off is supposed to be physical or comoving. Any comoving cut-off corresponds to a finite excitation above the Bunch-Davies state that immediately relaxes to the thermal vacuum, i.e. it inflates away as shown in section 7.3. Invoking a physical cut-off means that modes are being created all the time at the intro-

duced hypersurface corresponding to the cut-off. And it is this mode creation mechanism that may very well be questioned and considered unphysical. First, this mode pump requires some energy reservoir from which to excite the modes and one should ask why this is not already harmfully backreacting with the spacetime. Second, the pump needs to be stopped at the end of inflation[2]. But as already pointed out, since we are talking about fluctuation modes of the inflaton field, the fate of the mode creation mechanism is clearly related to the inflaton field itself. Thus the question as to what exactly happens will perhaps be answered when the final fate of the inflaton field is described and is thus not as fundamental as the backreaction issue for the mode creation mechanism. However, as pointed out already, mode creation means a growing Hilbert space which is not understood how to be realised.
With the above considerations, mechanisms to invoke a mixing of positive and negative frequency solutions a few e-folds before horizon exit as described in section 8.2 also seem to be unphysical.

All the arguments above clearly demonstrate that non-Bunch-Davies initial states are problematic to say the least within the framework of known physics. Then, one might however argue [53, 63] that the difficulties faced with alpha-vacua are natural and expected as this hints towards the need of trans-Planckian input for these states to make physical sense. It is suggested that all peculiarities and divergences will eventually be cured once a trans-Planckian theory or theory of quantum gravity is found. Even the authors of [54] who showed that non-local counter-terms arise in the renormalisation procedure conclude that a better understanding of high energy physics might help resolve some outstanding problems. In other words, alpha vacua are relevant due to a trans-Planckian cut-off and require this very cut-off in order to make physical sense, i.e. loop amplitudes need trans-Planckian input in order to yield well defined terms.

So does the question whether or not to allow non-Bunch-Davies initial states directly lead to the question of whether or not one simply believes if there will be a more fundamental theory that will overcome all the issues of our current formalism regarding alpha-vacua?
Before answering this, let us for a moment consider the following. As first formulated in [64] and now known as Haag's theorem [65], mathematically speaking the interaction picture does not exist in an interacting theory. However, quantum field theory seems to be able to describe nature with utmost precision as demonstrated by experiments. While at first this mathematical note seems unrelated to our discussion, it is important when raising the following point. Despite the mathematical inconsistency, QFT is undoubtedly successfully used by the practitioner. Perhaps this inconsistency might be overcome with a future formulation of the theory. However, the theory is applied as if fully consistent. So having this in mind, it might seem too pessimistic to simply exclude alpha-vacua from our focus just because they lead to mathematically ill-behaving results. On the contrary however, we are not yet in the position to experimentally test the predictions stemming from including alpha-vacua to our analysis. So perhaps when observing linear order corrections to the CMB, the mathematically inconsistent alpha-vacua will come into consideration again.

[2]The authors of [62] consider high-energy cosmic rays to result from trans-Planckian particle creation today, i.e. they assume the mode creation not to have stopped. They find strong constraints but do not exclude the possibility of trans-Planckian particle production being responsible for at least some part of ultra-high energy cosmic rays observed today.

This is however mere speculation at this point and there is no experimental indication that the formalism of alpha-vacua would accurately describe nature despite its formal shortcomings.
Let us now go back to the question posed at the beginning of this paragraph. When considering the history of physics, all underlying theories may be approximated by earlier theories in limiting scenarios. E.g. general relativity approaches Newtonian behaviour in a weak field limit and quantum mechanics yields classical behaviour for large quantum number n. In fact, if a supposedly underlying theory would yield a result inconsistent with the valid heuristic description, it would be regarded as not describing our reality despite its elegance or mathematical beauty. From this point of view, it seems that any trans-Planckian theory must therefore yield the low energy behaviour of our conventional approach when describing inflaton fluctuations. I.e. we have to be able to extract exactly the physics we have been using so far when being presented with a full theory of trans-Planckian physics. This is then the underlying reasoning explaining the applicability of the approach taken in section 8.2 where decoupling is assumed and any higher energy physics is integrated out. Hence the argument that alpha-vacua need trans-Planckian input in order to make physical sense seems to loose its appeal as any trans-Planckian physics should have the low energy behaviour we have been considering thus far within the conventional framework.

Relating to the heading of this chapter, we can finally conclude that the case for the Bunch-Davies vacuum is a very convincing one. The attempt to cure the behaviour of alpha-vacua with a cut-off immediately leads to the issue of what nature the cut-off has. So we are exchanging one challenge for another without even curing all problems such as the non-thermal behaviour. In conclusion, as long as the outstanding challenges to alpha-vacua, that had been noted as far back as [52, 66], remain, we cannot consider them in inflationary theory. Hence the most one can hope for are indeed corrections to the power spectrum of the order of H^2/Λ^2 as obtained when integrating out the high energy physics.
At last let us stress once again that the Bunch-Davies vacuum however requires the infinite past in order to be imposed and it is not obvious whether or not this is a realistic requirement.

Chapter 9

Conclusions

This work investigated the sensitivity of the inflationary paradigm to changes of the high energy behaviour of the theory. To carry out this investigation, we first derived the equation of motion for the inflaton fluctuations. It is this equation that the following considerations focused on.

We reviewed an ad-hoc modification of the dispersion relation in order to mimic any unknown high energy effects and found this approach to be of mere conceptual interest. We then introduced the idea of a cut-off scale Λ beyond which new physics has to be invoked. The influence of unknown physics is then parametrised in terms of boundary conditions at the cut-off on the equation of motion of the inflaton fluctuations and hence their future evolution. It is found that this approach leads to a modulation of the power spectrum of primordial fluctuations linear in H/Λ. More interestingly, we found that as soon as a cut-off scale denies access to the infinite past, one has to consider an α-vacuum to be the vacuum state of the theory. It is not possible to impose the Bunch-Davies vacuum at any time other than the infinite past.

However non-Bunch-Davies initial states contain an additional energy density. Thus it is of crucial importance to investigate their back-reaction on the inflationary background. Similar to the Minkowski vacuum, α-vacua diverge without some regularisation procedure. Yet we found that a physical cut-off scale as a regulator comes with the price of having to realise some mode creation mechanism, i.e. a time-dependent Hilbert space, whereas a comoving cut-off lets the excited states quickly relax to the standard vacuum.

We then investigated α-vacua further and found severe challenges. In contrast to the Bunch-Davies case they are not thermal. Furthermore, when considering an α-vacuum one always has to regularise the standard vacuum additionally to the contribution stemming from the alpha-vacuum in order to avoid divergences of the energy-momentum tensor whereas the regulator of the Bunch-Davies vacuum is also a regulator of flat space. Finally, considering alpha-vacua means relying on unknown trans-Planckian input in order for the argument to be meaningful. However, any unknown high energy physics should yield the known low energy behaviour, thus a theory of trans-Planckian physics should not predict deviations from established low energy physics. Therefore, applying effective field theory yields modulations of the power spectrum quadratic in H/Λ.

Observations have not yet discovered linear modulations, hence experimental input does not indicate that alpha-vacua are physical. Thus the theoretical and observational evidence make a strong case for the Bunch-Davies state and thus suggest that trans-Planckian effects in the sense considered within this work will not be observable.

At the very last, let us stress again that the Bunch-Davies vacuum however has to be imposed at the infinite past. So in inflationary models without an eternal history, the notion of a vacuum in the infinite past is ambiguous to say the least.

Appendix A

Quantum Field Theory in Curved Spacetime

Here, we will show explicitly how the creation and annihilation operators of two different expansions of a scalar field in a general background are related. Recall from chapter 2 that we may expand the mode functions of one expansion in terms of the other expansion's mode functions [19], i.e.

$$\bar{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right) \tag{A.0.1}$$

$$u_i = \sum_i \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right). \tag{A.0.2}$$

Considering

$$(u_i, u_j) = \delta_{ij},\ (u_i^*, u_j^*) = -\delta_{ij},\ (u_i, u_j^*) = 0, \tag{A.0.3}$$

and

$$\alpha_{ji} = (\bar{u}_j, u_k),\ \beta_{ij} = -(\bar{u}_i, u_j^*) \tag{A.0.4}$$

we may equate the two expansions of ϕ and write

$$\begin{aligned}
\sum_i (a_i u_i + a_i^\dagger u_i^*) &= \sum_j (\bar{a}_j \bar{u}_j + \bar{a}_j^\dagger \bar{u}_j^*)\ |\cdot u_k \\
\sum_i (a_i \underbrace{(u_i, u_k)}_{\delta_{ik}} + a_i^\dagger \underbrace{(u_i^*, u_k)}_{0}) &= \sum_j (\bar{a}_j (\bar{u}_j, u_k) + \bar{a}_j^\dagger (\bar{u}_j^*, u_k)) \\
a_k &= \sum_{j,i} [\bar{a}_j ((\alpha_{ji} u_i + \beta_{ji} u_i^*), u_k) \\
&+\ \bar{a}_j^\dagger (\left(\alpha_{ji}^* u_i^* + \beta_{ji}^* u_i\right), u_k)] \\
&= \sum_{j,i} [\bar{a}_j (\alpha_{ji} \underbrace{(u_i, u_k)}_{\delta_{ik}} + \beta_{ji} \underbrace{(u_i^*, u_k)}_{0}) \\
&+\ \bar{a}_j^\dagger (\alpha_{ji}^* \underbrace{(u_i^*, u_k)}_{0} + \beta_{ji}^* \underbrace{(u_i, u_k)}_{\delta_{ik}})] \\
&= \sum_j [\alpha_{jk} \bar{a}_j + \beta_{jk}^* \bar{a}_j^\dagger].
\end{aligned}$$

Now simply changing $k \to i$ and considering the same calculation for $\bar{a}_j$, we find the desired result

$$a_i = \sum_j [\alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}^\dagger_j] \tag{A.0.5}$$

$$\bar{a}_j = \sum_i [\alpha^*_{ji}a_i - \beta^*_{ji}a^\dagger_i]. \tag{A.0.6}$$

Appendix B

Mode Creation

In this work we have brought forward the argument that a physical cut-off requires a mechanism that creates fluctuation modes at the cut-off scale. For completeness, we will now outline such a mechanism based on [67, 68].

The argument presented starts by introducing the string or quantum gravity inspired modified uncertainty relation

$$\Delta x \Delta p \geq \frac{1}{2}\left(1 + \beta \left(\Delta p\right)^2 + ...\right), \tag{B.0.1}$$

where β is a positive constant. This implies a minimum uncertainty $\Delta x_{min} = \sqrt{\beta}$, thereby explicitly breaking local Lorentz invariance and β is taken such that Δx_{min} is the string or Planck scale. The above is equivalent to imposing modified commutation relations

$$[\mathbf{x}, \mathbf{p}] = i\left(1 + \beta \mathbf{p}^2 ...\right). \tag{B.0.2}$$

Then, the action for inflationary fluctuations, i.e. a minimally coupled massless real scalar field, is introduced in terms of proper time τ and proper, i.e. non-comoving coordinates x^i as

$$S = \int d\tau d^3x \frac{1}{2a}\left\{\left[\left(\partial_\tau + \frac{a'}{a}\partial_{x^i} x^i - 3\frac{a'}{a}\right)\phi\right]^2 - a^2\left(\partial_{x^i}\phi\right)^2\right\}, \tag{B.0.3}$$

where $' = \partial_\tau$ and summation convention is implied.

In the following, the ansatz of (B.0.1) is imposed on the above action (B.0.3). It is then possible to recast the action and deduce a Hamiltonian as well as a quantum field ϕ which only contain a k-mode *after* the mode's creation time when $k < a^2/\sqrt{\beta}$, i.e. when the physical wavelength of the k-mode becomes larger than λ_{min}. The action of $\hat{H}$ on the Hilbert space is then zero for any modes with k being larger than the cut-off.

This modification effectively resembles a mode creation mechanism for an expanding spacetime with a modified uncertainty principle. However, the above treatment does not uniquely specify the mode evolution once created as already pointed out in [67]. Furthermore, the mode's equation of motion is claimed to be singular *at* the creation time, thus some regularisation procedure will have to be added in order to make physically meaningful interpretations. At last, this mechanism requires that the uncertainty principle be modified in a certain way. Thus this proposal may be seen as a proof of concept but comes with yet to overcome challenges. Within the standard formalism we hence consider this behaviour unphysical.

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List of Figures

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Erklärung

Hiermit bestätige ich, dass die vorliegende Arbeit von mir selbständig verfasst wurde und ich keine anderen als die angegebenen Hilfsmittel - insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen - benutzt habe und die Arbeit von mir vorher nicht einem anderen Prüfungsverfahren eingereicht wurde. Die eingereichte schriftliche Fassung entspricht der auf dem elektronischen Speichermedium. Ich bin damit einverstanden, dass die Masterarbeit veröffentlicht wird.

Hamburg, August 22, 2013

Benedict Broy