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Analyticity of semigroups generated by higher order elliptic operators in spaces of bounded functions on C^1 domains

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Abstract

This paper shows the analyticity of semigroups generated by higher order divergence type elliptic operators in L^∞ spaces when the domain has only C^1 regularity. The domain can be unbounded. For this purpose we establish resolvent estimates in L^∞ spaces by a contradiction argument based on a blow up argument. Our results yield the L^∞ analyticity of solutions of parabolic equations for C^1 domains.

Keywords: Higher order elliptic operator, Analytic semigroup, L^∞ estimate, C^1 domain, Blow up argument.

Mathematics Subject Classification(2010). 35J40

1 Introduction

1.1 Main purpose of our study

The goal of this paper is to establish the analyticity of semigroups generated by divergence type elliptic operators of order $2m$ with the Dirichlet condition in L^∞ spaces on a domain with C^1 boundary. The analyticity results in L^∞ type spaces are often proved in a domain with C^{2m} regularity. The point of this paper is that we only assume that the domain has C^1 boundary no matter how the order of operator is high. Moreover, we give a proof of the resolvent estimates in L^∞ spaces without appealing to results in L^p spaces. Our argument is based on a blow up argument. Instead of stating results for general operators we first discuss the bi-Laplace operator Δ^2 as a simplest example. Let Ω be a C^1 domain in $\mathbb{R}^n$. We consider the resolvent equations

$$\begin{cases} (\lambda+\Delta^2)u = f & \text{in } \Omega \\ u = \partial_N u = 0 & \text{on } \partial\Omega. \end{cases}$$

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Let us state the main results on this operator. We define $M(u,\lambda)$ by

$$M(u,\lambda)=\sup_{x\in\Omega}(|\lambda||u(x)|+|\lambda|^{\frac{3}{4}}|\nabla u(x)|).$$

Theorem 1.1 (L^∞ apriori estimates for $\varDelta^2$ on uniformly C^1 domains). *Let $\Omega\subset\mathbb{R}^n$ be a domain with uniformly C^1 boundary $\partial\Omega$. (We allow Ω to be unbounded.) Then, there exist $\varepsilon>0$, $C>0$, $M>0$ such that*

$$M(u,\lambda)\le C\|f\|_{L^\infty(\Omega)} \tag{1}$$

for all $\lambda\in\Sigma_{\pi-\varepsilon}\cap\{|z|\ge M\}=\{\lambda\in\mathbb{C}:|\arg\lambda|<\pi-\varepsilon\}\cap\{|z|\ge M\}$ and $f\in L^\infty(\Omega)$ and weak solution $u_\lambda\in W^{2,p}_{0,loc}(\Omega)\cap W^{1,\infty}(\Omega)$ $(p>n)$ of the resolvent equation.

Let us illustrate our proof to establish a priori estimate. Our method is a contradiction argument based on a blow up argument. We have four steps to show a priori estimates. The crucial steps are (i) the compactness of a blow up sequence constructed by normalization and rescaling and (ii) the uniqueness of the blow up limit. We shall briefly explain each step.

Outline of the proof of Theorem1.1. We show L^∞ apriori estimates by a blow up argument which was first introduced to analyze non-linear partial differencial equations by E. De Giorgi. Let us explain how this method works.

Step1(Normalization)

First of all, we argue by contradiction, then we can take blow up sequence of weak solutions. In particular, there would exist $\varepsilon>0$ such that for all $k\in\mathbb{N}$ $\exists\lambda_k\in\Sigma_{\pi-\varepsilon}\cap\{|z|\ge k\}$, $f_k\in L^\infty(\Omega)$, $u_k\in W^{2,p}_{0,loc}(\Omega)\cap L^\infty(\Omega)(p>n)$ which is a weak solution of the resolvent equation

$$\begin{cases}(\lambda_k+\varDelta^2)u_k=f_k & \text{in }\Omega\\ u_k=\partial_N u_k=0 & \text{on }\partial\Omega\end{cases} \tag{2}$$

with L^∞ estimates $M(u_k,\lambda_k)>k\|f_k\|_\infty$. We set $v_k=|\lambda_k|u_k$ with $\lambda_k=|\lambda_k|e^{i\theta_k}$ and we normalize the resolvent equation as

$$\tilde{v_k}=\frac{v_k}{M(u_k,\lambda_k)},\ \tilde{f_k}=\frac{f_k}{M(u_k,\lambda_k)}.$$

Then, we get normalized resolvent equations in the weak sence

$$\begin{cases}(e^{i\theta_k}+\dfrac{\varDelta^2}{|\lambda_k|})\tilde{v_k}=\tilde{f_k} & \text{in }\Omega\\ \tilde{v_k}=\partial_N\tilde{v_k}=0 & \text{on }\partial\Omega\end{cases} \tag{3}$$

with the estimates $\frac{1}{k}>\|\tilde{f_k}\|_\infty$, $|\lambda_k|\ge k$, $|\arg\theta_k|\le\pi-\varepsilon$, and $M(\frac{\tilde{v}_k}{|\lambda_k|},\lambda_k)=1$.

Step2(Rescaling)

Secondly, we take a sequence of a point at which each solution takes a value close to its maximum. Note that $\tilde{v_k}$ is included in some Hölder space by the sobolev embedding theorem. Thus, there exists $x_k \in \Omega$ such that

$$|\tilde{v}_k(x_k)| + |\lambda_k|^{-\frac{1}{4}}|\nabla \tilde{v}_k(x_k)| > \frac{1}{2}.$$

Set the rescaled functions as

$$\tilde{w_k} = \tilde{v_k}(x_k + \frac{x}{\mid \lambda_k \mid^{\frac{1}{4}}}),\ \tilde{g_k} = \tilde{f_k}(x_k + \frac{x}{\mid \lambda_k \mid^{\frac{1}{4}}}).$$

Then the rescaled domain Ω_k of $\tilde{w_k}$ and $\tilde{g_k}$ is represented as $|\lambda_k|^{\frac{1}{4}}(\Omega - x_k)$. By changing the variables, we can show that $\tilde{w_k}$ is a weak solution of the rescaled resolvent equation

$$\begin{cases} (e^{i\theta_k} + \Delta^2)\tilde{w_k} = \tilde{g_k} & \text{in } \Omega_k \\ \tilde{w_k} = \partial_N \tilde{w_k} = 0 & \text{on } \partial\Omega_k \end{cases} \tag{4}$$

with $|\tilde{w}_k(0)| + |\nabla \tilde{w}_k(0)| > \frac{1}{2}$, $\tilde{M}(\tilde{w}_k, \lambda_k) = \sup_{x\in\Omega_k}(|\tilde{w}_k(x)| + |\nabla \tilde{w}_k(x)|) = 1$. Finally, we need compactness step and uniqueness step to get a contradiction.

Step3(Compactness)

This step needs local L^p a priori estimates up to boundary for the problem. The actual proof is very involved. We use C^1 regularity to derive such an estimates. In the compactness step, we show equicontinuity of $\{\tilde{w_k}\}_{k\in\mathbb{N}}$ on some open neighborhood near the origin. Take a smooth cut off function $\rho \in C_0^\infty(\mathbb{R}^n)$ and localize $\tilde{w_k}$ as $\tilde{w_k}^\rho = \rho\tilde{w_k}$. Then, $\tilde{w_k}^\rho$ is a weak solution of the localized resolvent equation

$$\begin{cases} (e^{i\theta_k} + \Delta^2)\tilde{w_k}^\rho = \tilde{g_k}\rho + (\text{some lower order terms of } \tilde{w_k}) & \text{in } \Omega_k \cap B_2(0) \\ \tilde{w_k}^\rho = \partial_N \tilde{w_k}^\rho = 0 & \text{on } \partial\Omega_k \text{ and near } \partial B_2(0) \end{cases} \tag{5}$$

In order to apply $W^{2,p}$ estimates, we modify the lower order term by Leibnitz's rule so that the lower order term is included in $W_0^{-2,p}(\Omega_k \cap B_2(0))$, and we have to mollify $\partial(\Omega_k \cap B_2(0))$ on some open neighborhood of $\partial(\Omega_k) \cap \partial(B_2(0))$ so that the boundary become uniformly C^1. By local $W^{2,p}$ estimate obtained by general results of $W^{m,p}$ estimate such as results of Y. Miyazaki [18], S-S. Byun[6], we can show that $\{\tilde{w_k}^\rho\} \subset W_0^{2,p}(\Omega_k \cap B_2(0))$ is uniformly bounded. By the zero extension from $\Omega_k \cap B_2(0)$ to $B_2(0)$ and a compact embedding to some Hölder space, there exists a subsequence $\{\tilde{w_{k_l}}^\rho\}$ of $\{\tilde{w_k}^\rho\}$ such that

$$\tilde{w_{k_l}}^\rho \to \exists w \quad \text{in } B_2(0)\ (l \to \infty) \text{ for some } w.$$

Since $|\tilde{w_{k_l}}^\rho(0)| + |\nabla \tilde{w}_{k_l}^\rho(0)| > \frac{1}{2}$ and $\{\tilde{w_{k_l}}^\rho\}$ converges locally uniformly, we can show that the limit function w of the blow up sequence satisfies

$$|w(0)| + |\nabla w(0)| \geq \frac{1}{2}.$$

Step4(Uniqueness)

By similar arguments in the compactness step, we can show that $\tilde{w_{k_l}} \to w \quad$ in $\Omega_\infty \cap M$ $(l \to \infty)$ for each compact set M, where Ω_∞ is determined by the way that the subsequence $\{x_{k_l}\}$ tends to $\partial\Omega$, i.e.,

$$\Omega_\infty = \begin{cases} \mathbb{R}^n & \text{if } \liminf_{l\to\infty} |\lambda_{k_l}|^{\frac{1}{4}} d(x_{k_l}, \partial\Omega) = \infty \\ \mathbb{R}^n_+ & \text{if } d = \limsup_{l\to\infty} |\lambda_{k_l}|^{\frac{1}{4}} d(x_{k_l}, \partial\Omega) < \infty \end{cases}$$

Let l tend to ∞, then the resolvent equation of $\tilde{w_{k_l}}$ tends to the limit equation

$$\begin{cases} (e^{i\theta_\infty} + \Delta^2)w = 0 & \text{in } \Omega_\infty \\ w = \partial_N w = 0 & \text{on } \partial\Omega_\infty \end{cases} \tag{6}$$

Then, by integrating by parts, we obtain

$$\int_{\Omega_\infty} w(e^{i\theta_\infty} + \Delta^2)\phi dx = 0 \text{ for all smooth test functions } \phi \in C_0^\infty(\Omega_\infty).$$

So, we consider the dual problem of this limit equation, and we can solve the dual problem by Fourier transformation or partial Fourier transformation. Then we can substitute ψ for $(e^{i\theta_\infty} - \Delta^2)\phi$ in the limit equation, and we get

$$\int_{\Omega_\infty} w\psi dx = 0 \text{ for all test function } \psi.$$

Therefore, we get the uniqueness result $w = 0$. This contradicts the result $|w(0)| + |\nabla w(0)| \geq \frac{1}{2}$ in the compactness step. □

1.2 Literature review of analyticity of semigroups and estimates in spaces of bounded functions

In K. Yosida[25], analyticity for second order elliptic operators in C_∞ space is first established when the domain is $(-\infty, \infty)$. K. Masuda[12] considered the case when the operator is a higher order elliptic operator on uniformly C^{2m} domains, and H. B. Stewart[20] improved this method. According to E.Davies[7], Gaussian estimate is valid for any domain when the operator is second order divergence type elliptic operators with L^∞ coefficients and the Dirichlet condition, and M. Hieber[9] established the analyticity in the case when the operator is an elliptic operator of order 2 and the domain is an arbitrary open set by means of Gaussian estimates. After these results, many reserchers dealt with this problem in order to relax the continuity assumption of coefficients. For example, M. Hieber[8] shows analyticity for higher order elliptic operators with VMO coefficients in $L^\infty(\mathbb{R}^n)$. In addition, some bibliographical remarks are seen in A. Lunardi[10]. As to other boundary value problems, K. Taira[22] considers boundary value problems of second order elliptic operators with various boundary conditions. In K. Abe[1], [2], [3], analyticity for Stokes operators in spaces of bounded functions on several domains is established. When we consider a higher order elliptic operator, Masuda-Stewart's method is a well-known method to show the analyticity of semigroups in L^∞ type spaces. In this method, we need $W^{2m,p}$estimates to obtain the analyticity, so we need the assumption that the boundary has uniformly $C^{2m-1,1}$ regularity. Therefore, the anlyticity results of semigroups is well-known when the boundary of domains is uniformly $C^{2m-1,1}$, and we want to know whether we can relax the regularity assumptions of boundaries. In our study, we apply $W^{m,p}$ estimates to get L^∞ estimates by a contradiction argument, and we relax the smoothness assumption of boundaries. For example, we can treat $\varDelta^2$ on a domain with uniformly C^1 boundary.

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2 Main results

Let the domain $\Omega \subset \mathbb{R}^n$ is a domain, L is a divergence type elliptic operator of order $2m$, N is the outward-pointing normal of Ω. More precisely, let m be a positive integer, $L = L_0 + L_1$ be a divergence type differential operator of order $2m$ with the leading term $L_0 = \sum_{|\alpha|,|\beta|=m} (-1)^{m+1} \partial^\beta a_{\alpha,\beta} \partial^\alpha$ of L and the lower order term $L_1 = \sum_{|\alpha|+|\beta| \le 2m-1} (-1)^{|\beta|+1} \partial^\beta a_{\alpha,\beta} \partial^\alpha$ of L. We mainly consider L^∞ resolvent a priori estimates and also consider the existence and uniqueness of the following resolvent equations;

$$\begin{cases} (\lambda - L)u = f & \text{in } \Omega \\ u = \partial_N u = \cdots = \partial_N^{m-1} u = 0 & \text{on } \partial\Omega \end{cases}$$

where the boundary conditions are of the Dirichlet type. We assume the following condition. Let coefficients $a_{\alpha,\beta}$ of L be complex-valued.
Let $b(x,\xi) = \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}(x) \xi^{\alpha+\beta}$ denote the principal symbol of L.

(E1) $a_{\alpha,\beta} \in \begin{cases} W^{1,\infty}(\Omega) & \text{if } |\alpha| = m \\ L^\infty(\Omega) & \text{if } |\alpha| \le m-1 \end{cases}$

(E2) L is uniformly storongly elliptic, i.e., there exists $\delta_L > 0$ such that

$$\mathrm{Re}(b(x,\xi)) \ge \delta_L |\xi|^{2m} \text{ for } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

Now we state resolvent estimates for higher order elliptic operators in L^∞ spaces. The definitions of uniformly C^1 domains and $W^{m,p}_{0,loc}(\Omega)$ are in section 3.1, and that of a weak solution of the resolvent equations is in section 3.3. We define a sectorial set to state L^∞ apriori estimates,

$$\Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\} = \{\lambda \in \mathbb{C} : |\arg \lambda| < \pi - \varepsilon\} \cap \{|z| \geq M\},$$

$$\kappa_L = \sup_{x\in\Omega} \sup_{\xi\in\mathbb{R}^n,\ \xi\neq 0} |\arg b(x,\xi)|.$$

$$N(u,\lambda) = \sup_{x\in\Omega}\Big(\sum_{|\alpha|=k\leq m-1} |\lambda|^{1-\frac{k}{2m}} |\partial^\alpha u(x)|\Big).$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an uniformly C^1 domain, L be a divergence type elliptic operator of order $2m$ with complex coefficients $a_{\alpha,\beta}$. Assume that L satisfy the condition (E1) of coefficients and the ellipticity condition (E3). Then, there exist $\kappa_L < \varepsilon < \frac{\pi}{2}$, $C > 0$, and $M > 0$ such that*

$$N(u,\lambda) \leq C\|f\|_{L^\infty(\Omega)} \tag{7}$$

for $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\}$, $f \in L^\infty(\Omega)$, and $u_\lambda \in W^{m,p}_{0,loc}(\Omega) \cap W^{m-1,\infty}(\Omega)$ $(p > n)$ which is a weak solution of the resolvent equation.

Remark 2.1. (1) If coefficients $a_{\alpha,\beta}$ of L be real-valued, $\kappa_L = 0$. Although the condition of smoothness of coefficients is not optimal in our results, the condition of smoothness of the boundary can be weakened from C^{2m} to C^1. Analyticit results in L^∞ spaces on non smooth domains, such as a Lipshitz domain, are still unknown.

(2) In previous works, non divergence type operators can be treated. Divergence type operator can be treated in our results, and operators of both types coincide when coefficients have C^m regularity. So, analyticity results of non divergence type operators in both L^p and L^∞ spaces are known when (a) coefficients have C^m regularity and boundary has C^1 regularity or (b) the boundary has C^{2m} regularity.

(3) Since we show this estimate by a contradiction argument, we don't know what variable a constant C depend on explicitly. We only know that a constant $C > 0$ is independent of λ, f, u_λ.

Now we state analyticity of semigroups generated by higher order elliptic operators. The definition of sectoriality is in section 4. In order to state L^∞ analyticities, we define the following notation.

$$D(L) = \{u \in \cap_{p>\frac{n}{m}} W^{m,p}_{0,loc}(\Omega) :\ u,\ Lu \in L^\infty(\Omega)\}.$$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain with uniformly C^1 boundary. Then, The operator L: $D(L) \to L^\infty(\Omega)$ is sectorial and generates an analytic semigroup in $L^\infty(\Omega)$.*

3 A priori estimates

3.1 Compactness

We start with the definition of a uniform C^1 domain and a Raifenberg flat domain. The definition of a C^1 domain is found in Y. Miyazaki[18], Chapter 6, and that of a Raifenberg flat domain is found in S-S. Byun[6], Chapter2.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain.*

(i) We say that Ω is a uniform C^1 domain if there exist a family of open sets $\{U_s\}_{s\in\Gamma}$ with countable index set Γ, $N \in \mathbb{N}_{>0}$, $d > 0$, $M_\Omega > 0$, and a non-decreasing function $\omega_\Omega : [0,\infty) \to [0, 2m_\Omega]$ satisfying $\lim_{x\to 0}\omega_\Omega(x) = 0$ such that the following conditions hold:

(a) Any $N+1$ distinct sets of $\{U_s\}_{s\in\Gamma}$ have an empty section.

(b) For each $x \in \partial\Omega$ there exists $s \in \Gamma$ such that

$$B_d(x) = \{y \in \mathbb{R}^n : \ |x-y|_{\mathbb{R}^n} < d\} \subset U_s.$$

(c) For each $s \in \Gamma$ there exist a transformation $T_s : \mathbb{R}^n \to \mathbb{R}^n$ which is composed of a rotation and a translation of a coodinate system, and a uniform C^1 function $\phi_s : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$|\partial_j \phi_s(x')| \le M_\Omega \ for \ x' \in \mathbb{R}^{n-1} \ and \ 1 \le j \le n,$$

$$|\partial_j \phi_s(x') - \partial_j \phi_s(y')| \le \omega_\Omega(|x'-y'|) \ for \ x', y' \in \mathbb{R}^{n-1} \ and \ 1 \le j \le n,$$

and that $T_s(U_s \cap \Omega) = T_s(U_s) \cap \{x = (x', x_n) \in \mathbb{R}^n : \ x' \in \mathbb{R}^{n-1}, \ x_n > \phi_s(x')\}$.

(ii) We say that Ω is a (δ, R) Raifenberg flat domain if there exist a family of open sets $\{U_s\}_{s\in\Gamma}$ with countable index set Γ, $\delta > 0$ and $R > 0$ such that the following conditions hold:

(a) Any $N+1$ distinct sets of $\{U_s\}_{s\in\Gamma}$ have an empty section.

(b) For each $x \in \partial\Omega$ and $0 < r \le R$ there exists $s \in \Gamma$, a transformation $T_s : \mathbb{R}^n \to \mathbb{R}^n$ which is composed of a rotation and a translation of a coodinate system a continuous function $\phi_s : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$B_r(x) \subset U_s, \ T_s(x) = 0, \ |\phi_s(x')| < \delta r \text{ for } x' \in \mathbb{R}^{n-1},$$

and that $T_s(U_s \cap \Omega) = T_s(U_s) \cap \{x = (x', x_n) \in \mathbb{R}^n : \ x' \in \mathbb{R}^{n-1}, \ x_n > \phi_s(x')\}$.

By a straight forward calculation, we get the following lemma.

Lemma 3.1. *Let Ω be a uniform C^1 domain. Then, Ω is a $(\omega_\Omega(R), R)$ Raifenberg flat domain.*

Definition 3.2. *We say that a function $u \in W^{m,p}_{loc}(\Omega)$ is in $W^{m,p}_{0,loc}(\Omega)$ if for all smooth function $\eta \in C_0^\infty(\mathbb{R}^n)$ u satisfies $u\eta \in W_0^{m,p}(\Omega)$.*

Remark 3.1. *the space* $W^{m,p}_{loc}(\Omega)$ *can be characterized similarly. A characterization of the space* $W^{m,p}_{loc}(\Omega)$ *is as follows:* u *is in* $W^{m,p}_{loc}(\Omega)$ *if and only if for all smooth function* $\eta \in C_0^\infty(\Omega)$ u *satisfies* $u\eta \in W^{m,p}(\Omega)$. *We need this space when we show the existence of weak solution.*

In the compactness step, we show the equicontinuity of the normalized blow up sequence $\{\tilde{w_k}\}_{k\in\mathbb{N}}$ on some open neighborhood near the origin. In order to show this, we need local $W^{m,p}$ estimates:

Definition 3.3. *We say that a pair of domain and operator* (Ω, L) *have a* $W^{m,p}$*estimate if there exists a constant* $C > 0$ *independent of* u *such that*

$$\|u\|_{W_0^{m,p}(\Omega)} \le C(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^{,p}(\Omega)}) \text{ for all } u \in W_0^{m,p}(\Omega).$$

Examples of pairs (Ω, L) are obtained by the following theorems. More detailed statements are seen in G. Simader[19], Y. Miyazaki[18], S-S. Byun[6]. In V. Maz'ya[14], the cases of higher order elliptic systems with bounded coefficients in bounded Lipshitz domains are considerd.

Theorem 3.1. (G. Simader[19]) *Let* $p \in (1, \infty)$, *and* $\Omega \subset \mathbb{R}^n$ *be a bounded* C^m *domain. Then, there exists* $C > 0$ *such that if* $Lu = f$, *then*

$$\|u\|_{W^{m,p}} \le C(\|f\|_{L^p} + \|u\|_{L^p}).$$

Theorem 3.2. (Y. Miyazaki[18]) *Let* $\Omega \subset \mathbb{R}^n$ *be a uniform* C^1 *domain,* $p \in (1, \infty)$, *and* $\epsilon \in (\kappa_L, \frac{\pi}{2})$. *Then, there exist* $M > 0$, $C > 0$ *such that for all* $\lambda \in \Sigma_{\pi-\epsilon} \cap \{|z| \ge M\}$ *the resolvent* $(\lambda - L)^{-1}$ *exists with the estimate*

$$\|(\lambda - L)^{-1}\|_{L(W^{-i,p}, W^{j,p})} \le C|\lambda|^{-1+\frac{i+j}{2m}}.$$

Theorem 3.3. (L^p version of S-S. Byun[6]) *Let* $c_0 > 0$, $c_1 > 0$, $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* L *be elliptic operator of order* $2m$ *with real coefficients* $a_{\alpha,\beta}$. *Assume that coefficients* $\{a_{\alpha,\beta}\}$ *satisfy* $\max_{a_{\alpha,\beta}} \|a_{\alpha,\beta}\|_\infty \le c_0$ *and the ellipticity constant* δ_L *satisfies* $\delta_L \le c_1$. *Then, there exists a small constant* $\delta = \delta(c_0, c_1, m, n, p) > 0$ *such that if* $a_{\alpha,\beta}$ *are* (δ, R)*vanishing, and* Ω *is* (δ, R) *Raifenberg flat domain, then for all* $f \in L^p(\Omega)$ *there exists the weak solution* $u_\lambda \in W_0^{m,p}(\Omega)$ *of the resolvent equation with the estimate*

$$\|u\|_{W^{m,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{8}$$

Remark 3.2. *In S-S. Byun[6], Estimates in the setting of Orlicz space are established. This* $W^{m,p}$ *estimates is obtained as a corollary of the results of S-S Byun[6] together with uniqueness results.*

In order to apply $W^{m,p}$ estimates to our compactness argument, we need a Caccioppoli type estimate. Our proof of this lemma is slightly based on arguments seen in Simader[19]. In A. Lunardi[10], such estimates of strong solutions are established when the operator is an non divergence type operator. Define the bilinear form $B[\cdot,\cdot]$ by

$$B[u,\phi] = \tilde{B}[u,\phi] + B_1[u,\phi]$$
$$= \sum_{|\alpha|,|\beta|\le m} (a_{\alpha,\beta}\partial^\alpha u^\rho, \partial^\beta \phi) + \sum_{|\alpha|+|\beta|\le 2m-1} (a_{\alpha,\beta}\partial^\alpha (u\rho_{\alpha,\beta}), \partial^\beta\phi).$$

For $R \ge 1$, $r \le 1$, $x_0 \in \mathbb{R}^n$, take $\rho,\ \rho_{\alpha,\beta} \in C_0^\infty(\mathbb{R}^n)$ which satisfy

$$\rho,\ \rho_{\alpha,\beta} = \begin{cases} 1 & \text{on } B_r(x_0) \\ 0 & \text{outside of } B_{(R+1)r}(x_0) \end{cases}, \ \|\rho\|_\infty \le 1,\ \|\rho_{\alpha,\beta}\|_{m,\infty} \le \frac{K}{Rr}.$$

Lemma 3.2. *Let $u \in W^{m,p}_{0,loc}(\Omega)$, $f \in L^\infty(\Omega)$. Assume that a pair $(L_0, B_r(x_0) \cap \Omega)$ of the leading order term L_0 with its weak form $\tilde{B}_0[u,\cdot]$ has a $W^{m,p}$ estimate with a constant C. Then, there exists $C'(m,p,N,C,a_{\alpha,\beta},K) > 0$ such that*

$$\|u\|_{W^{m,p}(B_r(0)\cap\Omega)}$$
$$\le C'(\sup_{\|\phi\|_{m,q}=1} |B[u,\phi]| + \frac{1}{Rr}\|u\|_{W^{m-1,p}(B_{(R+1)r}(x_0)\cap\Omega)} + \|u\|_{L^p(B_{(R+1)r}(x_0)\cap\Omega)}).$$

Proof. Set

$$\tilde{B}_0[u,\phi] = \sum_{|\alpha|=|\beta|=m} (a_{\alpha,\beta}\partial^\alpha u^\rho, \partial^\beta\phi),$$
$$\tilde{B}_1[u,\phi] = \sum_{|\alpha|+|\beta|\le 2m-1} (a_{\alpha,\beta}\partial^\alpha (u(\rho+\rho_{\alpha,\beta})), \partial^\beta\phi).$$

We estimate $|\tilde{B}_1[u,\phi]|$. If $|\alpha| = m$, we can take $0 \ne e_i < \alpha$ for some $1 \le i \le n$. Since $a_{\alpha,\beta} \in W^{1,\infty}(\Omega)$ for $|\alpha| = m$,

$$a_{\alpha,\beta}\partial^\alpha\{u(\rho+\rho_{\alpha,\beta})\} = \partial^{e_i}(a_{\alpha,\beta}\partial^{\alpha-e_i}\{u(\rho+\rho_{\alpha,\beta})\}) - (\partial^{e_i}a_{\alpha,\beta})(\partial^{\alpha-e_i}\{u(\rho+\rho_{\alpha,\beta})\}).$$

By Hölder inequality,

$$|\tilde{B}_1[u,\phi]|$$
$$\le \sum_{|\alpha|=m,|\beta|\le m-1} \|\partial^{e_i}(a_{\alpha,\beta}\partial^\beta\phi)\|_q \|\partial^{\alpha-e_i}(u(\rho_{\alpha,\beta}+\rho))\|_p$$
$$+ \sum_{|\alpha|\le m-1} \|a_{\alpha,\beta}\partial^\beta\phi\|_q \|\partial^\alpha(u(\rho_{\alpha,\beta}+\rho))\|_p$$
$$\le C_{m,N}\max_{\alpha,\beta}\|a_{\alpha,\beta}\|_{1,\infty}(\sum_{|\alpha|+|\beta|\le 2m-1}\|u\rho_{\alpha,\beta}\|_{m-1,p} + \|u^\rho\|_{m-1,p})\|\phi\|_{m,q}.$$

So we obtain

$$\sup_{\|\phi\|_{m,q}=1} |\tilde{B}_0[u,\phi]|$$
$$\le \sup_{\|\phi\|_{m,q}=1} |B[u,\phi]| + C_{m,N}\max_{\alpha,\beta}\|a_{\alpha,\beta}\|_{1,\infty}(\sum_{|\alpha|+|\beta|\le 2m-1}\|u\rho_{\alpha,\beta}\|_{m-1,p} + \|u^\rho\|_{m-1,p}). \tag{9}$$

Since supp $\rho \subset B_{(R+1)r}(x_0)$ is compact, without loss of generality, we may modify $\partial(B_{(R+1)r}(x_0) \cap \Omega)$ and can apply $W^{m,p}$ estimate to u^ρ.

Applying Hahn-Banach extension theorem to $\tilde{B}_0[u, \cdot]$, and an embedding $i : W^{m,q}(B_{(R+1)r}(x_0) \cap \Omega) \to \bigoplus_{|\alpha|\le m} L^q(B_{(R+1)r}(x_0) \cap \Omega)$ where $i(u) = \{\partial^\alpha u\}_\alpha$ and $\|\{u_\alpha\}_\alpha\| = (\sum_\alpha \|u_\alpha\|_{L^1}^q)^{\frac{1}{q}}$,

$$\exists f \in (\bigoplus_{|\alpha|\le m} L^q(B_{(R+1)r}(x_0) \cap \Omega))^* \text{ s.t. } f|_{W_0^{m,q}}(v) = \tilde{B}_0[u, v],$$

where $\|f\|_{(\bigoplus_{|\alpha|\le m} L^q(B_{(R+1)r}(x_0)\cap\Omega))^*} = \sup_{\|\phi\|_{m,q}=1} |\tilde{B}_0[u, \cdot]|$.

Since $\|f|_{W_0^{m,q}}\|_{W^{-m,p}} \le \|f\|_{(\bigoplus_{|\alpha|\le m} L^q(B_{(R+1)r}(x_0)\cap\Omega))^*}$, applying local $W^{m,p}$ estimates for $\tilde{B}_0$,

$$\|u^\rho\|_{W^{m,p}} \le C(\sup_{\|\phi\|_{m,q}=1} |\tilde{B}_0[u, \phi]| + \|u^\rho\|_{L^p}).$$

Thus, we obtain

$$\begin{aligned}
&\|u^\rho\|_{W^{m,p}} \\
&\le C(\sup_{\|\phi\|_{m,q}=1} |B[u, \phi]| + \|u^\rho\|_{L^p}) \\
&+ C_{C,m,N} \max_{\alpha,\beta} \|a_{\alpha,\beta}\|_{1,\infty}(\sum_{|\alpha|+|\beta|\le 2m-1} \|u\rho_{\alpha,\beta}\|_{m-1,p} + \|u^\rho\|_{m-1,p})
\end{aligned} \tag{10}$$

By the interpolation inequality on order of smoothness, for small $0 < s$,

$$\begin{aligned}
&\le C_{m,p,N,C,a_{\alpha,\beta}}(\sup_{\|\phi\|_{m,q}=1} |B[u, \phi]| + \frac{1}{rR}\|u\|_{W^{m-1,p}(B_{(R+1)r}(x_0)\cap\Omega)} \\
&+ s\|u^\rho\|_{m,p} + s^{-m+1}\|u^\rho\|_p).
\end{aligned}$$

Therefore, take s so that $C_{m,p,N,C,a_{\alpha,\beta}} s \le \frac{1}{2}$, we get results of our lemma

$$\begin{aligned}
&\|u\|_{W^{m,p}(B_r(0)\cap\Omega)} \le \|u^\rho\|_{W^{m,p}} \\
&\le C'(\sup_{\|\phi\|_{m,q}=1} |B[u, \phi]| + \frac{1}{Rr}\|u\|_{W^{m-1,p}(B_{(R+1)r}(x_0)\cap\Omega)} + \|u\|_{L^p(B_{(R+1)r}(x_0)\cap\Omega)}).
\end{aligned} \tag{11}$$

□

Now we state the equicontinuity in the compactness step. Assume that a rescaled uniformly C^1 domain Ω_k, rescaled sequences $\{\lambda_k\}_{k\in\mathbb{N}} = \{|\lambda_k|e^{i\theta_k}\}_{k\in\mathbb{N}}$, $\{\tilde{w_k}\}_{k\in\mathbb{N}}$, and $\{\tilde{g_k}\}_{k\in\mathbb{N}} \subset L^\infty(\Omega_k)$ satisfy the following properties:

(i) For each $k \in \mathbb{N}$ $\tilde{w_k} \in W^{m,p}_{0,loc}(\Omega_k) \cap W^{m-1,\infty}(\Omega_k)$ is a weak solution of the normalized resolvent equation

$$\begin{cases} (e^{i\theta_k} - L_k)\tilde{w_k} = \tilde{g_k} & \textit{in } \Omega_k \\ \tilde{w_k} = \partial_N \tilde{w_k} = \cdots = \partial_N^{m-1}\tilde{w_k} = 0 & \textit{on } \partial\Omega_k, \end{cases}$$

that is, for all $\phi \in C_0^\infty(\Omega_k)$ $\tilde{w_k}$ satisfies

$$e^{i\theta_k}(\tilde{w_k}, \phi)_{L^2(\Omega_k)} + \sum_{|\alpha|,|\beta| \le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}, \partial^\beta \phi)_{L^2(\Omega_k)} = (\tilde{g_k}, \phi)_{L^2(\Omega_k)},$$

where L_k is a uniformly elliptic operator.

(ii) $\{\tilde{w_k}\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega_k)$ and $\{\tilde{g_k}\}_{k\in\mathbb{N}} \subset L^\infty(\Omega_k)$ are uniformly bounded: that is, there exists $K > 0$ such that $\|\tilde{w_k}\|_{m-1,\infty} + \|\tilde{g_k}\|_\infty \le K$.

(iii) As k tends to ∞, $\{|\lambda_k|\}_{k\in\mathbb{N}}$ tends to ∞.

Proposition 3.1. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$, $\{\tilde{w_k}\}_{k\in\mathbb{N}}$, and $\{\tilde{g_k}\}_{k\in\mathbb{N}}$ satisfy the properties (i), (ii), (iii). Then, there exists a sub sequence $\{\tilde{w_{k_l}}\}_{l\in\mathbb{N}}$ of $\{\tilde{w_k}\}_{k\in\mathbb{N}}$ which converges to some function $\tilde{w}$ uniformly on some open neighborhood near the origin. Particulary, if $\{\tilde{w_k}\}_{k\in\mathbb{N}}$ satisfies the following additional condition:*

(iv) $|\tilde{w_k}(0)| + |\nabla\tilde{w_k}(0)| > \frac{1}{2}$.

Then, we get $|\tilde{w}(0)| + |\nabla\tilde{w}(0)| \ge \frac{1}{2}$.

Proof. Take a smooth cut off function $\rho \in C_0^\infty(\mathbb{R}^n)$ s.t. for $R \ge 1$, $r \le 1$, $x_0 \in \mathbb{R}^n$,

$$\rho = \begin{cases} 1 & \text{on } B_r(x_0) \\ 0 & \text{outside of } B_{(R+1)r}(x_0) \end{cases}, \ \|\rho\|_\infty \le 1,\ \|\partial^\alpha \rho\|_{m,\infty} \le \frac{K}{Rr} \text{ for } 1 \le |\alpha|.$$

Localize $\tilde{w_k}$ by setting $\tilde{w_k}^\rho = \rho\tilde{w_k}$. Then, $\tilde{w_k}^\rho$ is a weak solution of the localized resolvent equation

$$\begin{cases} (e^{i\theta_k} - L_k)\tilde{w_k}^\rho + I = \tilde{g_k}\rho & \text{in } \Omega_k \cap B_{(R+1)r}(x_0) \\ \tilde{w_k}^\rho = \partial_N\tilde{w_k}^\rho = \cdots = \partial_N^{m-1}\tilde{w_k}^\rho = 0 & \text{on } \partial\Omega_k \text{ and near } \partial B_{(R+1)r}(x_0) \end{cases} \tag{12}$$

where I is lower order terms of $\tilde{w_k}$. Now, we calculate the weak form of I more precisely. By Leibnitz's rule

$$\begin{aligned} &\sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha\tilde{w_k}^\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\ &= \sum_{|\alpha|,|\beta|\le m}\sum_{\gamma<\alpha} C^\alpha_\gamma |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\gamma\tilde{w_k}\partial^{\alpha-\gamma}\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\ &+ \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}(\partial^\alpha\tilde{w_k})\rho, \partial^\beta\phi)_{L^2(\Omega_k)}, \end{aligned}$$

where $C^\alpha_\gamma = \frac{\alpha!}{(\alpha-\gamma)!\gamma!}$. Similarly, we get

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}, \partial^\beta(\phi\rho))_{L^2(\Omega_k)} \\
&= \sum_{|\alpha|,|\beta|\le m}\sum_{\sigma<\beta} C^\beta_\sigma |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}, (\partial^\sigma\phi)(\partial^{\beta-\sigma}\rho))_{L^2(\Omega_k)} \\
&+ \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}, (\partial^\beta\phi)\rho)_{L^2(\Omega_k)}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&e^{i\theta_k}(\tilde{w_k}^\rho, \phi) + \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}^\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\
&= e^{i\theta_k}(\tilde{w_k}^\rho, \phi) + \sum_{|\alpha|,|\beta|\le m}\sum_{\gamma<\alpha} C^\alpha_\gamma |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\gamma \tilde{w_k}\partial^{\alpha-\gamma}\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\
&+ \sum_{|\alpha|,|\beta|\le m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}, \partial^\beta(\phi\rho))_{L^2(\Omega_k)} \\
&- \sum_{|\alpha|,|\beta|\le m}\sum_{\sigma<\beta} C^\beta_\sigma |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}, (\partial^\sigma\phi)(\partial^{\beta-\sigma}\rho))_{L^2(\Omega_k)} \\
&= (\tilde{g_k}\rho, \phi) + \sum_{|\alpha|,|\beta|\le m}\sum_{\gamma<\alpha} C^\alpha_\gamma |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\gamma \tilde{w_k}\partial^{\alpha-\gamma}\rho, \partial^\beta\phi)_{L^2(\Omega_k)} \\
&- \sum_{|\alpha|,|\beta|\le m}\sum_{\sigma<\beta} C^\beta_\sigma |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\alpha \tilde{w_k}, (\partial^\sigma\phi)(\partial^{\beta-\sigma}\rho))_{L^2(\Omega_k)}. \\
&= (\tilde{g_k}\rho, \phi) + I_1 - I_2 \qquad (13)
\end{aligned}$$

In order to apply $W^{m,p}$ estimates, we have to modify the lower order term $I_1 - I_2$ by Leibnitz's rule so that the lower order term is in $W^{-m,p}(\Omega_k \cap B_{(R+1)r}(0))$. By Leibnitz's rule, for functions u, v we get $u(\partial^j v) = \partial^j(uv) - (\partial^j u)v$ and inductively,

$$u(\partial^s v) = \sum_{t\le s} C^s_t (-1)^{s-t}\partial^t((\partial^{s-t}u)v).$$

So, we get

$$\begin{aligned}
I_1 &= \sum_{|\alpha|,|\beta|\le m}\sum_{\tau\le\gamma<\alpha} C^\alpha_\gamma C^\gamma_\tau (-1)^{\gamma-\tau}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\tau(\tilde{w_k}(\partial^{\alpha-\tau}\rho)), \partial^\beta\phi)_{L^2(\Omega_k)}, \\
I_2 &= \sum_{|\alpha|,|\beta|\le m}\sum_{\sigma<\beta}\sum_{\iota\le\alpha} C^\beta_\sigma C^\alpha_\iota (-1)^{\alpha-\iota}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta}\partial^\iota(\tilde{w_k}(\partial^{\alpha+\beta-\iota-\sigma}\rho), (\partial^\sigma\phi))_{L^2(\Omega_k)}.
\end{aligned}$$

Thus we get the weak form $-I_1 + I_2$. We also have to mollify $\partial(\Omega_k \cap B_{(R+1)r}(0))$ on some open neighborhood of $\partial(\Omega_k) \cap \partial(B_{(R+1)r}(0))$ so that the boundary become uniformly C^1. By local $W^{m,p}$ estimates and the previous lemma, there exists $C'' > 0$ such that

$$\begin{aligned}
&\|\tilde{w}_k^{\rho}\|_{W^{m,p}(B_r(x_0)\cap\Omega_k)} \\
&\leq C''\Big(\sup_{\|\phi\|_{m,q}=1}|B[u,\phi]| + \frac{1}{Rr}\|u\|_{W^{m-1,p}(B_{(R+1)r}(x_0)\cap\Omega_k)} + \|u\|_{L^p(B_{(R+1)r}(x_0)\cap\Omega_k)}\Big) \\
&\leq C'''(\|f\|_{L^p(B_{(R+1)r}(x_0))} + \|\tilde{w}_k\|_{W^{m-1,p}(B_{(R+1)r}(x_0))}) \\
&\leq C'''|B_{(R+1)r}|^{\frac{1}{p}}(\|f\|_{\infty} + \|\tilde{w}_k\|_{m-1,\infty}) \leq C'''|B_{(R+1)r}|^{\frac{1}{p}}K.
\end{aligned}$$

Therefore, the sequence $\{\tilde{w_k}^{\rho}\} \subset W_0^{m,p}(B_r(x_0)\cap\Omega_k)$ is uniformly bounded for $r \leq 1$, $x_0 \in \mathbb{R}^n$. By the zero extension from $\Omega_k \cap B_r(0)$ to $B_r(0)$, we get the uniformly boundedness of $\{\tilde{w_k}^{\rho}\} \subset W^{m,p}(B_1(0))$. By the Rellich's compactness theorem, there exists a subsequence $\{\tilde{w_{k_l}}^{\rho}\}$ of $\{\tilde{w_k}^{\rho}\}$ s.t.

$$\tilde{w_{k_l}}^{\rho} \to \exists w \quad \text{uniformly on } B_1(0)\ (l \to \infty).$$

Since $(\tilde{w_{k_l}}^{\rho})(0) > \frac{1}{2}$, we get $w(0) \geq \frac{1}{2}$. Thus we complete the Compactness step. □

3.2 Uniqueness of limit problem

In this chapter, we consider the uniqueness problem in the uniqueness step. Let $\tilde{a}_{\alpha,\beta} \in \mathbb{C}$, and we consider the limit resolvent equation of the following uniformly elliptic operator with constant coefficients:

$$\tilde{L}_0 = \sum_{|\alpha|,|\beta|=m} (-1)^{m+1}\tilde{a}_{\alpha,\beta}\partial^{\alpha+\beta}.$$

We assume that $\tilde{L}_0$ satisfy the ellipticity condition (E2). We have to consider the cases when $\Omega_\infty = \mathbb{R}^n$ or $\Omega_\infty = \mathbb{R}^n_+$. In the uniqueness step, we have to consider the following dual problem: We can solve this dual problem by the general facts written in such as H. Tanabe[24], but, in our case, we can solve our dual problem concretely.

Lemma 3.3 (Dual problem when Ω_∞ is the whole space $\mathbb{R}^n$)**.** *Let ψ be in $C_0^\infty(\mathbb{R}^n)$. Then there exists a solution ϕ in $S(\mathbb{R}^n)$ such that*

$$(e^{i\theta_\infty} - \tilde{L}_0)\phi = \psi \quad \text{in } \mathbb{R}^n \tag{14}$$

Proof. Let $\hat{\phi}$ denote the fourier transformation of ϕ. By Fourier transformation, we get

$$(e^{i\theta_\infty} - (-1)^{m+1}\sum_{|\alpha|,|\beta|=m}\tilde{a}_{\alpha,\beta}(i\xi)^{\alpha+\beta})\hat{\phi} = \hat{\psi} \quad \text{in } \mathbb{R}^n. \tag{15}$$

Since $\tilde{L}_0$ is uniformly elliptic,

$$e^{i\theta_\infty} + \sum_{|\alpha|,|\beta|=m}\tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta} \neq 0.$$

Thus, we get

$$\hat{\phi} = \frac{1}{e^{i\theta_\infty} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}\xi^{\alpha+\beta}} \hat{\psi} \in S(\mathbb{R}^n) \cdot S(\mathbb{R}^n) = S(\mathbb{R}^n).$$

Since $\hat{\phi} \in S(\mathbb{R}^n)$, we get $\phi \in S(\mathbb{R}^n)$. □

Lemma 3.4 (Dual problem when Ω_∞ is the half space $\mathbb{R}^n_+$)**.** *Let ψ be in $C_0^\infty(\mathbb{R}^n)$. Then there exists a solution ϕ in $S(\mathbb{R}^{n-1}) \times C^{2m}_\infty(\mathbb{R}_+)$ such that*

$$\begin{cases} (e^{i\theta_\infty} - \tilde{L_0})\phi = \psi & \text{in } \mathbb{R}^n_+ \\ \phi = \partial_N \phi = \cdots = \partial_N^{m-1}\phi = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

Proof. By the partial Fourier transformation with respect to the variable $x' = (x_1, \cdots, x_{n-1})$ in $\mathbb{R}^{n-1}$, we get the following ordinary differencial equation

$$\begin{cases} (e^{i\theta_\infty} - (-1)^{m+1} \sum_{|\alpha|,|\beta|=m} b_{\alpha,\beta}(0)(i\xi)^{(\alpha_1+\beta_1,\cdots,\alpha_{n-1}+\beta_{n-1},0)}\partial^{\alpha_n+\beta_n}\hat{\phi} = \hat{\psi} & \text{in } \mathbb{R}^n_+ \\ \hat{\phi} = \partial_N\hat{\phi} = \cdots = \partial_N^{m-1}\hat{\phi} = 0 & \text{on } \mathbb{R}^{n-1}\times\{0\}. \end{cases} \tag{16}$$

First of all, let ψ_0 be the zero extention of ψ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, and consider the whole space case. This case is already solved in the previous lemma and we get a solution

$$\phi_0 = F^{-1}\Big(\frac{1}{e^{i\theta_\infty} + \sum_{|\alpha|,|\beta|=m} b_{\alpha,\beta}(0)\xi^{\alpha+\beta}}\Big) * \psi_0 \in S(\mathbb{R}^n).$$

Secondly, let $h(x') = \phi_0(x', 0) \in S(\mathbb{R}^{n-1})$, define $\eta = \phi - \phi_0$, then we consider the following boundary value problem:

$$\begin{cases} (e^{i\theta_\infty} - \tilde{L_0})\eta = 0 & \text{in } \mathbb{R}^n_+ \\ \eta(x', 0) = -h(x') & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

We want to determine the characteristic roots of this ODE. The characteristic equation is

$$e^{i\theta_\infty} - (-1)^{m+1} \sum_{|\alpha|,|\beta|=m} b_{\alpha,\beta}(0)(i\xi)^{(\alpha_1+\beta_1,\cdots,\alpha_{n-1}+\beta_{n-1},0)}t^{\alpha_n+\beta_n} = 0.$$

Since L_0 is strongly uniformly elliptic,

$$e^{i\theta_\infty} - (-1)^{m+1} \sum_{|\alpha|,|\beta|=m} b_{\alpha,\beta}(0)(i\xi)^{(\alpha_1+\beta_1,\cdots,\alpha_{n-1}+\beta_{n-1},0)}(is)^{\alpha_n+\beta_n} \neq 0 \text{ for } s \in \mathbb{R},$$

$t = is$ are not characteristic roots for $s \in \mathbb{R}$. Thus, we get the characteristic roots

$$t_j = \pm p_j + iq_j \text{ for } 1 \leq j \leq m, \text{ and } p_j > 0,\ q_j \in \mathbb{R}.$$

So, we take $t_j = -p_j + iq_j$ for $1 \le j \le N$ and let $m_j \ge 1$ be the multiplicity of t_j. Then, $\hat{\eta}(\xi', x_n) = \displaystyle\sum_{j=1}^{N}\sum_{k=1}^{m_j} c_{k,j}(\xi')(x_n)^{k-1}e^{t_j(\xi')x_n}$ is a general solution which belongs to $C^{2m}_{\infty}(\mathbb{R}_+)$, and by the boundary condition we get

$$\hat{\eta}(\xi', 0) = \sum_{j=1}^{N}\sum_{k=1}^{m_j} c_{k,j}(\xi') = (\hat{g} - \hat{h})(\xi') \in S(\mathbb{R}^{n-1}).$$

Thus, we get

$$\hat{\phi} = \hat{\phi}_0 + \sum_{j=1}^{N}\sum_{k=1}^{m_j} c_{k,j}(\xi')(x_n)^{k-1}e^{t_j(\xi')x_n}.$$

We can show that the term $\sum_{j=1}^{N}\sum_{k=1}^{m_j} c_{k,j}(\xi')(x_n)^{k-1}e^{t_j(\xi')x_n}$ is in $S(\mathbb{R}^{n-1})$ and we get a desired solution. □

Now we consider the uniqueness step.

Lemma 3.5. *Let $|\theta_\infty| \le \pi - \epsilon$, $\tilde{w} \in W^{m,p}_{0,loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a weak solution of the normalized resolvent limit equation*

$$(e^{i\theta_\infty} - \tilde{L_0})\tilde{w} = 0 \quad \text{in } \mathbb{R}^n$$

that is, for all $\phi \in C_0^\infty(\mathbb{R}^n)$ $\tilde{w}$ satisfies

$$e^{i\theta_\infty}(\tilde{w}, \phi)_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(\partial^\alpha \tilde{w}, \partial^\beta \phi)_{L^2(\mathbb{R}^n)} = 0.$$

Then, $w = 0$.

Proof. Integrating by parts, we obtain

$$\int_{\mathbb{R}^n} \tilde{w}(e^{i\theta_\infty} - \tilde{L_0})\phi dx = 0 \text{ for all smooth test functions } \phi \in C_0^\infty(\mathbb{R}^n).$$

Since $C_0^\infty(\mathbb{R}^n)$ is dence in $S(\mathbb{R}^n)$, we can take ϕ in $S(\mathbb{R}^n)$ as a test function. So, we consider the dual problem of the limit equation. For all smooth ψ in $C_0^\infty(\mathbb{R}^n)$ we want to find a solution ϕ in $S(\mathbb{R}^n)$ s.t.

$$(e^{i\theta_\infty} - \tilde{L_0})\phi = \psi \quad \text{in } \mathbb{R}^n \tag{17}$$

Since we already solve this problem by previous lemmas, for all test function ψ in $C_0^\infty(\mathbb{R}^n)$ we can choose ϕ in $S(\mathbb{R}^n)$ s.t.

$$(e^{i\theta_\infty} - \tilde{L_0})\phi = \psi \quad \text{in } \mathbb{R}^n \tag{18}$$

and substitute ψ for $(e^{i\theta_\infty} - \tilde{L_0})\phi$ in the limit equation, then we get

$$\int_{\mathbb{R}^n} \tilde{w}\psi dx = 0 \text{ for all test function } \psi \in C_0^\infty(\mathbb{R}^n).$$

Finally, we approximate $\tilde{w}$ by convoluting with Friedrich's smooth mollifier ρ_ϵ, then the approximate sequence $\tilde{w}_\epsilon$ is equal to be $\int_{\mathbb{R}^n} \tilde{w}\rho_\epsilon dx = 0$ because we can take $\psi = \rho_\epsilon$ as a test function. As $0 = w_\epsilon$ tends to $\tilde{w}$ in $L^1(\mathbb{R}^n)$ space, we get $\tilde{w} = 0$ a.e. $\mathbb{R}^n$. Therefore, we get the uniqueness result $\tilde{w} = 0$ by continuity. □

Lemma 3.6. *Let $|\theta_\infty| \leq \pi - \epsilon$, $\tilde{w} \in W^{m,p}_{0,loc}(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ is a weak solution of the normalized resolvent limit equation*

$$\begin{cases} (e^{i\theta_\infty} - \tilde{L_0})\tilde{w} = 0 & in\ \mathbb{R}^n_+ \\ \tilde{w} = \partial_N \tilde{w} = \cdots = \partial_N^{m-1}\tilde{w} = 0 & on\ \partial\mathbb{R}^n_+, \end{cases}$$

that is, for all $\phi \in C_0^\infty(\mathbb{R}^n_+)$ $\tilde{w}$ satisfies

$$e^{i\theta_\infty}(\tilde{w}, \phi)_{L^2(\mathbb{R}^n_+)} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(\partial^\alpha \tilde{w}, \partial^\beta \phi)_{L^2(\mathbb{R}^n_+)} = 0.$$

Then, $w = 0$.

Proof. Integrating by parts, we obtain

$$\int_{\mathbb{R}^n_+} \tilde{w}(e^{i\theta_\infty} - \tilde{L_0})\phi dx = 0 \text{ for all smooth test functions } \phi \in C_0^\infty(\mathbb{R}^n_+).$$

Since $C_0^\infty(\mathbb{R}^n_+)$ is dence in $S(\mathbb{R}^{n-1}) \times C^{2m}_\infty(\mathbb{R}_+)$, we can take ϕ in $S(\mathbb{R}^n)$ as a test function. So, we consider the dual problem of the limit equation. For all smooth ψ in $C_0^\infty(\mathbb{R}^n)$ we want to find a solution ϕ in $S(\mathbb{R}^{n-1}) \times C_0^\infty(\mathbb{R}_+)$s.t.

$$\begin{cases} (e^{i\theta_\infty} - \tilde{L_0})\phi = \psi & \text{in } \mathbb{R}^n_+ \\ \phi = \partial_N \phi = \cdots = \partial_N^{m-1}\phi = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

Since we already solve this problem by previous lemmas, for all test function ψ in $C_0^\infty(\mathbb{R}^n_+)$ we can choose ϕ in $S(\mathbb{R}^{n-1}) \times C^{2m}_\infty(\mathbb{R}_+)$ s.t.

$$\begin{cases} (e^{i\theta_\infty} - \tilde{L_0})\phi = \psi & \text{in } \mathbb{R}^n_+ \\ \phi = \partial_N \phi = \cdots = \partial_N^{m-1}\phi = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

and substitute ψ for $(e^{i\theta_\infty} - \tilde{L_0})\phi$ in the limit equation, then we get

$$\int_{\mathbb{R}^n_+} \tilde{w}\psi dx = 0 \text{ for all test function } \psi \in C_0^\infty(\mathbb{R}^n_+).$$

Finally, take a arbitrary cut off function $\nu \in C_0^\infty(\mathbb{R}^n_+)$ and set $\tilde{\omega}^\nu = \tilde{\omega}\nu$ in $L^1(\mathbb{R}^n)$. Similarly in the proof of the previous lemma, we approximate $\tilde{w}^\nu$ by convoluting with Friedrich's smooth mollifier ρ_ϵ, then the approximate sequence $\tilde{w}^\nu_\epsilon$ is equal to be $\int_{\mathbb{R}^n_+} \tilde{w}(\nu\rho_\epsilon)dx = 0$ because we can take $\psi = \nu\rho_\epsilon$ as a test function. As $0 = \tilde{w}^\nu_\epsilon$ tends to $\tilde{w}^\nu$ in $L^1(\mathbb{R}^n)$ space, we get $\tilde{w}\nu = 0$ a.e. $\mathbb{R}^n$. Since ν is arbitrary, we get $\tilde{w} = 0$ a.e. $\mathbb{R}^n_+$. Therefore, we get the uniqueness result $\tilde{w} = 0$ in $\mathbb{R}^n_+$ by continuity. □

3.3 Proof of resolvent estimates in spaces of bounded functions

In order to prove resolvent estimates precisely, we define a weak solution of resolvent equation.

Definition 3.4. *Let m be a positive integer, $0 < \epsilon < \frac{\pi}{2}$, $M > 0$, $f \in L^\infty(\Omega)$, and $\lambda \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq M\}$ where $\Sigma_{\pi-\varepsilon} = \{\lambda \in \mathbb{C} : \ |\arg \lambda| < \pi - \varepsilon\}$. Then, we say that $u_\lambda \in W^{m,p}_{0,loc}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of the resolvent equation*

$$\begin{cases} (\lambda - L)u = f & \text{in } \Omega \\ u = \partial_N u = \cdots = \partial_N^{m-1} u = 0 & \text{on } \partial\Omega \end{cases}$$

if u_λ satisfies

$$\lambda(u_\lambda, \phi)_{L^2(\Omega)} + \sum_{|\alpha|,|\beta| \leq m} (a_{\alpha,\beta}\partial^\alpha u_\lambda, \partial^\beta \phi)_{L^2(\Omega)} = (f, \phi)_{L^2(\Omega)} \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Remark 3.3. *By Hölder inequality, Our definition of a weak solution is welldefined.*

proof of resolvent estimates. We argue by contradiction. Let us deny the L^∞estimates. Then we particularly obtain $\exists \varepsilon > 0$ s.t.

$$\forall k \in \mathbb{N}\ \exists \lambda_k \in \Sigma_{\pi-\varepsilon} \cap \{|z| \geq k\},\ f_k \in L^\infty(\Omega),\ u_k \in W^{m,p}_{0,loc}(\Omega) \cap L^\infty(\Omega) (p > n)$$

which is a weak solution of the resolvent equation

$$\begin{cases} (\lambda_k - L)u_k = f_k & \text{in } \Omega \\ u_k = \partial_N u_k = \cdots = \partial_N^{m-1} u_k = 0 & \text{on } \partial\Omega \end{cases} \tag{19}$$

with $N(u_k, \lambda_k) > k\|f_k\|_\infty$ where $N(u, \lambda) = \sup_{x \in \Omega}(\sum_{|\alpha|=k \leq m-1} |\lambda|^{1-\frac{k}{2m}} |\partial^\alpha u(x)|)$.

We set $v_k = |\lambda_k| u_k$ with $\lambda_k = |\lambda_k| e^{i\theta_k}$ and normalize the resolvent equation by setting $\tilde{v_k} = \frac{v_k}{N(u_k,\lambda_k)}$ and $\tilde{f_k} = \frac{f_k}{N(u_k,\lambda_k)}$. Then, we get the following normalized resolvent equations

$$\begin{cases} (e^{i\theta_k} - \dfrac{L}{|\lambda_k|})\tilde{v_k} = \tilde{f_k} & \text{in } \Omega \\ \tilde{v_k} = \partial_N \tilde{v_k} = \cdots = \partial_N^{m-1} \tilde{v_k} = 0 & \text{on } \partial\Omega \end{cases} \tag{20}$$

more precisely, for all $\phi \in C_0^\infty(\Omega)$ $\tilde{v_k}$ satisfies

$$e^{i\theta_k}(\tilde{v_k}, \phi)_{L^2(\Omega)} + \frac{1}{|\lambda_k|} \sum_{|\alpha|,|\beta| \leq m} (a_{\alpha,\beta}\partial^\alpha \tilde{v_k}, \partial^\beta \phi)_{L^2(\Omega)} = (\tilde{f_k}, \phi)_{L^2(\Omega)},$$

with the estimates $\frac{1}{k} > \|\tilde{f}_k\|_\infty$, $|\lambda_k| \geq k$, $|\arg\theta_k| \leq \pi - \varepsilon$, and $N(\frac{\tilde{v}_k}{|\lambda_k|}, \lambda_k) = 1$. Secondly, we want to take a sequence of a point at which each solution takes a value close to its maximum. Since $N(\frac{\tilde{v}_k}{|\lambda_k|}, \lambda_k) = 1$,

$$\exists x_k \in \Omega \ \text{ s.t. } \sum_{|\alpha|=k\leq m-1} |\lambda|^{-\frac{k}{2m}} |\partial^\alpha \tilde{v}_k(x_k)| > \frac{1}{2}. \tag{21}$$

Set the rescaled function as

$$\tilde{w_k} = \tilde{v_k}(x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}}) \text{ and } \tilde{g_k} = \tilde{f_k}(x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}}).$$

Then the rescaled domain Ω_k of $\tilde{w_k}$ and $\tilde{g_k}$ is represented as $|\lambda_k|^{\frac{1}{2m}}(\Omega - x_k)$. By changing the variables $x \in \Omega$ for $y = |\lambda_k|^{\frac{1}{2m}}(x - x_k)$, for all $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|\leq m} (a_{\alpha,\beta}\partial^\alpha \tilde{v_k}, \partial^\beta \phi)_{L^2(\Omega)} \\
&= \sum_{|\alpha|,|\beta|\leq m} \int_\Omega a_{\alpha,\beta}(x)\partial^\alpha \tilde{v_k}(x)\partial^\beta \phi(x)dx \\
&= \sum_{|\alpha|,|\beta|\leq m} \int_{\Omega_k} a_{\alpha,\beta}(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})(\partial^\alpha \tilde{v_k})(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})(\partial^\beta \phi)(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})\frac{1}{|\lambda_k|^{\frac{n}{m}}}dy \\
&= \sum_{|\alpha|,|\beta|\leq m} \int_{\Omega_k} b_{\alpha,\beta,k}(y)|\lambda_k|^{\frac{|\alpha|}{2m}}\partial^\alpha(\tilde{v_k}(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}}))|\lambda_k|^{\frac{|\beta|}{2m}}\partial^\beta(\phi(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}}))\frac{1}{|\lambda_k|^{\frac{n}{m}}}dy \\
&= \sum_{|\alpha|,|\beta|\leq m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m} - \frac{n}{m}}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w_k}, \partial^\beta \eta)_{L^2(\Omega_k)}
\end{aligned}$$

where $b_{\alpha,\beta,k}(y) = a_{\alpha,\beta}(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}})$, and $\eta(x) = \phi(x_k + \frac{y}{|\lambda_k|^{\frac{1}{2m}}}) \in C_0^\infty(\Omega_k)$. Similarly,

$$(\tilde{f_k}, \phi)_{L^2(\Omega)} = \frac{1}{|\lambda_k|^{\frac{n}{m}}}(\tilde{g_k}, \eta)_{L^2(\Omega_k)}, e^{i\theta_k}(\tilde{v_k}, \phi)_{L^2(\Omega)} = \frac{1}{|\lambda_k|^{\frac{n}{m}}} e^{i\theta_k}(\tilde{w_k}, \eta)_{L^2(\Omega_k)}.$$

Thus, $\tilde{w_k}$ is a weak solution of the following rescaled resolvent equation

$$\begin{cases} (e^{i\theta_k} - \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\beta|+1}|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(-1)^{m+1}\partial^\beta b_{\alpha,\beta,k}\partial^\alpha)\tilde{w_k} = \tilde{g_k} & \text{in } \Omega_k \\ \tilde{w_k} = \partial_N \tilde{w_k} = \cdots = \partial_N^{m-1}\tilde{w_k} = 0 & \text{on } \partial\Omega_k \end{cases} \tag{22}$$

more precisely, for all $\eta \in C_0^\infty(\Omega_k)$ $\tilde{w_k}$ satisfies

$$e^{i\theta_k}(\tilde{w_k}, \eta)_{L^2(\Omega_k)} + \sum_{|\alpha|,|\beta|\leq m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1}(b_{\alpha,\beta,k}\partial^\alpha \tilde{w_k}, \partial^\beta \eta)_{L^2(\Omega_k)} = (\tilde{g_k}, \eta)_{L^2(\Omega_k)},$$

with L^∞ estimates $\frac{1}{k} > \|\tilde{g_k}\|_\infty$, $\|\tilde{N}(\tilde{w_k})(x)\|_\infty = 1$, and $\tilde{N}(\tilde{w_k})(0) > \frac{1}{2}$ where $\tilde{N}(\tilde{w_k})(x) = \sum_{|\alpha|=k\leq m-1} |\partial^\alpha \tilde{w_k}(x)|$. Appling the proposition in the compactness step to our case, there exists a sub sequence $\{\tilde{w_{k_l}}\}_{l\in\mathbb{N}}$ of $\{\tilde{w_k}\}_{k\in\mathbb{N}}$ which converges to some function $\tilde{w}$ uniformly on some open neighborhood near the origin. Particulary, we get $\tilde{N}(\tilde{w})(0) \geq \frac{1}{4}$. Finally, we need the uniqueness result $w = 0$ to get a contradiction. Set $d_k = d_{\Omega_k}(0, \partial\Omega_k) = |\lambda_k|^{\frac{1}{2m}} d_\Omega(x_k, \partial\Omega)$. In order to apply the results in the uniqueness step, we have to show the convergence of each term of resolvent equations and the rescaled domain.

Case(i) $\tilde{d} = \liminf_{k\to\infty} d_k = \liminf_{k\to\infty} |\lambda_k|^{\frac{1}{2m}} d(x_k, \partial\Omega) = \infty$.

First of all, we show Ω_k tends to $\mathbb{R}^n$. For $r > 0$, there exists $k_0 \in \mathbb{N}$ s.t. $B_r(0) \subset \Omega_{k_0}$. So, for each smooth test function $\eta \in C_0^\infty(\mathbb{R}^n)$, there exists $k_\eta \in \mathbb{N}$ s.t. supp $\eta \subset \Omega_k$ for $k_\eta \leq k$. Secondly, we show the convergence of each rescaled terms. Since supp η is compact, we substitute a larger sub domain with C^1boundary for supp η. Then we apply similar argument in the compactness step to this case, there exists a subsequence $\{\tilde{w_{k_l}}\}$ of $\{\tilde{w_k}\}$ s.t.

$$\tilde{w_{k_l}} \to \exists w \quad \text{on } W^{m,p}(\text{supp } \eta)\ (l \to \infty).$$

Now we have to consider the convergence of the terms

$$|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} b_{\alpha,\beta,k}(x) = |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} a_{\alpha,\beta}(x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}}) \text{ on supp } \eta.$$

For $|\alpha| + |\beta| \leq 2m - 1$,

$$|\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} \|b_{\alpha,\beta,k}\|_\infty = |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} \|a_{\alpha,\beta}\|_\infty \to 0\ (k \to \infty).$$

For $|\alpha| = |\beta| = m$, since $a_{\alpha,\beta}$ are uniformly continuous, for $\epsilon > 0$ there exists $\delta > 0$ s.t. if $|x_k + \frac{x}{|\lambda_k|^{\frac{1}{2m}}} - x_k| = |\frac{x}{|\lambda_k|^{\frac{1}{2m}}}| < \delta$ then $|b_{\alpha,\beta,k}(x) - b_{\alpha,\beta,k}(0)| < \epsilon$. Since $x \in$ supp η and $|\lambda_k| \to \infty$, there exists k_0 s.t. $|\frac{x}{|\lambda_k|^{\frac{1}{2m}}}| < \delta$ for $k_0 \leq k$. Since $|a_{\alpha,\beta}(x_k)| \leq \|a_{\alpha,\beta}\|_\infty$, there exist a constant $\tilde{a}_{\alpha,\beta}$ and a subsequence $\{b_{\alpha,\beta,k_l}\}$ s.t. $b_{\alpha,\beta,k_l}(0) = a_{\alpha,\beta}(x_{k_l}) \to \tilde{a}_{\alpha,\beta}$ $(l \to \infty)$. Then, for $k_0 \leq k$

$$\begin{aligned} |b_{\alpha,\beta,k_l}(x) - \tilde{a}_{\alpha,\beta}| &\leq |b_{\alpha,\beta,k_l}(x) - b_{\alpha,\beta,k_l}(0)| + |b_{\alpha,\beta,k_l}(0) - \tilde{a}_{\alpha,\beta}| \\ &\leq \epsilon + |b_{\alpha,\beta,k_l}(0) - \tilde{a}_{\alpha,\beta}| \to \epsilon\ (l \to \infty). \end{aligned}$$

Since ϵ and $x \in$ supp η are arbitrary, we get $\|b_{\alpha,\beta,k_l}(x) - \tilde{a}_{\alpha,\beta}\|_\infty \to 0$ $(l \to \infty)$. As l tends to ∞, the rescaled equation

$$e^{i\theta_k}(\tilde{w_k}, \eta)_{L^2(\Omega_k)} + \sum_{|\alpha|,|\beta|\leq m} |\lambda_k|^{\frac{|\alpha|+|\beta|}{2m}-1} (b_{\alpha,\beta,k}\partial^\alpha \tilde{w_k}, \partial^\beta \eta)_{L^2(\Omega_k)} = (\tilde{g_k}, \eta)_{L^2(\Omega_k)}$$

tends to

$$e^{i\theta_\infty}(w, \eta)_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|=|\beta|=m} \tilde{a}_{\alpha,\beta}(\partial^\alpha w, \partial^\beta \eta)_{L^2(\mathbb{R}^n)} = 0,$$

where $|\theta_\infty| \leq \pi - \epsilon$. Therefore, w is a weak solution of the limit resolvent equation

$$(e^{i\theta_\infty} - \tilde{L_0})\tilde{w} = 0 \qquad in\ \mathbb{R}^n,$$

where $\tilde{L}_0 = \sum_{|\alpha|=|\beta|=m} (-1)^{m+1}\tilde{a}_{\alpha,\beta}\partial^{\alpha+\beta}$. So, we can apply the uniqueness step and we get $w = 0$.

Case(ii) $\tilde{d} = \liminf_{k\to\infty} d_k = \liminf_{k\to\infty}|\lambda_k|^{\frac{1}{2m}} d(x_k, \partial\Omega) < \infty$.

In this case, we show Ω_k tends to $\mathbb{R}^n_+$. There exists a subsequence $\{d_{k_l}\}$ of $\{d_k\}$ s.t. $\lim_{l\to\infty} d_{k_l} = \tilde{d}$. Since rotations and translations preserve the ellipticity and Ω is uniformly C^1, without loss of generality, we may assume that the perpendicular from x_{k_l} to $\partial\Omega$ coincides with the x_n-axis and $\tilde{d} = 0$, i.e., we may assume Ω_{k_l} tends to $\mathbb{R}^n_+$ as $l \to \infty$. Let y_l be the point of this intersection. For each smooth test function $\eta \in C_0^\infty(\mathbb{R}^n_+)$, $d(\text{supp }\eta, \partial\mathbb{R}^n_+) = d_\eta > 0$. Since $y_l \to 0$, we can take large $R_\eta > 0$ so that for all large $l \in \mathbb{N}$ $B_{R_\eta}(y_l) \supset$supp η. By Lemma 2.1., Ω is a $(\omega_\Omega(R_\eta),\ R_\eta)$ Reifenberg flat domain where ω_Ω is the nondecreasing function in Definition 2.1.. Since $\Omega_l = |\lambda_l|^{\frac{1}{2m}}(x_l - \Omega)$, Ω_l is a $(\omega_\Omega(\frac{R_\eta}{|\lambda_l|^{\frac{1}{2m}}}),\ R_\eta)$ Reifenberg frat domain. By the Reifenberg frat condition,

$$B_{R_\eta}(y_l) \cap \Omega_k \supset B_{R_\eta}(y_l) \cap \{x_n > \omega_\Omega(\frac{R_\eta}{|\lambda_l|^{\frac{1}{2m}}})R_\eta\}$$

Since $\omega_\Omega(|h|) \to 0$ as $|h| \to 0$, we can take large l_0 so that for $l \geq l_0$

$$\begin{aligned} &\supset B_{R_\eta}(y_l) \cap \{x_n > d_\eta\} \\ &\supset B_{R_\eta}(y_l) \cap \text{supp }\eta. \end{aligned}$$

As a consequence, for $l_0 \leq l$ supp η is included in Ω_l. So, we can apply the similar argument in case (i), there exists l_0 s.t. for $l_0 \leq l$ supp $\eta \subset \Omega_l$, and furthermore w is a weak solution of the limit resolvent equation

$$\begin{cases} (e^{i\theta_\infty} - \tilde{L_0})\tilde{w} = 0 & in\ \mathbb{R}^n_+ \\ \tilde{w} = \partial_N\tilde{w} = \cdots = \partial_N^{m-1}\tilde{w} = 0 & on\ \partial\mathbb{R}^n_+, \end{cases}$$

that is, for all $\phi \in C_0^\infty(\mathbb{R}^n_+)$ $\tilde{w}$ satisfies

$$e^{i\theta_\infty}(\tilde{w}, \phi)_{L^2(\mathbb{R}^n_+)} + \sum_{|\alpha|,|\beta|=m} \tilde{a}_{\alpha,\beta}(\partial^\alpha\tilde{w}, \partial^\beta\phi)_{L^2(\mathbb{R}^n_+)} = 0.$$

Then, by the uniqueness step, we get $w = 0$. □

Remark 3.4. *If Ω is unbounded, we need Ω to be uniformly C^1 when Ω_k tends to Ω_∞.*

4 Construction of weak solutions

The definition of sectorial is as follows:

Definition 4.1. *Let X be a complex Banach space, with norm $\|\cdot\|$, and $L : D(L) \subset X \to X$ be a linear operator, with not necessarily dence domain. Then, we say L is sectorial if there exist constants $\omega \in \mathbb{R}, 0 \leq \epsilon \leq \frac{\pi}{2},\ C > 0$ such that*

(i) The resolvent set $\rho(L)$ of L contains

$$S_{\pi-\epsilon,\omega} = \{\lambda \in \mathbb{C} :\ \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \pi - \epsilon\},$$

(ii)

$$\|(\lambda - L)\|_{L(X)} \leq \frac{C}{|\lambda - \omega|}\ for\ all\ \lambda \in S_{\pi-\epsilon,\omega}.$$

Now we prove the sectoriality of semigroups. In order to show the sectoriality of L, we have to get uniqueness and existence of the resolvent equation. Our construction of weak solutions is based on an approximation method seen in A. Lunardi[10]. In A. Lunardi[10], strong solutions of resolvent equations are constructed when the operator is a non divergence type operator.

proof of the existence and uniqueness of weak solution. First of all, we use a approximation method to get a weak solution of the resolvent equation. For $k \in \mathbb{N}$, let $\phi_k \in C_0^\infty(\mathbb{R}^n)$ be a cut off function s.t.

$$0 \leq \phi_k \leq 1,\ \phi_k = \begin{cases} 1 & \text{in } B_k(0) \\ 0 & \text{outside of } B_{2k}(0) \end{cases}$$

For arbitrary $f \in L^\infty(\Omega)$, we define

$$f_k = \phi_k f \in L^p(\Omega)\ (1 \leq p \leq \infty).$$

By $W^{m,p}$ estimates, we get a weak solution $u_k \in W_0^{m,p}(\Omega)$ of

$$\begin{cases} (\lambda - L)u_k = f_k & \text{in } \Omega \\ u_k = \partial_N u_k = \cdots = \partial_N^{m-1} u_k = 0 & \text{on } \partial\Omega \end{cases}$$

with the estimate $\|u_k\|_{W^{m,p}(\Omega)} \leq C_p \|f_k\|_{L^p(\Omega)}$. Since $W_0^{m,p} \hookrightarrow W^{m-1,\infty}$ by Sobolev embedding type theorems, u_k is in $W_{0,loc}^{m,p}(\Omega) \cap W^{m-1,\infty}(\Omega)$. So, we can apply L^∞ estimates to

$$\begin{cases} (\lambda - L)u_k = f_k & \text{in } \Omega \\ u_k = \partial_N u_k = \cdots = \partial_N^{m-1} u_k = 0 & \text{on } \partial\Omega \end{cases}$$

We obtain

$$N(u_k, \lambda) \leq C\|f_k\|_{L^\infty} \leq C\|f\|_{L^\infty}. \tag{23}$$

Therefore, $\{u_k\} \subset W^{m-1,\infty}(\Omega)$ is uniformly bounded. Furthermore, we also can show that $\{u_k\} \subset W^{m,p}(M)$ is uniformly bounded for each compact set $M \subset \Omega$ by almost same way as in the compactness step. So, there exists a subsequence $\{u_{k_j}\}$ in $W^{m-1,\infty} \cap C^{m-1,-\frac{n}{p}+1}(\Omega)$ s.t.

$u_{k_j} \to \exists u \in W^{m-1,\infty}(\Omega) \cap C^{m-1,-\frac{n}{p}+1}(M)$ uniformly on each compact set M.

with the estimates $N(u,\lambda) \leq C\|f\|_{L^\infty}$. We are going to show that u is in $W^{m,p}_{0,loc}(\Omega)$ and is a weak solution of

$$\begin{cases} (\lambda - L)u = f & \text{in } \Omega \\ u = \partial_N u = \cdots = \partial_N^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix any closed ball $B_R(0)$ with $R \geq 4|\lambda|^{-\frac{1}{2}}$. Apply the previous lemma in $B_R(0) \cap \Omega$, $\{u_{k_j}\} \subset W^{m,p}(B_R(0) \cap \Omega)$ is uniformly bounded, so that $u \in W^{m,p}(B_R(0) \cap \Omega)$. Since R is arbitrary, u is in $W^{m,p}_{loc}(\Omega)$. Furthermore, for each smooth function $\phi \in C_0^\infty(\mathbb{R}^n)$ $\phi u_k \in W_0^{m,p}(\Omega)$. Since ϕ has compact support, ϕu_k converges to ϕu in $W_0^{m,p}(\Omega)$. So $u \in W^{m,p}_{0,loc}(\Omega)$.Take large $j, l \in \mathbb{N}$, then we get

$$\begin{cases} (\lambda - L)(u_{k_j} - u_{k_l}) = 0 & \text{in } B_R(0) \cap \Omega \\ u_{k_j} - u_{k_l} = \partial_N(u_{k_j} - u_{k_l}) = \cdots = \partial_N^{m-1}(u_{k_j} - u_{k_l}) = 0 & \text{on } \partial(B_R(0) \cap \Omega). \end{cases}$$

By the local $W^{m,p}$ estimates as in the compactness step we get for $x_0 \in B_{\frac{R}{2}}(0)$,

$$\|u_{k_j} - u_{k_l}\|_{W^{m,p}(B_{|\lambda|^{-\frac{1}{2}}}(x_0) \cap \Omega)} \leq C(\lambda)\|u_{k_j} - u_{k_l}\|_{m-1,\infty} \to 0 \quad (j, l \to \infty). \tag{24}$$

Covering $B_{\frac{R}{2}}(0)$ by a finite number of balls $B_{|\lambda|^{-\frac{1}{2}}}(x_0)$, we get

$$u_{k_j} \to u \quad \text{in } W^{m,p}(B_{\frac{R}{2}}(0) \cap \Omega) \tag{25}$$

As j tends to ∞, the resolvent equation tends to

$$(\lambda - L)u = f \quad \text{in } B_{\frac{R}{2}}(0) \cap \Omega.$$

Since R is arbitrary,

$$(\lambda - L)u = f \quad \text{in } \Omega, \text{ and } u \in W^{m,p}_{0,loc}(\Omega).$$

Thus, we get a weak solution of resolvent equation. We get uniqueness by linearity of resolvent equation and L^∞ apriori estimates. So, L is sectorial. □

Proof of analyticity. Let us define e^{tL} as follows;

$$\begin{cases} e^{0L}x = x & \text{for all } x \in L^\infty(\Omega) \\ e^{tL}x = \dfrac{1}{2\pi i}\displaystyle\int_\gamma e^{t\lambda}(\lambda - L)^{-1}x d\lambda & \text{for all } t > 0 \text{ and } x \in L^\infty(\Omega), \end{cases}$$

where $r > M > 0$, $\theta < \epsilon$, and γ is the curve

$$(\Sigma_{\pi-\theta} \cap \{|\lambda \geq r\}) \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \geq \theta, |\lambda| = r\},$$

oriented counter clockwise. Since L is sectorial, e^{tL} is an analytic semigroup in $L^\infty(\Omega)$ by applying abstruct semigroup theories. □

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