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INTERIOR ERROR ESTIMATES FOR LEAST–SQUARES MIXED FINITE ELEMENT METHODS

A. I. PEHLIVANOV* AND G. F. CAREY†

Abstract. In a previous paper [see Numer. Math. **72**(1996), 501–522] we developed finite element error estimates for the least-squares mixed formulation of second order elliptic boundary-value problems. These estimates were established under appropriate regularity assumptions and confirmed in supporting numerical experiments. In the present work we extend the analysis to develop interior estimates on subdomains under weaker global regularity assumptions.

Key words. least–squares mixed finite elements, interior error estimates

AMS subject classification. 65N30

1. Introduction. There is an increasing research interest in mixed finite element methods for first order elliptic systems primarily because the flux enters explicitly in the formulation. Hence the method is capable of generating more accurate flux approximation and this may be of value in certain applications such as flow through porous media [2]. Most of the attention has focused on mixed Galerkin finite element methods [5, 11, 26], see also [12]. However, since this corresponds to a saddle-point problem these schemes are subject to the consistency requirements of the associated inf-sup condition [1, 4, 15]. Moreover, this mixed formulation frequently leads to nonsymmetric systems that are indefinite. More recently, least-squares mixed finite element schemes have been proposed as a possible alternative [3, 8, 9, 10, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24, 25, 6]. While there are still several open questions regarding both the theoretical properties and practical viability of this approach, it is not subject to the previous LBB requirement and also leads to a symmetric positive definite system.

Based on our previous works [23, 24, 21], optimal error estimates were developed in [25] for the prototype second order elliptic problem under certain global regularity assumptions. Several different variants of the formulation were studied in which an additional curl term and flux boundary constraint were considered in order to obtain improved estimates.

In general, we do not have the required global smoothness. However, the solution is usually smooth in subdomains away from the singularities of the data. The present work continues in the manner of our previous studies to develop interior estimates on such subdomains. Local error estimates for Galerkin finite element methods were developed by Nitsche and Schatz [20], Schatz and Wahlbin [27, 28], Wahlbin [29], and our approach follows the same basic strategy. However, the least-squares mixed method imposes specific problems which had to be overcome.

To fix ideads, let Ω be the domain of interest and Ω_0, Ω_1 be compact subdomains of Ω such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. Consider finite element approximations u_h and $\boldsymbol{\sigma}_h$ to the primary solution u and the flux $\boldsymbol{\sigma}$. Assume that the finite element spaces for u_h and $\boldsymbol{\sigma}_h$ consist of piecewise polynomials of degree k and r, respectively. When we use piecewise polynomials of equal degree, i.e. $k = r$, the analysis follows the

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general approach of Nitsche and Schatz [20], Schatz and Wahlbin [27], Wahlbin [29]. For example, the L^2-error estimate is

$$\begin{aligned} \|u-u_h\|_{0,\Omega_0}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega_0} \le\ & Ch^2\left(\|u-u_h\|_{1,\Omega_1}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,\Omega_1}\right)\\ &+C\left(\|u-u_h\|_{-1,\Omega_1}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,\Omega_1}\right)\\ &+Ch^{k+1}\left(\|u\|_{k+1,\Omega_1}+\|\boldsymbol{\sigma}\|_{k+1,\Omega_1}\right)\ . \end{aligned} \tag{1.1}$$

Note that, as usual, we have terms of optimal order plus the error in weaker norms on Ω_1. The latter terms control the rate of convergence provided the solution is sufficiently smooth on Ω_1.

However, when finite element spaces of different polynomial degrees are employed, significant technical difficulties arise. Such difficuties are not present in previous works. Then we have to use the Galerkin projection u_h^* for a specific auxiliary problem, see further (3.3).

Let $k+1=r$. Then, of course, we have the same estimate for $\|u-u_h\|_{0,\Omega_0}$ as in (1.1) above. When $k+1=r$, $k>1$,

$$\begin{aligned} \|u-u_h\|_{-1,\Omega_0}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega_0} \le\ & Ch^3\left(\|u-u_h\|_{1,\Omega_1}+\|u-u_h^*\|_{1,\Omega_1}\right)\\ &+Ch\left(\|u-u_h\|_{-1,\Omega_1}+\|u-u_h^*\|_{-1,\Omega_1}\right)\\ &+C\left(\|u-u_h\|_{-2,\Omega_1}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,\Omega_1}\right)\\ &+Ch^2\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,\Omega_1}\\ &+Ch^{r+1}\left(\|u\|_{r,\Omega_1}+\|\boldsymbol{\sigma}\|_{r+1,\Omega_1}\right)\ . \end{aligned} \tag{1.2}$$

The reader is referred to Section 3 for these and other error estimates.

The structure of the paper is as follows: In section 2 we present the least-squares mixed formulation of the problem and define the associated spaces. Section 3 contains the main results. The proofs are given in Section 4.

2. Least–squares formulation. Let Ω be a bounded domain in $\mathbb{R}^n$, $n=2,3$, with boundary Γ. Consider the second order boundary–value problem

$$-\operatorname{div}(A\operatorname{grad}u)-\boldsymbol{b}\cdot\operatorname{grad}u+c(x)u=f \quad \text{in}\ \ \Omega\ , \tag{2.1}$$

$$u=0 \quad \text{on}\ \ \Gamma\ , \tag{2.2}$$

where $A=(a_{ij}(x))_{i,j=1}^n$, $x\in\bar{\Omega}$, is a symmetric positive definite matrix of coefficients, $\boldsymbol{b}=(b_1(x),\ldots,b_n(x))^T$. Introducing $\boldsymbol{\sigma}=-A\operatorname{grad}u$, $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_n)$, we obtain the following system of first-order differential equations for u and $\boldsymbol{\sigma}$

$$\operatorname{div}\boldsymbol{\sigma}+\boldsymbol{b}^TA^{-1}\boldsymbol{\sigma}+cu=f \quad \text{in}\ \ \Omega\ , \tag{2.3}$$

$$\boldsymbol{\sigma}+A\operatorname{grad}u=0 \quad \text{in}\ \ \Omega\ , \tag{2.4}$$

$$u=0 \quad \text{on}\ \ \Gamma\ . \tag{2.5}$$

Since $\operatorname{grad}u=-A^{-1}\boldsymbol{\sigma}$, applying the curl–operator we get

$$\operatorname{curl}A^{-1}\boldsymbol{\sigma}=0 \quad \text{in}\ \ \Omega\ , \tag{2.6}$$

see Neittaanmäki and Křížek [16]. Also, from the boundary condition $u = 0$ on Γ, it follows that $\boldsymbol{n} \wedge \operatorname{grad} u = 0$ where $\wedge$ denotes the exterior product. This implies the property

$$\boldsymbol{n} \wedge A^{-1}\boldsymbol{\sigma} = 0 \quad \text{on} \quad \Gamma \,. \tag{2.7}$$

Next, define the spaces

$$V = \{v \in H^1(\Omega) : v = 0 \quad \text{on} \quad \Gamma\} \,, \tag{2.8}$$

$$\begin{aligned} \boldsymbol{W} = \{\boldsymbol{q} \in L^2(\Omega)^n &: \operatorname{div} \boldsymbol{q} \in L^2(\Omega) \\ &\operatorname{curl} A^{-1}\boldsymbol{q} \in L^2(\Omega)^{2n-3} \,, \ \boldsymbol{n} \wedge A^{-1}\boldsymbol{q} = 0 \ \text{ on } \ \Gamma\} \end{aligned} \tag{2.9}$$

with norms

$$\|v\|_{1,\Omega}^2 \equiv \|v\|_{0,\Omega}^2 + \|\operatorname{grad} u\|_{0,\Omega}^2 \,,$$
$$\|\boldsymbol{q}\|_{H(\operatorname{div},\operatorname{curl})}^2 \equiv \|\boldsymbol{q}\|_{0,\Omega}^2 + \|\operatorname{div} \boldsymbol{q}\|_{0,\Omega}^2 + \|\operatorname{curl} A^{-1}\boldsymbol{q}\|_{0,\Omega}^2 \,.$$

We specify a least–squares minimization problem for (2.3), (2.4), and (2.6): find $u \in V$, $\boldsymbol{\sigma} \in \boldsymbol{W}$ such that

$$J(u, \boldsymbol{\sigma}) = \inf_{v \in V, \boldsymbol{q} \in \boldsymbol{W}} J(v, \boldsymbol{q}) \,,$$

where

$$\begin{aligned} J(v, \boldsymbol{q}) &= \left(\operatorname{curl} A^{-1}\boldsymbol{q}, \operatorname{curl} A^{-1}\boldsymbol{q}\right)_{0,\Omega} \\ &+ \left(\operatorname{div} \boldsymbol{q} + \boldsymbol{b}^T A^{-1}\boldsymbol{q} + cv - f, \operatorname{div} \boldsymbol{q} + \boldsymbol{b}^T A^{-1}\boldsymbol{q} + cv - f\right)_{0,\Omega} \\ &+ \left(\boldsymbol{q} + A \operatorname{grad} v, A^{-1}(\boldsymbol{q} + A \operatorname{grad} v)\right)_{0,\Omega} \,. \end{aligned} \tag{2.10}$$

Taking variations leads to the weak statement: find $u \in V$, $\boldsymbol{\sigma} \in \boldsymbol{W}$ such that

$$a(u, \boldsymbol{\sigma}; v, \boldsymbol{q}) = \left(f, \operatorname{div} \boldsymbol{q} + \boldsymbol{b}^T A^{-1}\boldsymbol{q} + cv\right)_{0,\Omega} \quad \text{for all} \quad v \in V, \boldsymbol{q} \in \boldsymbol{W}, \tag{2.11}$$

where

$$\begin{aligned} a(u, \boldsymbol{\sigma}; v, \boldsymbol{q}) &= \left(\operatorname{curl} A^{-1}\boldsymbol{\sigma}, \operatorname{curl} A^{-1}\boldsymbol{q}\right)_{0,\Omega} \\ &+ \left(\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b}^T A^{-1}\boldsymbol{\sigma} + cu, \operatorname{div} \boldsymbol{q} + \boldsymbol{b}^T A^{-1}\boldsymbol{q} + cv\right)_{0,\Omega} \\ &+ \left(\boldsymbol{\sigma} + A \operatorname{grad} u, A^{-1}(\boldsymbol{q} + A \operatorname{grad} v)\right)_{0,\Omega} \,. \end{aligned} \tag{2.12}$$

For any compact subdomain G of Ω, i.e. $G \subset\subset \Omega$, define

$$V(G) = \{v \in H^1(G) : v = 0 \quad \text{on} \quad \partial G\} \,, \tag{2.13}$$

$$\begin{aligned} \boldsymbol{W}(G) = \{\boldsymbol{q} \in L^2(G)^n &: \operatorname{div} \boldsymbol{q} \in L^2(G) \\ &\operatorname{curl} A^{-1}\boldsymbol{q} \in L^2(G)^{2n-3} \,, \ \boldsymbol{q} = 0 \ \text{ on } \ \partial G\} \,. \end{aligned} \tag{2.14}$$

Let Ω_1 be a fixed compact subdomain of Ω with sufficiently smooth boundary. If $v \in V(\Omega_1)$, $\boldsymbol{q} \in \boldsymbol{W}(\Omega_1)$ then extending v and $\boldsymbol{q}$ by zero outside Ω_1 we conclude that the solution $(u, \boldsymbol{\sigma})$ of (2.11) satisfies

$$a(u, \boldsymbol{\sigma}; v, \boldsymbol{q}) = \left(f, \operatorname{div} \boldsymbol{q} + \boldsymbol{b}^T A^{-1}\boldsymbol{q} + cv\right)_{0,\Omega_1} \tag{2.15}$$

for all $v \in V(\Omega_1)$, $\boldsymbol{q} \in \boldsymbol{W}(\Omega_1)$.

Let us now consider the finite element approximation problem. First, introduce a partition $\mathcal{T}_h$ of Ω into finite elements. Let $P_k(\Sigma)$, $\Sigma \subset \mathbb{R}^n$, be the set of polynomials of degree k on Σ and let $\hat{K}$ denote the master element. Suppose that for any element $K \in \mathcal{T}_h$ there exists a mapping $F_K : \hat{K} \to K$, $F_K(\hat{K}) = K$, with components $(F_K)_i \in P_s(\hat{K})$, $i = 1, \ldots, n$; i.e., these components are polynomials of degree s. As usual, we have the correspondence $v_h(x) = \hat{v}_h(\hat{x})$, $\boldsymbol{q}_h(x) = \hat{\boldsymbol{q}}_h(\hat{x})$ for any $x = F_K(\hat{x})$, $\hat{x} \in \hat{K}$, and any functions $\hat{v}_h, \hat{\boldsymbol{q}}_h$ on $\hat{K}$.

Let V_h and $\boldsymbol{W}_h$ denote the finite element spaces corresponding to V and $\boldsymbol{W}$, respectively. The discrete approixmation to problem (2.11) then becomes: find $u_h \in V_h$, $\boldsymbol{\sigma}_h \in \boldsymbol{W}_h$ such that

$$a(u_h, \boldsymbol{\sigma}_h; v_h, \boldsymbol{q}_h) = \left(f, \operatorname{div} \boldsymbol{q}_h + \boldsymbol{b}^T A^{-1} \boldsymbol{q}_h + c v_h\right)_{0,\Omega} \tag{2.16}$$

for all $v_h \in V_h$, $\boldsymbol{q}_h \in \boldsymbol{W}_h$.

Denote

$$\Omega_1^h = \{K \in \mathcal{T}_h \ : \ K \subset \Omega_1\} , \tag{2.17}$$

that is, Ω_1^h consists of all elements $K \subset \Omega_1$. Now we specify that the restrictions of the finite element spaces V_h and $\boldsymbol{W}_h$ on Ω_1^h consist of piecewise polynomials of degree k and r, respectively. More specifically,

$$\begin{aligned} V_h(\Omega_1) = \{v_h \in C^0(\Omega_1) : v_h|_K = \hat{v}_h|_{\hat{K}} \in P_k(\hat{K}) \ \ \forall K \in \mathcal{T}_h, K \subset \Omega_1, \\ v_h = 0 \ \text{ on } \ \partial\Omega_1^h \ \text{ and outside } \ \Omega_1^h\} , \end{aligned} \tag{2.18}$$

$$\begin{aligned} \boldsymbol{W}_h(\Omega_1) = \{\boldsymbol{q}_h \in C^0(\Omega_1)^n : (\boldsymbol{q}_h)_i|_K = (\hat{\boldsymbol{q}}_h)_i|_{\hat{K}} \in P_r(\hat{K}) \ \ \forall K \in \mathcal{T}_h, K \subset \Omega_1, \\ i = 1, \ldots, n, \ \boldsymbol{q}_h = 0 \ \text{ on } \ \partial\Omega_1^h \ \text{ and outside } \ \Omega_1^h\} . \end{aligned} \tag{2.19}$$

Then the solution u_h, $\boldsymbol{\sigma}_h$ to (2.16) also satisfies

$$a(u_h, \boldsymbol{\sigma}_h; v_h, \boldsymbol{q}_h) = \left(f, \operatorname{div} \boldsymbol{q}_h + \boldsymbol{b}^T A^{-1} \boldsymbol{q}_h + c v_h\right)_{0,\Omega_1} \tag{2.20}$$

for all $v_h \in V_h(\Omega_1)$, $\boldsymbol{q}_h \in \boldsymbol{W}_h(\Omega_1)$. Using (2.15), (2.20) and the inclusions $V_h(\Omega_1) \subset V(\Omega_1)$, $\boldsymbol{W}_h(\Omega_1) \subset \boldsymbol{W}(\Omega_1)$ we derive the following "interior" orthogonality property

$$a(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h; v_h, \boldsymbol{q}_h) = 0 \quad \text{for all} \ \ v_h \in V_h(\Omega_1) \ , \ \boldsymbol{q}_h \in \boldsymbol{W}_h(\Omega_1) . \tag{2.21}$$

Remark. We use Dirichlet boundary conditions for problem (2.1)–(2.2) for clarity of exposition. Since the definition of $V_h(\Omega_1)$ and $\boldsymbol{W}_h(\Omega_1)$ is not affected by the boundary conditions, the theory presented in the next sections covers the general case. □

In [25] we specified certain conditions on the coefficients of the boundary-value problem (see inequalities (2.4), (2.5), (2.7), and (2.10) in [25]). Here we require that these inequalities are satisfied in the fixed subdomain Ω_1. Under these conditions we have the following coercivity estimate:

$$C\left(\|v\|_{1,\Omega_1}^2 + \|\boldsymbol{q}\|_{H(\mathrm{div},\mathrm{curl})}^2\right) \le a(v, \boldsymbol{q}; v, \boldsymbol{q}) \tag{2.22}$$

for all $v \in V(\Omega_1)$, $\boldsymbol{q} \in \boldsymbol{W}(\Omega_1)$. Moreover, taking into account the Friedrichs inequality

$$\|\boldsymbol{q}\|_{1,\Omega_1} \le C\|\boldsymbol{q}\|_{H(\mathrm{div},\mathrm{curl})} \quad \text{for all} \quad q \in \boldsymbol{W}(\Omega_1) , \tag{2.23}$$

we obtain a coercivity estimate in H^1-norm; i.e.

$$C\left(||v||^2_{1,\Omega_1} + ||\boldsymbol{q}||^2_{1,\Omega_1}\right) \le a(v, \boldsymbol{q}; v, \boldsymbol{q}) \ , \tag{2.24}$$

where the constant C depends only on the coefficients of equation (2.1) and the Poincaré–Friedrichs ineqality constant for Ω_1.

3. Error Estimates. In this section we present the main results. The proofs are given in the next section.

First, we state the result for the case of equal polinomial degrees in the finite element spaces $V_h(\Omega_1)$ and $\boldsymbol{W}_h(\Omega_1)$.

THEOREM 3.1. *Let $k = r$ and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. The following estimates hold:*

$$\begin{aligned} ||u - u_h||_{1,\Omega_0} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{1,\Omega_0} \le\ & Ch\left(||u - u_h||_{1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{1,\Omega_1}\right) \\ & + C\left(||u - u_h||_{-1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{-1,\Omega_1}\right) \\ & + Ch^k\left(||u||_{k+1,\Omega_1} + ||\boldsymbol{\sigma}||_{k+1,\Omega_1}\right) \ , \end{aligned} \tag{3.1}$$

$$\begin{aligned} ||u - u_h||_{0,\Omega_0} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\Omega_0} \le\ & Ch^2\left(||u - u_h||_{1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{1,\Omega_1}\right) \\ & + C\left(||u - u_h||_{-1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{-1,\Omega_1}\right) \\ & + Ch^{k+1}\left(||u||_{k+1,\Omega_1} + ||\boldsymbol{\sigma}||_{k+1,\Omega_1}\right) \ . \ \square \end{aligned} \tag{3.2}$$

Note that the third terms on the right-hand sides of (3.1) and (3.2) are of optimal order. Also, note that h and h^2, respectively, appear in the first terms. Hence the terms which involve negative norms will actually control the rate of convergence. Since the error is measured in weaker norms, we expect to achieve optimal convergence rate in Ω_0, especially when some care is taken of the singularities away from Ω_1.

In order to present the results for the case of differing polynomial degrees, let $u_h^* \in V_h$ be such that

$$(A \operatorname{grad}(u - u_h^*), \operatorname{grad} v_h)_{0,\Omega} + (c(u - u_h), cv_h)_{0,\Omega} = 0 \ \text{ for all } \ v_h \in V_h \ . \tag{3.3}$$

Note that u_h^* is similar to a Galerkin finite element projection.

THEOREM 3.2. *Let $k + 1 = r$ and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. The following estimates hold:*

$$\begin{aligned} ||u - u_h||_{1,\Omega_0} \le\ & Ch\left(||u - u_h||_{1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{1,\Omega_1}\right) \\ & + C\left(||u - u_h||_{-1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{-1,\Omega_1}\right) \\ & + Ch^k\left(||u||_{k+1,\Omega_1} + ||\boldsymbol{\sigma}||_{k+1,\Omega_1}\right) \ , \end{aligned} \tag{3.4}$$

$$\begin{aligned} ||u - u_h||_{0,\Omega_0} \le\ & Ch^2\left(||u - u_h||_{1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{1,\Omega_1}\right) \\ & + C\left(||u - u_h||_{-1,\Omega_1} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{-1,\Omega_1}\right) \\ & + Ch^{k+1}\left(||u||_{k+1,\Omega_1} + ||\boldsymbol{\sigma}||_{k+1,\Omega_1}\right) \ , \end{aligned} \tag{3.5}$$

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{1,\Omega_0} \le Ch^2\left(||u - u_h||_{1,\Omega_1} + ||u - u_h^*||_{1,\Omega_1}\right)$$

$$
\begin{aligned}
&+C\left(\|u-u_h\|_{-1,\Omega_1}+\|u-u_h^*\|_{-1,\Omega_1}\right)\\
&+C\left(h\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,\Omega_1}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,\Omega_1}\right)\\
&+Ch^r\left(\|u\|_{r,\Omega_1}+\|\boldsymbol{\sigma}\|_{r+1,\Omega_1}\right) .
\end{aligned}
\tag{3.6}
$$

If $k+1=r,\ k>1$, *then*

$$
\begin{aligned}
\|u-u_h\|_{-1,\Omega_0}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega_0} \le\ & Ch^3\left(\|u-u_h\|_{1,\Omega_1}+\|u-u_h^*\|_{1,\Omega_1}\right)\\
&+Ch\left(\|u-u_h\|_{-1,\Omega_1}+\|u-u_h^*\|_{-1,\Omega_1}\right)\\
&+C\left(\|u-u_h\|_{-2,\Omega_1}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,\Omega_1}\right)\\
&+Ch^2\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,\Omega_1}\\
&+Ch^{r+1}\left(\|u\|_{r,\Omega_1}+\|\boldsymbol{\sigma}\|_{r+1,\Omega_1}\right) . \qquad \square
\end{aligned}
\tag{3.7}
$$

Again, we would like to emphasize that the error in the corresponding norms on Ω_0 is bounded by terms of optimal order plus the error in weaker norms on Ω_1.

4. Error Analysis. Let $\varphi \in H^1(\Omega)$. Denote

$$
\operatorname{curl}\varphi = (-\partial_2\varphi, \partial_1\varphi) \quad \text{when} \quad \Omega \subset \mathbb{R}^2 , \tag{4.1}
$$

$$
\operatorname{curl}\varphi = \begin{pmatrix} 0 & -\partial_3\varphi & \partial_2\varphi \\ \partial_3\varphi & 0 & -\partial_1\varphi \\ -\partial_2\varphi & \partial_1\varphi & 0 \end{pmatrix} \quad \text{when} \quad \Omega \subset \mathbb{R}^3 . \tag{4.2}
$$

For $\boldsymbol{q} = (q_1, \ldots, q_n) \in H^1(\Omega)^n$, $v \in H^1(\Omega)$ the following relations hold:

$$
\operatorname{div}(\varphi\boldsymbol{q}) = \operatorname{grad}\varphi \cdot \boldsymbol{q} + \varphi \operatorname{div}\boldsymbol{q} , \tag{4.3}
$$

$$
\operatorname{curl}(\varphi\boldsymbol{q}) = \varphi \operatorname{curl}\boldsymbol{q} + \operatorname{curl}\varphi\ \boldsymbol{q} , \tag{4.4}
$$

$$
\operatorname{grad}(\varphi v) = v \operatorname{grad}\varphi + \varphi \operatorname{grad} v , \tag{4.5}
$$

where

$$
\begin{aligned}
&\operatorname{curl}\boldsymbol{q} = \partial_1 q_2 - \partial_2 q_1 \quad \text{when} \quad \Omega \subset \mathbb{R}^2 ,\\
&\operatorname{curl}\boldsymbol{q} = (\partial_2 q_3 - \partial_3 q_2, \partial_3 q_1 - \partial_1 q_3, \partial_1 q_2 - \partial_2 q_1) \quad \text{when} \quad \Omega \subset \mathbb{R}^3 .
\end{aligned}
$$

Throughout this section we shall use subdomains of Ω_1. We assume that all these subdomains have smooth boundaries. Recall that k and r denote the element polynomial degrees for u_h and $\boldsymbol{\sigma}_h$, respectively.

LEMMA 4.1. *Let* $G_0 \subset\subset G \subset\subset \Omega_1$. *If* $k=r$

$$
\begin{aligned}
\|u-u_h\|_{0,G_0}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,G_0} \le\ & Ch\left(\|u-u_h\|_{1,G}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G}\right)\\
&+C\left(\|u-u_h\|_{-1,G}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,G}\right) .
\end{aligned}
\tag{4.6}
$$

If $k+1=r,\ k>1$,

$$
\begin{aligned}
\|u-u_h\|_{-1,G_0}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,G_0} \le\ & C\left(h^2\|u-u_h\|_{1,G}+h\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G}\right)\\
&+C\left(\|u-u_h\|_{-2,G}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,G}\right) .
\end{aligned}
\tag{4.7}
$$

Proof. For $G_0 \subset\subset \widetilde{G} \subset\subset G$ let ω be a cut–off function such that $\omega \in C_0^\infty(\widetilde{G})$, $\omega = 1$ on G_0. Denote $\varepsilon = u - u_h$, $\boldsymbol{e} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, $\widetilde{\varepsilon} = \omega\varepsilon$, $\widetilde{\boldsymbol{e}} = \omega\boldsymbol{e}$. Consider the auxiliary problem: find $\xi \in V(G)$, $\boldsymbol{\eta} \in \boldsymbol{W}(G)$ such that

$$a(\xi,\boldsymbol{\eta};v,\boldsymbol{q}) = (E,v)_{0,G} + (\boldsymbol{F},\boldsymbol{q})_{0,G} \tag{4.8}$$

for all $v \in V(G)$, $\boldsymbol{q} \in \boldsymbol{W}(G)$. The functions E and $\boldsymbol{F}$ will be specified later. Setting $v = \widetilde{\varepsilon} \in V(G)$, $\boldsymbol{q} = \widetilde{\boldsymbol{e}} \in \boldsymbol{W}(G)$, and using (4.3)–(4.5),

$$\begin{aligned}
a(\xi,\boldsymbol{\eta};\widetilde{\varepsilon},\widetilde{\boldsymbol{e}}) &= (\operatorname{curl} A^{-1}\boldsymbol{\eta}, \operatorname{curl} A^{-1}(\omega\boldsymbol{e})_{0,G} \\
&\quad + (\operatorname{div}\boldsymbol{\eta} + \boldsymbol{b}^T A^{-1}\boldsymbol{\eta} + c\xi, \operatorname{div}(\omega\boldsymbol{e}) + \boldsymbol{b}^T A^{-1}\omega\boldsymbol{e} + c\omega\varepsilon)_{0,G} \\
&\quad + (\boldsymbol{\eta} + A\operatorname{grad}\xi, A^{-1}(\omega\boldsymbol{e}) + \operatorname{grad}(\omega\varepsilon))_{0,G} \\
&= (\operatorname{curl} A^{-1}\boldsymbol{\eta}, \operatorname{curl}\omega\cdot(A^{-1}\boldsymbol{e}))_{0,G} + (\operatorname{curl} A^{-1}\boldsymbol{\eta}, \omega\operatorname{curl} A^{-1}\boldsymbol{e})_{0,G} \\
&\quad + (\operatorname{div}\boldsymbol{\eta} + \boldsymbol{b}^T A^{-1}\boldsymbol{\eta} + c\xi, \omega(\operatorname{div}\boldsymbol{e} + \boldsymbol{b}^T A^{-1}\boldsymbol{e} + c\varepsilon))_{0,G} \\
&\quad + (\operatorname{div}\boldsymbol{\eta} + \boldsymbol{b}^T A^{-1}\boldsymbol{\eta} + c\xi, \boldsymbol{e}\cdot\operatorname{grad}\omega)_{0,G} \\
&\quad + (\boldsymbol{\eta} + A\operatorname{grad}\xi, \omega A^{-1}(\boldsymbol{e} + A\operatorname{grad}\varepsilon))_{0,G} \\
&\quad + (\boldsymbol{\eta} + A\operatorname{grad}\xi, \varepsilon\operatorname{grad}\omega)_{0,G}\,.
\end{aligned}$$

Similarly, for $\widetilde{\xi} = \omega\xi$, $\widetilde{\boldsymbol{\eta}} = \omega\boldsymbol{\eta}$,

$$\begin{aligned}
a(\widetilde{\xi},\widetilde{\boldsymbol{\eta}};\varepsilon,\boldsymbol{e}) &= (\omega\operatorname{curl} A^{-1}\boldsymbol{\eta}, \operatorname{curl} A^{-1}\boldsymbol{e})_{0,G} + (\operatorname{curl}\omega\cdot(A^{-1}\boldsymbol{\eta}), \operatorname{curl} A^{-1}\boldsymbol{e})_{0,G} \\
&\quad + (\omega(\operatorname{div}\boldsymbol{\eta} + \boldsymbol{b}^T A^{-1}\boldsymbol{\eta} + c\xi), \operatorname{div}\boldsymbol{e} + \boldsymbol{b}^T A^{-1}\boldsymbol{e} + c\varepsilon)_{0,G} \\
&\quad + (\boldsymbol{\eta}\cdot\operatorname{grad}\omega, \operatorname{div}\boldsymbol{e} + \boldsymbol{b}^T A^{-1}\boldsymbol{e} + c\varepsilon)_{0,G} \\
&\quad + (\omega(\boldsymbol{\eta} + A\operatorname{grad}\xi, A^{-1}\boldsymbol{e} + \operatorname{grad}\varepsilon))_{0,G} \\
&\quad + (\xi\operatorname{grad}\omega, \boldsymbol{e} + A\operatorname{grad}\varepsilon)_{0,G}\,.
\end{aligned}$$

Hence

$$a(\xi,\boldsymbol{\eta};\widetilde{\varepsilon},\widetilde{\boldsymbol{e}}) = a(\widetilde{\xi},\widetilde{\boldsymbol{\eta}};\varepsilon,\boldsymbol{e}) + \mathcal{L}(\xi,\boldsymbol{\eta};\varepsilon,\boldsymbol{e}) \tag{4.9}$$

where

$$\begin{aligned}
\mathcal{L}(\xi,\boldsymbol{\eta};\varepsilon,\boldsymbol{e}) &= (\operatorname{curl} A^{-1}\boldsymbol{\eta}, \operatorname{curl}\omega\cdot(A^{-1}\boldsymbol{e}))_{0,G} - ((\operatorname{curl}\omega)A^{-1}\boldsymbol{\eta}, \operatorname{curl} A^{-1}\boldsymbol{e})_{0,G} \\
&\quad + (\operatorname{div}\boldsymbol{\eta} + \boldsymbol{b}^T A^{-1}\boldsymbol{\eta} + c\xi, \boldsymbol{e}\cdot\operatorname{grad}\omega)_{0,G} \\
&\quad - (\boldsymbol{\eta}\cdot\operatorname{grad}\omega, \operatorname{div}\boldsymbol{e})_{0,G} - (\boldsymbol{\eta}\cdot\operatorname{grad}\omega, \boldsymbol{b}^T A^{-1}\boldsymbol{e} + c\varepsilon)_{0,G} \\
&\quad + (\boldsymbol{\eta} + A\operatorname{grad}\xi, \varepsilon\operatorname{grad}\omega)_{0,G} \\
&\quad - (\xi\operatorname{grad}\omega, \boldsymbol{e})_{0,G} - (\xi\operatorname{grad}\omega, A\operatorname{grad}\varepsilon)_{0,G} \\
&= ((\operatorname{curl}\omega)A^{-1}\operatorname{curl} A^{-1}\boldsymbol{\eta}, \boldsymbol{e})_{0,G} + (A^{-1}\operatorname{curl}((\operatorname{curl}\omega)A^{-1}\boldsymbol{\eta}), \boldsymbol{e})_{0,G} \\
&\quad + ((\operatorname{div}\boldsymbol{\eta} + \boldsymbol{b}^T A^{-1}\boldsymbol{\eta} + c\xi)\operatorname{grad}\omega, \boldsymbol{e})_{0,G} \\
&\quad + (\operatorname{grad}(\boldsymbol{\eta}\cdot\operatorname{grad}\omega), \boldsymbol{e})_{0,G} - ((\boldsymbol{\eta}\cdot\operatorname{grad}\omega)A^{-1}\boldsymbol{b}, \boldsymbol{e})_{0,G} \\
&\quad - ((\boldsymbol{\eta}\cdot\operatorname{grad}\omega)c, \varepsilon)_{0,G} + ((\boldsymbol{\eta} + A\operatorname{grad}\xi)\operatorname{grad}\omega, \varepsilon)_{0,G} \\
&\quad - (\xi\operatorname{grad}\omega, \boldsymbol{e})_{0,G} + (\operatorname{div}(\xi A\operatorname{grad}\omega, \varepsilon)_{0,G}
\end{aligned} \tag{4.10}$$

Integration by parts and the fact that $\omega \in C_0^\infty(\widetilde{G})$ were used at the last step.

First, we consider the case $k = r$. The following a priori estimate for problem (4.8) holds:

$$\|\xi\|_{2,G} + \|\boldsymbol{\eta}\|_{2,G} \le C\left(\|E\|_{0,G} + \|\boldsymbol{F}\|_{0,G}\right) . \tag{4.11}$$

Let $\widetilde{\xi}_I$ and $\widetilde{\boldsymbol{\eta}}_I$ be the standard interpolants of $\widetilde{\xi}$ and $\widetilde{\boldsymbol{\eta}}$. Taking into account the orthogonality condition (2.21) and the bound (4.11),

$$\begin{aligned} a(\widetilde{\xi}, \widetilde{\boldsymbol{\eta}}; \varepsilon, \boldsymbol{e}) &= a(\widetilde{\xi} - \widetilde{\xi}_I, \widetilde{\boldsymbol{\eta}} - \widetilde{\boldsymbol{\eta}}_I; \varepsilon, \boldsymbol{e}) \\ &\le Ch\left(\|\xi\|_{2,G} + \|\boldsymbol{\eta}\|_{2,G}\right)\left(\|\varepsilon\|_{1,G} + \|\boldsymbol{e}\|_{1,G}\right) \\ &\le Ch\left(\|E\|_{0,G} + \|\boldsymbol{F}\|_{0,G}\right)\left(\|\varepsilon\|_{1,G} + \|\boldsymbol{e}\|_{1,G}\right) . \end{aligned} \tag{4.12}$$

Furthermore, bounding the respective terms in (4.10) and again applying (4.11),

$$\begin{aligned} \mathcal{L}(\xi, \boldsymbol{\eta}; \varepsilon, \boldsymbol{e}) &\le Ch\left(\|\xi\|_{2,G} + \|\boldsymbol{\eta}\|_{2,G}\right)\left(\|\varepsilon\|_{-1,G} + \|\boldsymbol{e}\|_{-1,G}\right) \\ &\le C\left(\|E\|_{0,G} + \|\boldsymbol{F}\|_{0,G}\right)\left(\|\varepsilon\|_{-1,G} + \|\boldsymbol{e}\|_{-1,G}\right) . \end{aligned} \tag{4.13}$$

Recalling $\omega \in C_0^\infty(\widetilde{G})$, $G_0 \subset\subset \widetilde{G}$ and $\omega = 1$ on G_0,

$$\begin{aligned} \|u - u_h\|_{0,G_0} &\le \|\omega(u - u_h)\|_{0,G} \\ &\le \sup_{E \in L^2(G)} \frac{|(\omega(u - u_h), E)_{0,G}|}{\|E\|_{0,G}} . \end{aligned} \tag{4.14}$$

First, let $E \in L^2(G)$ be arbitrary but fixed. Consider problem (4.8) with $\boldsymbol{F} = 0$, $v = \widetilde{\varepsilon}$, $\boldsymbol{q} = \widetilde{\boldsymbol{e}}$. Then

$$\begin{aligned} (\omega(u - u_h), E)_{0,G} &= a(\xi, \boldsymbol{\eta}; \widetilde{\varepsilon}, \widetilde{\boldsymbol{e}}) \\ &\le C\|E\|_{0,G}\left(h\left(\|\varepsilon\|_{1,G} + \|\boldsymbol{e}\|_{1,G}\right) + \|\varepsilon\|_{-1,G} + \|\boldsymbol{e}\|_{-1,G}\right) . \end{aligned} \tag{4.15}$$

Hence the upper bound for $\|u - u_h\|_{0,G}$ in (4.15) follows from (4.9), (4.12), (4.13), and (4.14).

Similarly,

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,G_0} &\le \|\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,G} \\ &\le \sup_{\boldsymbol{F} \in L^2(G)^n} \frac{|(\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \boldsymbol{F})_{0,G}|}{\|\boldsymbol{F}\|_{0,G}} . \end{aligned} \tag{4.16}$$

Let $\boldsymbol{F} \in L^2(G)^n$ be arbitrary but fixed and consider problem (4.8) with $E = 0$. Then

$$\begin{aligned} (\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \boldsymbol{F})_{0,G} &= a(\xi, \boldsymbol{\eta}; \widetilde{\varepsilon}, \widetilde{\boldsymbol{e}}) \\ &\le C\|\boldsymbol{F}\|_{0,G}\left(h\left(\|\varepsilon\|_{1,G} + \|\boldsymbol{e}\|_{1,G}\right) + \|\varepsilon\|_{-1,G} + \|\boldsymbol{e}\|_{-1,G}\right) \end{aligned} \tag{4.17}$$

which completes the proof of estimate (4.6).

Next, we consider the case $k+1=r$, $k>1$. We have the a priori estimate

$$\|\xi\|_{3,G}+\|\boldsymbol{\eta}\|_{2,G}\le C\left(\|E\|_{1,G}+\|\boldsymbol{F}\|_{0,G}\right)\,. \tag{4.18}$$

where the constant C does not depend on E and $\boldsymbol{F}$. Then following the same general approach as before

$$\begin{aligned} a(\widetilde{\xi},\widetilde{\boldsymbol{\eta}};\varepsilon,\boldsymbol{e}) &= a(\widetilde{\xi}-\widetilde{\xi}_I,\widetilde{\boldsymbol{\eta}}-\widetilde{\boldsymbol{\eta}}_I;\varepsilon,\boldsymbol{e})\\ &\le C\left(\|\xi\|_{3,G}+\|\boldsymbol{\eta}\|_{2,G}\right)\left(h^2\|\varepsilon\|_{1,G}+h\|\boldsymbol{e}\|_{1,G}\right)\\ &\le Ch\left(\|E\|_{1,G}+\|\boldsymbol{F}\|_{0,G}\right)\left(h^2\|\varepsilon\|_{1,G}+h\|\boldsymbol{e}\|_{1,G}\right)\,. \end{aligned} \tag{4.19}$$

Also,

$$\begin{aligned} \mathcal{L}(\xi,\boldsymbol{\eta};\varepsilon,\boldsymbol{e}) &\le Ch\left(\|\xi\|_{3,G}+\|\boldsymbol{\eta}\|_{2,G}\right)\left(\|\varepsilon\|_{-2,G}+\|\boldsymbol{e}\|_{-1,G}\right)\\ &\le C\left(\|E\|_{1,G}+\|\boldsymbol{F}\|_{0,G}\right)\left(\|\varepsilon\|_{-2,G}+\|\boldsymbol{e}\|_{-1,G}\right)\,. \end{aligned} \tag{4.20}$$

We have

$$\begin{aligned} \|u-u_h\|_{-1,G_0} &\le \|\omega(u-u_h)\|_{-1,G}\\ &\le \sup_{E\in H_0^1(G)}\frac{|(\omega(u-u_h),E)_{0,G}|}{\|E\|_{1,G}}\,. \end{aligned} \tag{4.21}$$

Let $E\in H_0^1(G)$ be arbitrary but fixed. Using (4.8) with $\boldsymbol{F}=0$, (4.9), (4.19), and (4.20), we obtain the estimate for $\|u-u_h\|_{-1,G_0}$. The estimate for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,G_0}$ follows in the same way. This concludes the proof. □

LEMMA 4.2. *Let $G_0\subset\subset G\subset\subset\Omega_1$ and $k=r$. Then*

$$\begin{aligned} \|u-u_h\|_{1,G_0}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G_0} &\le Ch\left(\|u-u_h\|_{1,G}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G}\right)\\ &\quad + C\left(\|u-u_h\|_{0,G}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,G}\right)\\ &\quad + Ch^k\left(\|u\|_{k+1,G}+\|\boldsymbol{\sigma}\|_{k+1,G}\right) \end{aligned} \tag{4.22}$$

Proof. Denote $\varepsilon_h=u_I-u_h$, $\boldsymbol{e}_h=\boldsymbol{\sigma}_I-\boldsymbol{\sigma}_h$, where u_I and $\boldsymbol{\sigma}_I$ are the standard interpolants of u and $\boldsymbol{\sigma}$. Define a projection operator R such that $Rw\in V_h(G)$, $R\boldsymbol{p}\in\boldsymbol{W}_h(G)$ for $w\in V(G)$, $\boldsymbol{p}\in\boldsymbol{W}(G)$, and

$$a(w-Rw,\boldsymbol{p}-R\boldsymbol{p};v_h,\boldsymbol{q}_h)=0 \quad \text{for all} \quad v_h\in V_h(G),\boldsymbol{q}_h\in\boldsymbol{W}_h(G)\,. \tag{4.23}$$

Let $\widetilde{\varepsilon}_h=\omega\varepsilon_h$, $\widetilde{\boldsymbol{e}}_h=\omega\boldsymbol{e}_h$, where $\omega\in C_0^\infty(\widetilde{G})$, $\omega=1$ on G_0, $G_0\subset\subset\widetilde{G}\subset\subset G$. Then

$$\|\varepsilon_h\|_{1,G_0}\le\|\widetilde{\varepsilon}_h\|_{1,G}\le\|\widetilde{\varepsilon}_h-R\widetilde{\varepsilon}_h\|_{1,G}+\|R\widetilde{\varepsilon}_h\|_{1,G} \tag{4.24}$$

and

$$\|\widetilde{\boldsymbol{e}}_h\|_{1,G_0}\le\|\widetilde{\boldsymbol{e}}_h\|_{1,G}\le\|\widetilde{\boldsymbol{e}}_h-R\widetilde{\boldsymbol{e}}_h\|_{1,G}+\|R\widetilde{\boldsymbol{e}}_h\|_{1,G}\,. \tag{4.25}$$

Taking into account the coercivity estimate (2.24) and using the projection property (4.23),

$$\begin{aligned}\|\widetilde{\varepsilon}_h - R\widetilde{\varepsilon}_h\|_{1,G}^2 + \|\widetilde{e}_h - R\widetilde{e}_h\|_{1,G}^2 &\le Ca(\widetilde{\varepsilon}_h - R\widetilde{\varepsilon}_h, \widetilde{e}_h - R\widetilde{e}_h; \widetilde{\varepsilon}_h - R\widetilde{\varepsilon}_h, \widetilde{e}_h - R\widetilde{e}_h)\\ &= Ca(\widetilde{\varepsilon}_h - R\widetilde{\varepsilon}_h, \widetilde{e}_h - R\widetilde{e}_h; \widetilde{\varepsilon}_h - (\widetilde{\varepsilon}_h)_I, \widetilde{e}_h - (\widetilde{e}_h)_I)\\ &\le C\left(\|\widetilde{\varepsilon}_h - R\widetilde{\varepsilon}_h\|_{1,G} + \|\widetilde{e}_h - R\widetilde{e}_h\|_{1,G}\right)\\ &\quad \times \left(\|\widetilde{\varepsilon}_h - (\widetilde{\varepsilon}_h)_I\|_{1,G} + \|\widetilde{e}_h - (\widetilde{e}_h)_I\|_{1,G}\right)\end{aligned} \tag{4.26}$$

For any element $K \subset G$,

$$\begin{aligned}\|\widetilde{\varepsilon}_h - (\widetilde{\varepsilon}_h)_I\|_{1,K} &\le Ch^k\|\omega\varepsilon_h\|_{k+1,K}\\ &\le Ch^k\|\varepsilon_h\|_{k,K}\\ &\le Ch\|\varepsilon_h\|_{1,K}\,.\end{aligned} \tag{4.27}$$

Similarly,

$$\begin{aligned}\|\widetilde{e}_h - (\widetilde{e}_h)_I\|_{1,K} &\le Ch^r\|\omega e_h\|_{r+1,K}\\ &\le Ch^r\|e_h\|_{r,K}\\ &\le Ch\|e_h\|_{1,K}\,.\end{aligned} \tag{4.28}$$

From (4.26), (4.27), and (4.28),

$$\|\widetilde{\varepsilon}_h - R\widetilde{\varepsilon}_h\|_{1,G} + \|\widetilde{e}_h - R\widetilde{e}_h\|_{1,G} \le Ch\left(\|\varepsilon_h\|_{1,G} + \|e_h\|_{1,G}\right)\,. \tag{4.29}$$

Now we estimate the terms $\|R\widetilde{\varepsilon}_h\|_{1,G}$ and $\|R\widetilde{e}_h\|_{1,G}$. We have

$$\begin{aligned}C\left(\|R\widetilde{\varepsilon}_h\|_{1,G}^2 + \|R\widetilde{e}_h\|_{1,G}^2\right) &\le a(R\widetilde{\varepsilon}_h, R\widetilde{e}_h; R\widetilde{\varepsilon}_h, R\widetilde{e}_h)\\ &= a(\widetilde{\varepsilon}_h, \widetilde{e}_h; R\widetilde{\varepsilon}_h, R\widetilde{e}_h)\\ &= a(\varepsilon_h, e_h; \omega R\widetilde{\varepsilon}_h, \omega R\widetilde{e}_h) + \mathcal{L}(R\widetilde{\varepsilon}_h, R\widetilde{e}_h; \varepsilon_h, e_h)\\ &= a(\varepsilon_h, e_h; \omega R\widetilde{\varepsilon}_h - (\omega R\widetilde{\varepsilon}_h)_I, \omega R\widetilde{e}_h - (\omega R\widetilde{e}_h)_I)\\ &\quad + a(\varepsilon_h, e_h; (\omega R\widetilde{\varepsilon}_h)_I, (\omega R\widetilde{e}_h)_I)\\ &\quad + \mathcal{L}(R\widetilde{\varepsilon}_h, R\widetilde{e}_h; \varepsilon_h, e_h)\end{aligned} \tag{4.30}$$

Each of these terms can be bounded as follows: first, following the same reasoning as in (4.27) and (4.28), we have

$$\|\omega R\widetilde{\varepsilon}_h - (\omega R\widetilde{\varepsilon}_h)_I\|_{1,G} \le Ch\|R\widetilde{\varepsilon}_h\|_{1,G}\,, \tag{4.31}$$

$$\|\omega R\widetilde{e}_h - (\omega R\widetilde{e}_h)_I\|_{1,G} \le Ch\|R\widetilde{e}_h\|_{1,G}\,. \tag{4.32}$$

Also,

$$\begin{aligned}\|(\omega R\widetilde{\varepsilon}_h)_I\|_{1,G} &\le \|(\omega R\widetilde{\varepsilon}_h)_I - \omega R\widetilde{\varepsilon}_h\|_{1,G} + \|\omega R\widetilde{\varepsilon}_h\|_{1,G}\\ &\le C\|R\widetilde{\varepsilon}_h\|_{1,G}\,,\end{aligned} \tag{4.33}$$

and, similarly,

$$\|(\omega R\widetilde{e}_h)_I\|_{1,G} \le \|R\widetilde{e}_h\|_{1,G}\,. \tag{4.34}$$

Next, recalling the definition of ε_h and $\boldsymbol{e}_h$,

$$\begin{aligned} & a(u_I - u_h, \boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h; (\omega R\widetilde{\varepsilon}_h)_I, (\omega R\widetilde{\boldsymbol{e}}_h)_I) \\ & \quad = a(u_I - u, \boldsymbol{\sigma}_I - \boldsymbol{\sigma}; (\omega R\widetilde{\varepsilon}_h)_I, (\omega R\widetilde{\boldsymbol{e}}_h)_I) \\ & \quad \leq Ch^k \left(\|u\|_{k+1,G} + \|\boldsymbol{\sigma}\|_{k+1,G}\right) \left(\|R\widetilde{\varepsilon}_h\|_{1,G} + \|R\widetilde{\boldsymbol{e}}_h\|_{1,G}\right) . \end{aligned} \tag{4.35}$$

Finally, for the last term in (4.30),

$$\mathcal{L}(R\widetilde{\varepsilon}_h, R\widetilde{\boldsymbol{e}}_h; \varepsilon_h, \boldsymbol{e}_h) \leq C\left(\|\varepsilon_h\|_{0,G} + \|\boldsymbol{e}_h\|_{0,G}\right)\left(\|R\widetilde{\varepsilon}_h\|_{1,G} + \|R\widetilde{\boldsymbol{e}}_h\|_{1,G}\right) . \tag{4.36}$$

Estimate (4.22) follows from (4.24)–(4.36). □

Now we are ready to prove estimates (3.1) and (3.2).

Proof of Theorem 3.1. Let $\Omega_0 \subset\subset G_1 \subset\subset G_2 \subset\subset \Omega_1$. Applying (4.22) with Ω_0 and G_1, and (4.6) with G_1 and G_2, we get (3.1). Applying Lemma 4.1 again, we get (3.2). □

The analysis below concerns the case $k+1=r$. Recall that u_h^* is defined in (3.3).

LEMMA 4.3. *Let $G_0 \subset\subset G \subset\subset \Omega_1$ and $k+1=r$. Then*

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,G_0} \leq\; & C\left(\|u - u_h\|_{0,G} + \|u - u_h^*\|_{0,G}\right) \\ & + C\left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,G} + h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,G}\right) \\ & + Ch^r\|\boldsymbol{\sigma}\|_{r+1,G} \end{aligned} \tag{4.37}$$

Proof. Denote $\varepsilon_h = u_h^* - u_h$, $\boldsymbol{e}_h = \boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h$. Let $\omega \in C_0^\infty(\widetilde{G})$, $\omega = 1$ on G_0, $G_0 \subset\subset \widetilde{G} \subset\subset G$. Denote $\widetilde{\varepsilon}_h = \omega\varepsilon_h$, $\widetilde{\boldsymbol{e}}_h = \omega\boldsymbol{e}_h$. Using the projection operator R defined in (4.23) with $w = \widetilde{\varepsilon}_h$, $\boldsymbol{p} = \widetilde{\boldsymbol{e}}_h$, we have

$$\|\boldsymbol{e}_h\|_{1,G_0} \leq \|\widetilde{\boldsymbol{e}}_h\|_{1,G} \leq \|\widetilde{\boldsymbol{e}}_h - R\widetilde{\boldsymbol{e}}_h\|_{1,G} + \|R\widetilde{\boldsymbol{e}}_h\|_{1,G} . \tag{4.38}$$

Analogously to (4.26),

$$\|\widetilde{\varepsilon}_h - R\widetilde{\varepsilon}_h\|_{1,G} + \|\widetilde{\boldsymbol{e}}_h - R\widetilde{\boldsymbol{e}}_h\|_{1,G} \leq C\left(\|\widetilde{\varepsilon}_h - (\widetilde{\varepsilon}_h)_I\|_{1,G} + \|\widetilde{\boldsymbol{e}}_h - (\widetilde{\boldsymbol{e}}_h)_I\|_{1,G}\right), \tag{4.39}$$

where $(\cdot)_I$ means the standard interpolant. For any element $K \subset G$,

$$\begin{aligned} \|\widetilde{\varepsilon}_h - (\widetilde{\varepsilon}_h)_I\|_{1,K} & \leq Ch^k\|\omega\varepsilon_h\|_{k+1,K} \\ & \leq Ch^k\|\varepsilon_h\|_{k,K} \\ & \leq C\|\varepsilon_h\|_{0,K} \end{aligned}$$

which leads to

$$\|\widetilde{\varepsilon}_h - (\widetilde{\varepsilon}_h)_I\|_{1,G} \leq \|\widetilde{\varepsilon}\|_{0,G} . \tag{4.40}$$

Substituting the above estimate and (4.28) into (4.39),

$$\begin{aligned} \|\widetilde{\boldsymbol{e}}_h - R\widetilde{\boldsymbol{e}}_h\|_{1,G} & \leq C\left(\|u_h^* - u_h\|_{0,G} + h\|\boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h\|_{1,G}\right) \\ & \leq C\left(\|u - u_h^*\|_{0,G} + \|u - u_h^*\|_{0,G}\right) \\ & \quad + Ch\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,G} + Ch^{r+1}\|\boldsymbol{\sigma}\|_{r+1,G} . \end{aligned} \tag{4.41}$$

Using the coercivity property (2.24),

$$(4.42)\qquad \begin{aligned} C\left(\|R\widetilde{\varepsilon}_h\|_{1,G}^2 + \|R\widetilde{e}_h\|_{1,G}^2\right) &\le a(R\widetilde{\varepsilon}_h, R\widetilde{e}_h; R\widetilde{\varepsilon}_h, R\widetilde{e}_h) \\ &= a(\widetilde{\varepsilon}_h, \widetilde{e}_h; R\widetilde{\varepsilon}_h, R\widetilde{e}_h) \\ &= a(\varepsilon_h, e_h; \omega R\widetilde{\varepsilon}_h, \omega R\widetilde{e}_h) + \mathcal{L}(R\widetilde{\varepsilon}_h, R\widetilde{e}_h; \varepsilon_h, e_h) \ , \end{aligned}$$

where (4.10) was used. For the second term on the right–hand side of (4.42) we have

$$(4.43)\qquad \begin{aligned} \mathcal{L}(R\widetilde{\varepsilon}_h, R\widetilde{e}_h; \varepsilon_h, e_h) &\le C\left(\|\varepsilon_h\|_{0,G} + \|e_h\|_{0,G}\right)\left(\|R\widetilde{\varepsilon}_h\|_{1,G} + \|R\widetilde{e}_h\|_{1,G}\right) \\ &\le C\Big(\|u - u_h^*\|_{0,G} + \|u - u_h\|_{0,G} \\ &\qquad + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,G} + h^r\|\boldsymbol{\sigma}\|_{r+1,G}\Big) \\ &\quad \times \left(\|R\widetilde{\varepsilon}_h\|_{1,G} + \|R\widetilde{e}_h\|_{1,G}\right) . \end{aligned}$$

Define a projection operator S such that $Sw \in V_h(G)$ for $w \in V(G)$ and [NOTE: it is enough to assume that $w \in H^1(G)$ – check!]

$$(4.44)\qquad (A\,\mathrm{grad}\,(w - Sw), \mathrm{grad}\, v_h)_{0,G} + (c(w - Sw), cv_h)_{0,G} = 0$$

for all $v_h \in V_h(G)$. Then

$$(4.45)\qquad \begin{aligned} &a(\varepsilon_h, e_h; \omega R\widetilde{\varepsilon}_h, \omega R\widetilde{e}_h) \\ &\quad = a(u_h^* - u_h, \boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h; \omega R\widetilde{\varepsilon}_h - S(\omega R\widetilde{\varepsilon}_h), \omega R\widetilde{e}_h - (\omega R\widetilde{e}_h)_I) \\ &\qquad + a(u_h^* - u_h, \boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h; S(\omega R\widetilde{\varepsilon}_h), (\omega R\widetilde{e}_h)_I) \ . \end{aligned}$$

It is easy to see that

$$(4.46)\qquad \|\omega R\widetilde{\varepsilon}_h - S(\omega R\widetilde{\varepsilon}_h)\|_{1,G} \le Ch\|R\widetilde{\varepsilon}_h\|_{1,G} \ .$$

Then (4.46), (4.32), and integration by parts lead to

$$(4.47)\qquad \begin{aligned} &a(u_h^* - u_h, \boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h; \omega R\widetilde{\varepsilon}_h - S(\omega R\widetilde{\varepsilon}_h), \omega R\widetilde{e}_h - (\omega R\widetilde{e}_h)_I) \\ &\quad \le C\left(\|u_h^* - u_h\|_{0,G} + \|\boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h\|_{1,G}\right) \\ &\qquad \times \left(\|\omega R\widetilde{\varepsilon}_h - S(\omega R\widetilde{\varepsilon}_h)\|_{1,G} + \|\omega R\widetilde{e}_h - (\omega R\widetilde{e}_h)_I\|_{1,G}\right) \\ &\quad \le C\left(\|u - u_h\|_{0,G} + \|u - u_h^*\|_{0,G} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,G} + h^r\|\boldsymbol{\sigma}\|_{r+1,G}\right) \\ &\qquad \times\ h\left(\|R\widetilde{\varepsilon}_h\|_{1,G} + \|R\widetilde{e}_h\|_{1,G}\right) \end{aligned}$$

For the second term on the right–hand side of (4.45) we use the orthogonality property (2.21). Then

$$(4.48)\qquad \begin{aligned} &a(u_h^* - u_h, \boldsymbol{\sigma}_I - \boldsymbol{\sigma}_h; S(\omega R\widetilde{\varepsilon}_h), (\omega R\widetilde{e}_h)_I) \\ &\quad = a(u_h^* - u, \boldsymbol{\sigma}_I - \boldsymbol{\sigma}; S(\omega R\widetilde{\varepsilon}_h), (\omega R\widetilde{e}_h)_I) \\ &\quad \le C\left(\|u - u_h^*\|_{0,G} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_I\|_{1,G}\right)\left(\|S(\omega R\widetilde{\varepsilon}_h)\|_{1,G} + \|(\omega R\widetilde{e}_h)_I\|_{1,G}\right) \\ &\quad \le C\left(\|u - u_h^*\|_{0,G} + h^r\|\boldsymbol{\sigma}\|_{r+1,G}\right)\left(\|R\widetilde{\varepsilon}_h\|_{1,G} + \|R\widetilde{e}_h\|_{1,G}\right) \ . \end{aligned}$$

At the last step we used inequalities similar to (4.33) and (4.34). Combining (4.42), (4.43), (4.45), (4.47), and (4.48),

$$\begin{aligned}\|R\widetilde{e}_h\|_{1,G} &\le C\left(\|u-u_h\|_{0,G}+\|u-u_h^*\|_{0,G}\right)\\ &+C\left(\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,G}+h\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G}\right)\\ &+Ch^r\|\boldsymbol{\sigma}\|_{r+1,G}\,.\end{aligned} \tag{4.49}$$

Hence the desired result follows from (4.38), (4.41), and (4.49). □

The next lemma concerns interior error estimates for the projection u_h^* defined in (3.3). The proof is given in the fundamental work of Nitsche and Schatz [20].

LEMMA 4.4. *Let $G_0 \subset\subset G \subset\subset \Omega_1$. Then*

$$\begin{aligned}\|u-u_h^*\|_{0,G_0} &\le Ch^2\|u-u_h^*\|_{1,G}+C\|u-u_h^*\|_{-1,G}\\ &+Ch^{k+1}\|u\|_{k+1,G}\,. \quad \square\end{aligned} \tag{4.50}$$

Now we are ready to prove estimates (3.4)–(3.7).

Proof of Theorem 3.2. Estimates (3.4) and (3.5) follow in the same manner as (3.1) and (3.2). It remains to prove (3.6) and (3.7).

Let G_1 and G_2 be subdomains of Ω_1 such that $\Omega_0 \subset\subset G_1 \subset\subset G_2 \subset\subset \Omega_1$. Applying Lemma 4.3 for Ω_0 and G_1,

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,\Omega_0} &\le C\left(\|u-u_h\|_{0,G_1}+\|u-u_h^*\|_{0,G_1}\right)\\ &+C\left(\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,G_1}+h\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G_1}\right)\\ &+Ch^r\|\boldsymbol{\sigma}\|_{r+1,G_1}\,.\end{aligned} \tag{4.51}$$

Next, we use inequality (4.6) from Lemma 4.1 for G_1 and G_2

$$\begin{aligned}\|u-u_h\|_{0,G_1}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,G_1} &\le Ch\left(\|u-u_h\|_{1,G_2}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G_2}\right)\\ &+C\left(\|u-u_h\|_{-1,G_2}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,G_2}\right)\,.\end{aligned} \tag{4.52}$$

Lemma 4.4 for G_1 and G_2 leads to

$$\|u-u_h^*\|_{0,G_1} \le Ch^2\|u-u_h^*\|_{1,G_2}+C\|u-u_h^*\|_{-1,G_2}+Ch^r\|u\|_{r,G_2}\,. \tag{4.53}$$

From (4.51), (4.52), and (4.53),

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,\Omega_0} &\le Ch^2\|u-u_h\|_{1,G_2}+Ch^2\|u-u_h^*\|_{1,G_2}\\ &+C\left(\|u-u_h\|_{-1,G_2}+\|u-u_h^*\|_{-1,G_2}\right)\\ &+C\left(h\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{1,G_2}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{-1,G_2}\right)\\ &+Ch^r\left(\|u\|_{r,G_2}+\|\boldsymbol{\sigma}\|_{r+1,G_2}\right)\,.\end{aligned} \tag{4.54}$$

In order to obtain the desired estimate (3.6) we have to apply inequality (3.1) for G_2 and Ω_1 to the term $\|u-u_h\|_{1,G_2}$ in (4.54).

Next, we use inequality (4.7) from Lemma 4.1 for Ω_0 and G_1:

$$\begin{aligned}\|u - u_h\|_{-1,\Omega_0} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega_0} \le C \left(h^2\|u - u_h\|_{1,G_1} + h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,G_1}\right) \\ + C\left(\|u - u_h\|_{-2,G_1} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{-1,G_1}\right) .\end{aligned} \tag{4.55}$$

In order to bound the terms $\|u - u_h\|_{1,G_1}$ and $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{1,G_1}$ inequalities (3.4) and (3.6) for G_1 and Ω_1 are used. This concludes the proof of estimate (3.7). □

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