AUCTIONS WITH ENDOGENOUS OPTING-OUT FEES AND RECURSIVE WINNING PROCEDURES FROM THE TALMUD

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ABSTRACT

In this article I analyze auctions with uncertain value, cooling-off right in return to endogenously determined opting-out fee and recursive winning procedure. I show that although bidding strategies in these auctions are less aggressive compared to auction with costless withdrawals (Asker, 2000), expected revenues are usually higher. In addition I show that these auctions are spurious-bidding robust and almost shill-bidding robust.

Keywords and Phrases: Auctions, Withdrawal rights, Cooling-off rights.

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1. Introduction

The Latin *caveat emptor* doctrine stipulated that buyer's obligations and commitments are binding and irreversible, except in cases of fraud. This doctrine was not generally accepted even in ancient eras[1], since even fully rational agent may regret when the real value of the item is state-dependent and revealed post-factum[2]. Modern legislation in many countries allows parties to certain types of contracts to withdraw from their liabilities after signing the contract within a specified period[3]. The withdrawal right has many names in various legal contexts; consumer's withdrawal right is known as *cooling-off* right. Nevertheless, most classical economic analyses of auctions implicitly adopted the *caveat emptor* doctrine and assumed that bidding is equivalent to writing a call option; the bidder undertakes to buy the auctioned item for his bid (in case of winning) and withdrawals are either precluded or penalized. The auctioneer's call option expires, however, once a higher bid is submitted. It is usually assumed that if the winner defaults the auction is canceled and default fee (if imposed) is paid by the winner only.

In a pioneering research, Asker (2000) studied the effect of costless cooling-off right in first price sealed-bid auctions with uncertain value, and showed that it raises the auctioneer's expected revenue if having obliged to buy the auctioned item under the least desired contingency incurs negative utility to the winner.

Asker tested his model experimentally and found some support to his theoretical predictions. Nevertheless, its fairness and applicability is questionable. Regarding fairness, the replacement of the *caveat emptor* doctrine by a no less radical *bona fide* doctrine which totally exempts buyers from any responsibility and burdens the entire default risk on the seller could be justified as a device against fraud, but hardly when the real value of the auctioned item is unknown in real time to the seller as well. Regarding practical applicability, Asker's analysis is based on the unrealistic assumption of risk-neutrality. Indeed, the risk-neutrality assumption is indispensable for analytic solvability of the model[4], and having no choice I am going to use it too.

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[1] See for example, Leviticus 25, 14-17.

[2] Posterior regret due prior irrationality or myopia was recognized in ancient eras as the moral basis for repentance and annulment of vows. See for example Jeremiah Ch. 3 Vs. 14, 22, and Ch. 31, Vs. 17, 21.

[3] See Asker (2000) for a brief survey.

[4] The effect of risk-aversion on contestants' behavior is ambiguous. See for example, Hilmann and Katz (1984), Hilmann and Samet (1987), Van Long & Vousde (1987), Skaperdas and Gan (1995), Konrad & Schlesinger (1997), Corenes and Hartley (2003), (2010), Van Long (2013). The same ambiguity applies also regarding the effect risk-aversion on bidding, unless certain types of utility

Nevertheless, it is crucial to be aware of the potential affect of risk-aversion on the model's equilibrium. For example, Asker's radical *bona fide* attitude may deter risk-averse sellers from using his auction apparatus. In addition, due to the introduction of costless cooling-off right Asker's model contains spurious asymmetric equilibria. Based on theoretical speculations Asker conjectured that spurious equilibria are implausible, but honestly reported observed spurious bids in some of his experiments. These observations cannot so easily be discarded as curiosity based on Asker's theoretical speculations, because his laboratory experiments ignored real world phenomena of collusion (e.g., bid-rigging and shill-bidding).

In this article I analyze an alternative auction model with uncertain value, recursive winning procedure and cooling-off right in return to endogenously determined opting-out fee. According to the Mishnah[5] this kind of auction was practiced in the Temple of the Jerusalem for redemption of sacred items that no longer could be used for ritual purposes. The Mishnah states:

> If one said: 'I will acquire it for ten *dinars* and another '[for] twenty' and another 'for thirty' and another 'for forty' and another 'for fifty' and he [that bid] fifty recanted, they take pledges from his property up to ten *dinars*. If he [that bid] forty recanted, they take pledges from his possession up to ten *dinars*. If he [that bid] thirty recanted, they take pledges from his possessions up to ten *dinars*. If he that bid twenty recanted they take pledges from his possession up to ten *dinars*. If he that bid ten recanted they sell [the asset] for what it is worth, and collect what remains from him who bid ten[6].

The Mishnah rules that: (a) bidder's liability does not expire at the submission of a higher bid, implying that bidding means guaranteeing a minimum price for the auctioned item; (b) The Mishnah establishes a *recursive winning procedure*. That is, if the winner recants the auctioned item is offered to the next highest bidder, and the withdrawer pays an opting-out fee to compensate for the auctioneer's loss; (c) The Mishnah rules that the opting-out fee is endogenously determined such that the total receipts of the auctioneer equals the original winning bid.

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function are assumed (Eső & WhiteSource, 2004).

[5] The Mishnah (from the Hebrew מִשְׁנָה, "study by repetition") is the earliest major written codex of the Jewish Rabbinic "oral law" tradition, redacted by R. Yehuda HaNasi around 200 CE. The Talmud (from the Hebrew תַּלְמוּד "study") contains collections of rabbinical traditions, mainly interpretations of the Mishnah. The earlier version of the Talmud is the Jerusalem Talmud (or the Palestinian Talmud), redacted around 400CE. The more prevailing version is the Babylonian Talmud, redacted around 500 CE. The Mishnah is written in Hebrew while both *Talmudim* are written in Aramaic.

[6] Mishnah, tractate Arakhin Ch. 8 §2.

The Talmud[7] quotes R. Hisda's ruling that this Mishnah applies to the case of *sequential* withdrawals, but if bidders recant *simultaneously* "we divide it [the compensation] among them." Unfortunately, the Talmud did not define the terms *sequential* and *simultaneous* withdrawals, did not explain why sequential and simultaneous withdrawals should be treated differently[8], and specified no sharing rule for the case of simultaneous withdrawals. Medieval commentators of the Talmud suggested 4 alternative sharing rules: Equal Award[9], Proportional *pro-rata* distribution[10], Recursive Incremental Allocation (RI)[11] and the Talmud Rule (Aumann & Maschler, 1985)[12].

I suggest that R. Hisda's distinction between sequential and simultaneous withdrawal refers to extensive and normal (strategic) game-forms, respectively. Sequential withdrawals relate to open ascending auctions while simultaneous withdrawals relate to sealed-bid auctions. This interpreteation of the Talmudic ruling is perhaps unorthodox, but it should be emphasized that while this study is inspired by the Talmud it is definetly *not* a Talmudic research[13].

The Talmudic auction model adresses both problems associated above with Asker's model. First, due to continuous bdder's liability, recursive winning procedure and endogenous opting-out fee, it is fairer because it divides the risk between all bidders. On the one hand it assures the the item will be sold for the highest bid, while on the other hand it spreads the posterior value uncertainty among all bidders who collectively and endogenously determine their opting-out fees. These legal novelties are particularly relevant for modern buisness environment characterized with contingent posterior values, consumer's remorse and winners' defaults (e.g. auctions of drilling concessions, confiscated assets etc.). Second, it can be hardly believed that

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[7] Babylonian Talmud, tractate Arakhin 27b.

[8] The classical commentators of the Talmud also did not address this question, probably because it was unpractical after the destruction of the Temple. For a peculiar explanation see R. Y. Abramski commentary on the Tosefta, *Chazon Yehezkel*, (Ararkhin 4, 8).

[9] Maimonides, (Laws of matrimony Ch. 17 §8 and Laws of Arachin and Haramin Ch. 8 §4).

[10] R. Hananel (990-1053, quoted in Tosfot Kethuboth 93a, starting at *Rabbe*).

[11] Rashi (bArachin 27b starting at *Meshalshin*), Rabad (Glosses on Rif Kethuboth 93a), Abraham Ibn-Ezra, (*Sefer Ha-Mispar*, with German translation by M. Silberberg), *Kaufmann Verlag*, Frankfurt a. M. (1895) pp. 57. This allocation corresponds to the Shapley (1953) value, see Littlechild & Owen (1973) and O'neill (1982).

[12] R. Gershom, (bArakhin 27b starting at *Tania*), see Lipschütz & Schwarz (2015).

[13] Indeed, I am unaware of historical evidence for using sealed-bid auctions in ancient eras. Neverhteless, there are plenty of indications that Talmudic Sages were familiar with the notions of simultaneous moves and information sets. See for example bMegilah 9a, Rashbam on Baba-Batra 107b (starting at *ubedin*), mSanhedrin (3, 6) and more.

an inferior or inefficient auction model could sustain in the Jerusalem Temple thoughout centuries[14], indicating that the Talmudic auction model is applicable.

As shown below, bidders in Talmudic open ascending auction with recursive winning procedure and endogenous opting-out fee bid exactly as in a regular English auction with certain value while bidders in Talmudic sealed-bid auction tend to bid less aggresively then bidders in Asker's auction with costless cooling-off right. Nevertheless, usually, expected revneue in the Talmudic sealed-bid auction exceeds expected revenues in both regular sealed-bid auction with no cooling-off right and Asker's auction. Finally, unlike Asker's auction the Talmudic sealed-bid auction is spurious bidding free and if the auction is suficiently competitive it is also shill-bidding robust. In other words, as in Bulow and Klemperer (2002), a seller in a Talmudic sealed-bid auction can even benefit from limiting the number of bidders.

The article proceeds as follows. Section 2 contains a brief survey of related literature. Section 3 analyzes the benchmark models of auctions with no cooling-off right. Section 4 reanalyzes the model of open ascending auction assuming costless cooling-off right and presents Asker's results regarding effect of costless withdrawals on sealed bid auction. Section 5 analyzes Talmudic open ascending and sealed-bid auctions equilibria. Section 6 compares expected revenues of various auction types. Section 7 discusses the possibilities of spurious equilibria and shill-bidding in a Talmudic sealed-bid auction. Section 8 concludes and summarizes. The proofs of all propositions are relegated to the appendix (section 99).

2. RELATED LITERATURE

As mentioned above, the economic literature usually assumed that a bid is binding unless a higher bid is submitted, and in case of default the auction is canceled and a default penalty (if imposed) is paid by the withdrawer only. A notable exceptional branch of this literature studied a special sort of consumer's remorse in common value auctions known as *winner's cours*, overbidding due to exaggerated optimism regarding the real value of the auctioned item[15]. The effect of contingent

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[14] No doubt that the Second Temple was destructed in 70 CE. According to the rabbinic tradition it was constructed in 350 BCE (*Seder Olam Rabbah,* Ch. 29), implying that it stood for 420 years. Contemporary historians believe that it was constructed in 516 BCE, implying that it stood for 586 years (Goldwurm, 1982).

[15] For example, Bazerman and Samuelson (1983) explained that usually the estimate closest to the true value of the auctioned item is either the average or the median, while the winning bid is, of course, the highest, and the amount of overestimation often exceeds the difference between the winning bidder's estimate and his bid. See also Case (1979), Oren & Williams (1975), Rothkopf (1980), Winkler &

winner's curse on bidders' behavior has been studied theoretically and empirically, and several potential anomalies associated with auctions have been attributed to the winner's curse effect[16].

Many authors claimed that since the history of bidders' drop-out in open English auctions is common knowledge, winner's course is expected in sealed-bid auctions only. Defaults, however, occur in English auctions as well[17] because winner's curse is one of six potential causes of winner's default documented in the literature[18], indicating that the exposure of bidders to consumer's remorse risk is not limited to common value auctions and defaults are not necessarily related to winner's course. Bidding may be a win-win strategy for bidders on the edge of bankruptcy. If the realized value of the auctioned object exceeds expectations, the bidder may be saved from bankruptcy and make even a nice profit. Otherwise, the bidder goes bankrupt but this was anyway expected[19]. Moreover, in contract auctions bidders on the edge of bankruptcy do not have to take the risks of the project into account and thus may be incentivized to bid more competitively[20]. In certain types of auctions the bidder is incentivized to submit multiple bids and withdraw a winning bid in order to buy the auctioned item for his own lower bid[21].

Two remedies were suggested in the literature for consumer's remorse (especially for winner's curse), second-price sealed-bid auction[22] and weighted sum sealed-bid auction[23]. However, as far as I know these suggestions remained theoretical exercises and have never been applied in practice. Common remedies against consumer's remorse are mainly precautionary bidding strategies and cooling-off rights and the common remedies for consumer's remorse risk in regular market transactions are seller's reputation and warranties[24]. Stipulated damage payments for winner's default, (*liquidated damage* clauses) are the most commonly used hedges for sellers. Other measures taken by sellers are performance bonds, third party guarantees, like letters of

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Brooks (1980), Kagel & Levin (1986), Thaler (1988), Lind & Plott (1991) to mention only few.

[16] E.g. upward sloping demand curve (Bulow & Klemperer, 2002), and the turn of winner's curse to "winner's blessing" for bidders characterized by a DARA utility function (Eső & WhiteSource, 2004).

[17] Lamping (2007).

[18] See for example Harstad & Rothkopf (1995), Zheng (2001), Calveras, Ganuza, & Hauk (2004).

[19] Klemperer (2002), Borad (2007).

[20] Zheng (2001), Klemperer (2002).

[21] Rothkopf (1991).

[22] Vickery (1961).

[23] Riley (1988).

[24] Roberts (2011).

credit and surety bonds and, in certain auctions, ruling out "extreme" or "suspicious" bids. However, liquidated damage clauses in common law states are limited to the ex-ante expected damage which may be lower than the realized damage[25].

Asker (2000) showed that under certain assumptions costless cooling-off right increases expected revenues but as he indicated, the effect of costless cooling-off right on bidders' behavior has not been widely explored. In fact, Asker's reference list contains a single relevant item, von Ungern-Sternberg (1991), who studied contract auctions in which the winner may consider withdrawal due to winning in another (more attractive) contract auction where the winnings in the two auctions are uncorrelated, and showed that if withdrawals are allowed, bidders may bid more aggressively. Most studies have assumed that withdrawals are either costless or incur an arbitrary predetermined fixed penalty which is burdened on the winning withdrawer, and in case of withdrawal the auction is canceled[26]. Other studies applied a mechanism design approach to determine the optimal penalty in case of contract breach[27], but usually these mechanisms are inapplicable by standard auctions format.

This article is related to Harstad and Rothkopf (1995) who studied a sealed-bid auction with withdrawable bids in return for "compensation penalties," where in case of cooling-off the item is awarded to the subsequent bidder. They showed that on the average the bid-taker is better off with this scheme than if withdrawals are not allowed. They considered two compensation penalties schemes: (a) the difference between the withdrawn bid and its subsequent one; (b) a predetermined fraction of the withdrawn bid. The authors emphasized that the first penalty scheme is attractive in theory but may not be practical. Bidders may prefer the second penalty scheme over a withdrawal option for a penalty which is unknown at the bidding time. The Talmudic auction model addresses this shortcoming, as shown below.

3. BENCHMARK AUCTIONS WITH NO COOLING-OFF RIGHT

3.1. Basic sett-up and Notation

Consider a set N of risk-neutral bidders competing in an auction over an item which its posterior value for bidder $i \in N$ is state dependent. In state 1, with probability $(1-p)$, agent i evaluates the item by v_i which is uniformly distributed

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[25] Chillemi & Mezzetti (2014).
[26] E.g. Waehrer (1995), Spulber (1990) and more.
[27] Chillemi and Mezzetti (2014).

over a support normalized to $[0,1]$[28]. In state 2, with probability p, the value of the item is $z\ \forall i \in N$. The value of the item for the seller is assumed zero in all states. Bidder i's bidding strategy in auction of type α is denoted by $b_i^\alpha = \beta_i^\alpha(v_i)$, the bids' vector by $\mathbf{b}^\alpha = \{b_i^\alpha\}_{i=1}^n$ and the values' vector by $\mathbf{v} = \{v_i\}_{i=1}^n$. For convenience, $\mathbf{b}^k$ and $\mathbf{v}$ are arrayed in ascending order, namely $b_i^\alpha \le b_{i+1}^\alpha$ and $v_i \le v_{i+1}$, implying that $b_n^\alpha = \max \mathbf{b}^\alpha$ and $v_n = \max \mathbf{v}$. Following the literature since Vickery (1961), the analysis is confined to symmetric equilibria.

3.2. Auctions with Certain Value

The best response strategy of bidder $i \in N$ in a regular open ascending English auction with certain value, is to stay active as long as $b_i^E \le v_i$. It follows that if the current bid is $b_{n-1}^E = v_{n-1}$ where $v_{n-1} = \max \mathbf{v}_{-n}$, bidder n can win the auction with certainty by bidding $b_n^E = b_{n-1}^E + \varepsilon$, $(\varepsilon > 0)$. Letting $\varepsilon \to 0$ implies,

(1) $$b_n^E = v_{n-1}.$$

Bidder i's prior expected payoff function in a regular first price sealed-bid auction with certain value is,

(2) $$\pi_i^{SB} = \left[V\left(b_i^{SB}\right)\right]^{n-1}\left(v_i - b_i^{SB}\right),\ \forall i \in N$$

where $b_i^{SB} = \beta_i^{SB}(v_i)$ and $\left[V\left(b_i^{SB}\right)\right]^{n-1} = \Pr\left(b_i = \max \mathbf{b}\right)$ is bidder i's winning probability. Since the analysis is confined to symmetric equilibria, we may replace $V\left(b_i^{SB}\right)$ by v_i. Rationality constraint implies that bidder $i \in N$ is active in the auction if and only if $\pi_i^{SB} \ge 0$. Assuming that the sole constraint imposed on bidders is $b_i^{SB} \ge 0$ implies that this condition is fulfilled if and only if $v_i \ge v^* = 0$.

Differentiating (2) with respect to b_i^{SB} yields the first order condition:

(3) $$\frac{\partial \pi_i^{SB}}{\partial b_i^{SB}} = (n-1)v_i^{n-2}\left(v_i - b_i^{SB}\right) - v_i^{n-1} = 0,\ \forall i \in N.$$

b_i^{SB} can be derived directly from (3), or indirectly using the envelope theorem, which is the prevailing method in the literature and will be applied throughout this article. By the envelope theorem, if b_i^{SB} is the maximization variable and v_i is a

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[28] That is, the original support $[v_{min}, v_{max}]$ is normalized to $\left[\frac{v_{min}-v_{min}}{v_{max}-v_{min}}, \frac{v_{max}-v_{min}}{v_{max}-v_{min}}\right] = [0,1]$.

varied parameter $\pi_i'^{SB}(v_i) = \left[v\left(b_i^{SB}\right)\right]^{n-1}$. Therefore,

(4) $$\pi_i^{SB}(v_i) = \pi_i^{SB}(0) + \int_0^{v_i} x^{n-1} dx .$$

Since $\pi_i^{SB}(0) = 0$ by definition, equating (2) with (4) and solving for b_i^{SB} yields,

(5) $$b_i^{SB} = \left(\frac{n-1}{n}\right) v_i .$$

3.3. Auctions with Uncertain Value

Define bidder i's prior expected payoff function in an open ascending English auction with uncertain value and no cooling-off right,

(6) $$E\left(\pi_i^I\right) = (1-p)v_i + pz - b_i^I ,$$

Bidder i's best response is to stay active in the auction until $E\left(\pi_i^I\right) = 0$. The rationality constraint implies that bidder i is active in this auction if and only if $E\left(\pi_i^I\right) \geq 0$. This condition is satisfied for all $z \geq 0$, and for $z < 0$ if and only if $v_i \geq v_i^* = \frac{-pz}{1-p}$. Thus,

(7) $$b_i^I = \begin{cases} (1-p)v_i + pz & 0 \leq z \text{ or } z < 0 \text{ and } v_i \geq \frac{-pz}{1-p} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the current bid is $b_{n-1}^I = (1-p)v_{n-1} + pz$. Bidder n can win with certainty by bidding $b_{n-1}^I + \varepsilon$. Letting $\varepsilon \to 0$ implies,

(8) $$b_n^I = \begin{cases} (1-p)v_{n-1} + pz & 0 \leq z \text{ or } z < 0 \text{ and } v_{n-1} \geq \frac{-pz}{1-p} \\ 0 & \text{otherwise.} \end{cases}$$

Bidder i's posterior expected payoff function in a first price sealed-bid auction with uncertain value and no cooling-off right (conditional on winning) is

(9) $$E\left(V_i^{II}\right) = (1-p)v_i + pz - b_i^{II} ,$$

where $b_i^{II} = \beta_i^{II}(v_i)$ is bidder i's bidding strategy in this auction. The rationality constraint implies that bidder i is active in this auction if either $z \geq 0$ or $v_i \geq v_i^* = \frac{-pz}{1-p}$ if $z < 0$. Bidder i's prior expected payoff function is,

(10) $$E\left[\pi_i^{II}(v_i)\right] = \left[V\left(b_i^{II}\right)\right]^{n-1}\left[(1-p)v_i + pz - b_i^{II}\right], \forall i \in N .$$

Solving for b_i^{II} using the indirect derivation method described above yields,

$$(11)\qquad b_i^{II} = \begin{cases} (1-p)b_i^{SB} + pz & z \ge 0 \\ (1-p)\left[b_i^{SB} + \frac{1}{nv_i^{n-1}}\left(\frac{-pz}{1-p}\right)^n\right] + pz & z < 0 \text{ and } v_i \ge \frac{-pz}{1-p} \\ 0 & \text{otherwise.} \end{cases}.$$

Define $\Delta_{b_i^E, b_i^I} = b_i^E - b_i^I$ and $\Delta_{b_i^{SB}, b_i^{II}} = b_i^{SB} - b_i^{II}$.

Proposition 1:

a. $\Delta_{b_i^E, b_i^I} \ge 0 \Leftrightarrow v_i \ge z$.

b. $z < 0$ and $v_i \ge \frac{-pz}{1-p} \Rightarrow \Delta_{b_i^{SB}, b_i^{II}} \ge 0 \Leftrightarrow p \ge \dfrac{v_i\left[\frac{n(z-nb_i^{SB})}{z}\right]^{\frac{1}{n-1}}}{v_i\left[\frac{n(z-nb_i^{SB})}{z}\right]^{\frac{1}{n-1}} - z}$.

Verbally, Proposition 1 states that in auctions with no cooling-off right, uncertainty regarding the value of the auctioned item generally decreases bidding.

4. Auctions with Costless Cooling-Off Right

With the introduction of costless cooling-off right, in state 2 the winner can simply cool-off with no cost. Assuming that rational bidder will not spend time and effort to submit a spurious bid just for the fun of cooling-off, implies that bidders' behavior in an open ascending auction with costless cooling-off right is identical to their behavior in a regular English auction. Namely, the winning price in this auction is given by (1).

The expected payoff of bidder $i \in N$ in Asker (2000) sealed-bid auction with costless cooling-off right is,

$$(12)\quad E\left[\pi_i^{SA}(v_i)\right] = \begin{cases} \left[V\left(b_i^{II}\right)\right]^{n-1}\left[(1-p)\left(v_i - b_i^{SA}\right) + p\left(z - b_i^{SA}\right)\right] & b_i^{SA} \le v_i \le z \\ 0 & \text{otherwise} \end{cases}, \forall i \in N.$$

As indicated by Asker (2000), z must be the absolute lower bound of any bidding strategy, since even if $v_i < z$, the bidder is incentivized to bid at least z because in case of winning the bidder's payoff is strictly non-negative and any lower bid will be bettered by other bidders. If $v_i > z$ and $z < 0$ the winner will surely cool-off if state 2 eventuates, therefore all bidders may ignore this possibility and bid according to Vickery (1961) equilibrium bidding function $b_i^{SA} = b_i^{SB} = \left(\frac{n-1}{n}\right)v_i$. If $z \ge 0$ then bidders characterized by $v_i \le z$ shall bid $b_i^{SA} = z$, which is the lower bidding bound. The problem facing bidders with $v_i > z$, is isomorphic with the problem of bidders in an auction with a reserve price of z, implying that in this case $b_i^{SA} = b_i^{SB} + \frac{z^n}{nv^{n-1}}$. To summarize, the equilibrium bidding function in Asker (2000) sealed-bid auction with costless cooling-off right assuming no spurious equilibria is,

$$b_i^{SA} = \begin{cases} z & v_i \le z \\ b_i^{SB} + \dfrac{z^n}{nv_i^{n-1}} & 0 \le z < v_i \\ b_i^{SB} & z < 0 \end{cases} \tag{13}$$

Define $\Delta_{b_i^{SA}, b_i^{II}} = b_i^{SA} - b_i^{II}$. Although Asker (2000) proved that $\Delta_{b_i^{SA}, b_i^{II}} \ge 0$, allowing costless cooling-off is not necessarily the auctioneer's dominant strategy, because what matters for the auctioneer is not the effect of costless cooling-off right on equilibrium bidding functions but its effect on expected revenues. (See section 6).

5. Talmudic Auctions

The Talmudic auction law provides that: (a) bidding means guaranteeing a minimum price for the auctioned item; (b) if the winner recants the auctioned item is offered to the subsequent bidder, and the withdrawer pays an opting-out fee; (c) the opting-out fee is endogenously determined. The Talmud also rules that if the bidders withdraw *sequentially* the opting-out fee of withdrawer i is $\delta_i = b_i - b_{i-1}$, and if the bidders withdraw *simultaneously* the auctioneer's loss is burdened on all bidders according to a certain sharing rule.

As mentioned in the introduction, I suggest that *sequential withdrawals* refer to open ascending auctions and *simultaneous withdrawals* refer to sealed-bid auctions. An open ascending auction corresponds to an extensive game-form with complete information, because the history of biddings and withdrawals is common knowledge. In particular, in an open ascending auction every bidder knows $\delta_i = b_i - b_{i-1}$ exactly and in real time (at the submission of b_i). Therefore, it seems fair enough to burden δ_i on a withdrawer in an open ascending auction. A sealed-bid auction, on the other hand, corresponds to a normal (strategic) game form with incomplete information. At the submission of b_i, δ_i is an unknown random variable from bidder i's point of view. Therefore, the fairness of burdening bidder i with the posterior realization of δ_i in case of cooling-off is not self evident[29].

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[29] A fairer approach would allocate the burden of compensation between the withdrawers' set according to a certain sharing rule which fairly weighs each withdrawer prior expected contribution to the auctioneer's loss. As mentioned above, medieval Talmudic scholars suggested four alternative

5.1. Talmudic Open ascending Auction

Consider a set N of risk-neutral bidders in a Talmudic open ascending auction with uncertain value, and denote bidder i's bidding strategy in this auction by $b_i^{TE} = \beta_i^{TE}(v_i)$. According to the Talmudic law, if bidder n (the winner) cools-off, the item is sold to bidder $n-1$ for b_{n-1}^{TE} and bidder n pays an opting-out fee $\delta_n = b_n^{TE} - b_{n-1}^{TE}$. If bidder $n-1$ cools-off too, the item is sold to bidder $n-2$ and bidder $n-1$ pays $\delta_{n-1} = b_{n-1}^{TE} - b_{n-2}^{TE}$ and so on. It follows that in a Talmudic open ascending auction, bidder i may get the auctioned item either if $b_i^{TE} = \max \mathbf{b}^{TE}$ or if all higher bidders have cooled-off. If a subset $K = \{j | j = k,...,n\}$ of bidders recant and the auctioned item is finally sold for $t = \max(0, b_{k-1}^{TE})$, the auctioneer's loss is $L = b_n^{TE} - t$ and the opting-out fees are,

(14) $$\delta_i = \begin{cases} b_1^{TE} & i = 1 \\ b_i^{TE} - b_{i-1}^{TE} & i > 1. \end{cases}$$

The best response strategy of bidder $i \in N$ in a Talmudic open ascending auction with recursive winning procedure, is to stay active in the auction as long as $E(\pi_i^{TE}) = (1-p)(v_i - b_i^{TE}) - p\delta_i \geq 0$, implying that

(14) $$b_i^{TE} = v_i - \left(\frac{p}{1-p}\right)\delta_i,$$

and it follows that,

(15) $$b_n^{TE} = v_{n-1} - \left(\frac{p}{1-p}\right)\delta_{n-1} + \delta_n.$$

Letting $\delta_i \to 0 \ \forall i \in N$ implies that,

(16) $$b_n^{TE} = b_n^E = v_{n-1}.$$

Namely, the winning bid in a Talmudic open ascending auction with recursive winning procedure and endogenous opting-out fee equals winning bid in a regular English auction with uncertain value and costless cooling-off right. Intuitively, like the costless cooling-off right, the Talmudic mechanism removes uncertainty and thus restores the bidding strategy to its full certainty level. The significant difference between the two auction mechanisms is in risk allocation and expected revenue. In a

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sharing rules. The disputation between these scholars was focused on the question how to do this the fairest way. See Lipschütz & Schwarz (2015).

regular English auction with costless cooling-off right the auctioneer bears the entire default risk, therefore expected revenue is lower. By splitting the risk between the auctioneer and the bidders, the Talmudic auction mechanism manages to equalize the auctioneer's expected revenue with the auctioneer's revenue in a regular English auction with certain value (see section 6.1).

5.2. Talmudic Sealed Bid Auction

Consider a set N of risk-neutral bidders compete in a Talmudic first price sealed-bid auction with uncertain value, and denote bidder i's bidding strategy in this auction by $b_i^{TS}=\beta_i^{TS}\left(v_i\right)$. Suppose that if a subset $C\subseteq N$ of bidders cool-off, the auctioned item is finally sold for $t=\max\left(0,b_j^{TS}\right)$ where $b_j^{TS}=\max \mathbf{b}_{N\setminus C}^{TS}$ and the auctioneer's loss is $L=b_n^{TS}-t$. Define $K=\left\{j\middle|i\in C \text{ and } b_j^{TS}\geq b_i^{TS}\Rightarrow j\in C\right\}$, as the set of *effective withdrawers* namely, withdrawers who contributed positively to the auctioneer's loss. Suppose that R. Hisda's ruling "we divide it [the compenation] between them" refers to effective withdrawers only. That is, a withdrawer has to pay opting-out fee if and only if his withdrawal is effective[30].

Legally, distributing L fairly among the members of K is a problem of allocating common liability among *multiple toartfeasors*. Mathematically, this is a *dual bankruptcy problem*, $B^K=\left\langle K,L,\mathbf{g}\right\rangle$, where K is the effective withdrawers set, $L=b_n^{TS}-t$ is the auctioneer's loss, $g_i=b_i^{TS}-t$ is the maximum liability of withdrawer $i\in K$ and $\sum_{i\in K}g_i>L$. The collection of all dual bankruptcy problems is $\mathbb{B}$. A *solution* or *sharing rule* is a mapping $\varphi:\mathbb{B}\to\mathbb{R}^N$ satisfying $0\leq\varphi_i\left(B^K\right)\leq g_i,\ \forall i\in N$ and $\sum_{i\in N}\varphi_i\left(B^K\right)=L$, and the set of all sharing rules is $\mathbf{S}$. By definition, $\varphi_i\left(B^K\right)$ increases monotonically with L, which increases monotonically with b_n^{TS} and k.

The expected posterior payoff of bidder $i\in N$ in a Talmudic sealed-bid auction is,

$$E\left(\pi_i^{TS}\right)=\left(1-p\right)\left(v_i-b_i^{TS}\right)-p\min\left(b_i^{TS}-z,E\left[\varphi_i\left(B^K\right)\right]\right). \tag{17}$$

At first glance, the model seems insolvable because b_i^{TS} is probably a function of $E\left[\varphi_i\left(B^K\right)\right]$ which is apparently a function of b_i^{TS}. Proposition 2 provides a sufficient

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30 For example, suppose that $N=\{1,2,3\}$ and bidders 1 and 3 recant. In this case $K=\{3\}$. The withdrawal of bidder 1 is ineffective because the item is sold to bidder 2.

condition for analytical solvability of b_i^{TS}.

Proposition 2:

In a Talmudic sealed-bid auction, if $z \le b_i^{TS} - E\left[\varphi_i\left(B^K\right)\right], \forall i \in K, \ \forall K \subseteq N$ *then:*

(a) In state 2 $C = K = N$.

(b) $E\left[\varphi_i\left(B^N\right)\right] = \left(\frac{n+1-\left[V\left(b_i^{TS}\right)\right]^{n-1}}{n^2}\right) b_i^{TS}, \ \forall i \in N, \ \forall \varphi \in \mathbf{S}$.

Proposition 2 establishes that if z is sufficiently small, then in state 2 all bidders cool-off and bidder i's expected opting-out fee is independent of the auctioneer's choice of sharing rule.

There are two alternative settings regarding withdrawals in this auction:

(a) the bids' profile is published with the declaration of the winner. Under this setting L, $\mathbf{b}$ and $\boldsymbol{\delta}$ are common knowledge when a bidder has to decide whether to buy the auctioned item or cool-off.

(b) after the closure of the auction and the realization of the state, the winner is announced and has to decide whether to buy the auctioned item for b_n^{TS} or cool-off. If the winner cools-off, the second highest bidder is announced and required to make his decision and so on. Under this setting the bidder makes a decision under uncertainty regarding L, $\mathbf{b}$ and $\boldsymbol{\delta}$.

Proposition 3:

In subgame perfect equilibrium of a Talmudic sealed-bid auction, settings (a) and (b) are strategically equivalent[31].

Henceforth assume that $z \le b_i^{TS} - E\left[\varphi_i\left(B^K\right)\right], \forall i \in K, \ \forall K \subseteq N$. By Proposition 2, under this assumption $t = 0$, and (17) can be rewritten as,

(18) $$E\left(\pi_i^{TS}\right) = (1-p)\left(v_i - b_i^{TS}\right) - p\left(\frac{n+1-v_i^{n-1}}{n^2}\right) b_i^{TS},$$

By definition, $\varphi_i\left(B_{-i}^N, 0\right) = 0 \ \forall i \in N$, thus assuming that the sole constraint imposed on bidders is $b_i^{TS} \ge 0 \ \forall i \in N$ implies that the rationality constraint for bidder $i \in N$ is $v_i^* = 0, \ \forall i \in N$. The prior expected payoff of bidder $i \in N$ in this auction is,

(19) $$E\left[\pi_i^{TS}\left(v_i\right)\right] = \left[V\left(b_i^{TS}\right)\right]^{n-1}(1-p)\left(v_i - b_i^{TS}\right) - p\left(\frac{n+1-\left[V\left(b_i^{TS}\right)\right]^{n-1}}{n^2}\right) b_i^{TS}, \forall i \in N$$

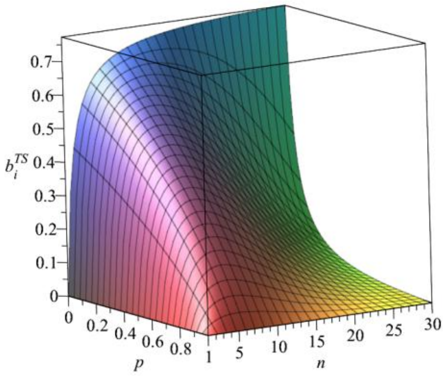
[31] It should be emphasized that Proposition 3 refers to strategic equivalence only. From moral and legal points of view, however, these two settings are substantially different.

where $\left[V\left(b_i^{TS}\right)\right]^{n-1} = \Pr\left(b_i^{TS} = \max \mathbf{b}^{TS}\right)$. Solving for b_i^{TS} using the indirect method described above yields,

(20) $$b_i^{TS} = \frac{(1-p)n(n-1)v_i^n}{\left[(1-p)n^2 - p\right]v_i^{n-1} + p(n+1)}$$ [32].

Figure 1 presents a simulation of b_i^{TS} calibrated for $v_i = 0.8$.

Figure 1



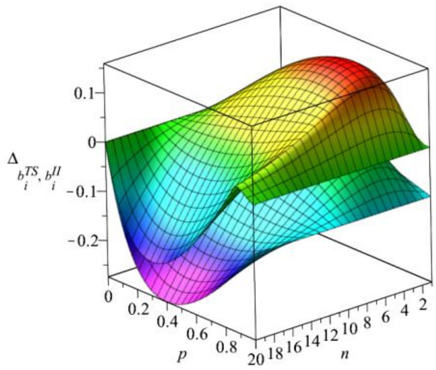
Notice that in Figure 1 b_i^{TS} is not monotonically increasing with n. The implications of this feature of b_i^{TS} are discussed in section 7.

Define $\Delta_{b_i^{TS}, b_i^{II}} = b_i^{TS} - b_i^{II}$. By (20) and (11)

(21) $$\Delta_{b_i^{TS}, b_i^{II}} = \frac{n(n-1)(1-p)v_i^n}{\left[(1-p)n^2 - p\right]v_i^{n-1} + p(n+1)} - \begin{cases} (1-p)b_i^{SB} + pz & z \geq 0 \\ (1-p)\left[b_i^{SB} + \frac{1}{nv_i^{n-1}}\left(\frac{-pz}{1-p}\right)^n\right] + pz & z < 0 \text{ and } v_i \geq \frac{-pz}{1-p} \\ 0 & \text{otherwise} \end{cases}$$

$\operatorname{sgn}\left(\Delta_{b_i^{TS}, b_i^{II}}\right)$ is generally indeterminate. Figure 2 presents 3D simulations of $\Delta_{b_i^{TS}, b_i^{II}}$ calibrated for $v_i = 0.8$, $z = -0.1$ (the upper graph) and $z = 0.1$ (the lower graph).

Figure 2



[32] Notice that, as expected, $\lim_{p \to 0} b_i^{TS} = b_i^{SB}$ and $\lim_{p \to 1} b_i^{TS} = 0$.

Figure 2 demonstrates that as expected from the ambiguous impact of n on b_i^{TS}, for relatively small values of n and p, bidders in a Talmudic sealed-bid auction tend to bid more aggressively compared with bidders in a regular sealed-bid auction with uncertain prior value and no cooling-off right.

Similarly, define $\Delta_{b^{SA},b^{TS}} = b^{SA} - b^{TS}$.

Proposition 4:

$\Delta_{b_i^{TS},b_i^{SA}} < 0$.

It should be emphasized that from the auctioneer's point of view both $\Delta_{b_i^{TS},b_i^{II}}$ and $\Delta_{b_i^{TS},b_i^{SA}}$ are irrelevant for comparisons of the Talmudic sealed-bid auction and Asker's auction, because what counts for the auctioneer is not equilibrium bidding strategies, but expected revenues (See section 6).

6. Expected Revenues

The expected revenue of the auctioneer in auction of type α is given by,

(22) $$E\left(R^{\alpha}\right) = \sum_{i=1}^{n} \int_{v^*}^{1} \beta_i^{\alpha}\left(v_i\right)\left[v\left(b_i^{\alpha}\right)\right]^{n-1} \mathrm{d}v_i .$$

6.1. Open Ascending Auctions

Inserting (1), (5), and (16) into (22) yields

(23) $$E\left(R^{E}\right) = E\left(R^{TE}\right) = \sum_{i=1}^{n-1} \int_{0}^{1} v_i \cdot v_i^{n-1} dv_i = E\left(R^{SB}\right) = \sum_{i=1}^{n} \int_{0}^{1} \left(\frac{n-1}{n}\right) v_i \cdot v_i^{n-1} dv_i = \frac{n-1}{n+1},$$

which is a well known result (Krishna, 2009). On the other hand, inserting (7) into (22) yields,

(24) $$E\left(R^{I}\right) = \begin{cases} (1-p)\left(\frac{n-1}{n+1}\right) + p\left(\frac{n-1}{n}\right)z & z \geq 0 \\ (1-p)\left(\frac{n-1}{n+1}\right) + pz\left[\left(\frac{n-1}{n}\right) - \frac{1}{n}\left(\frac{n-1}{n+1}\right)\left(\frac{-pz}{1-p}\right)^{n}\right] & z < 0 \text{ and } \frac{-pz}{1-p} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $\Delta_{R^E,R^I} = E\left(R^E\right) - E\left(R^I\right)$.

Proposition 5:

$\Delta_{R^E,R^I} \geq 0 \ \forall z < \frac{n}{n+1}$.

Table 1 summarizes this section.

Table 1: Expected Revenues across Open-Ascending Auctions

Value of z	(1) Open Ascending Auction with Uncertain Value and Costless Cooling-Off Right	(2) Relative Magnitude	(3) All Other Types of Open Ascending Auctions
$z < 0$ and $\frac{-pz}{1-p} > 1$	0	$<$	$\frac{n-1}{n+1}$
$z < 0$ and $\frac{-pz}{1-p} \le 1$	$(1-p)\left(\frac{n-1}{n+1}\right) + pz\left[\left(\frac{n-1}{n}\right) - \frac{1}{n}\left(\frac{n-1}{n+1}\right)\left(\frac{-pz}{1-p}\right)^n\right]$	$<$	$\frac{n-1}{n+1}$
$z \ge 0$	$(1-p)\left(\frac{n-1}{n+1}\right) + p\left(\frac{n-1}{n}\right)z$	$\le^*$	$\frac{n-1}{n+1}$

Notes: *For $z < \frac{n}{n+1}$.

6.2. Sealed-Bid Auctions

By inserting (11) into (22) we obtain that the expected revenue in a regular sealed-bid auction with uncertain value and no cooling-off right is,

(25)
$$E\left(R^{II}\right) = \begin{cases} pz\left[1-\left(\dfrac{-pz}{1-p}\right)^n\right] + (1-p)\left(\dfrac{n-1}{n+1}\right)\left[1-\left(\dfrac{-pz}{1-p}\right)^{n+1}\right] \\ \qquad + (1-p)\left(\dfrac{-pz}{1-p}\right)^n\left(1-\dfrac{pz}{1-p}\right) & z \le 0 \\ pz + (1-p)(1-z)\left(\dfrac{n-1}{n+1} + z^n\right) & 0 < z \le 1 \end{cases}$$

Inserting (13) into (22) yields the expected auctioneer's revenue in Asker's sealed-bid auction with costless cooling-off right,

(26)
$$E\left(R^{SA}\right) = \begin{cases} (1-p)\left(\dfrac{n-1}{n+1}\right) & z \le 0 \\ pz + (1-p)(1-z)\left(\dfrac{n-1}{n+1} + z^n\right) & 0 < z \le 1 \\ pz & 1 < z. \end{cases}$$

Asker (2000) proved that $\Delta_{R^{SA},R^{II}} = E\left(R^{SA}\right) - E\left(R^{II}\right) \ge 0 \Leftrightarrow z < 0$.

Inserting (20) into (22) yields the expected revenue of the auctioneer in a Talmudic sealed-bid auction,

(27)
$$E\left(R^{TS}\right) = \frac{n^2(n-1)(1-p)\left[p(n+1)^2 \mathrm{H}_0 + (2-3p)n^2 - p(2n+3)\right]}{2(n+1)\left[p-(1-p)n^2\right]^2},$$

where $\mathrm{H}_0 = \text{hypergeom}\left(\left[1, \frac{2}{n-1}\right], \left[\frac{n+1}{n-1}\right], \frac{-(1-p)n^2+p}{p(n+1)}\right)$ is the corresponding hypergeometric function. Notice that $E\left(R^{TS}\right)\big|_{p=0} = \frac{n-1}{n+1}$ and $E\left(R^{TS}\right)\big|_{p=1} = 0$.

Define $\Delta_{R^{TS},R^{II}} = E\left(R^{TS}\right) - E\left(R^{II}\right)$.

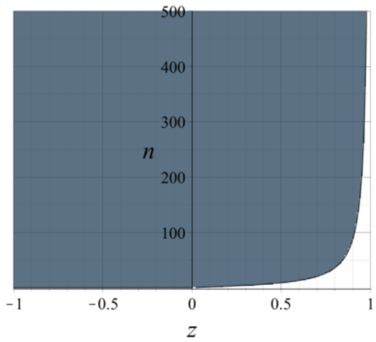
Proposition 6:

$\forall z \le p^{-1}\ \exists \underline{n} \Rightarrow \Delta_{R^{TS},R^{II}} \ge 0\ \forall n > \underline{n}$.

The shaded area in Figure 3 contains combinations of z and n satisfying

$\Delta_{R^{TS},R^{II}} \geq 0$ assuming $p = 0.2$.

Figure 3



Define $\Delta_{R^{TS},R^{SA}} = E\left(R^{TS}\right) - E\left(R^{SA}\right)$.

Proposition 7:

$\forall z < 0$ and $\forall z \in \left[1, \frac{1}{p}\right]$ and $\begin{cases} \frac{p}{2p-1} \leq z \leq 1 & p \leq \frac{1}{2} \\ 0 \leq z \leq \frac{p}{2p-1} & p > \frac{1}{2} \end{cases}$, $\exists \underline{n} \Rightarrow \Delta_{R^{TS},R^{SA}} \geq 0 \ \forall n > \underline{n}$.

The shaded area in Figure 4 contains combinations of n and z satisfying $\Delta_{R^{TS},R^{SA}} \geq 0$ assuming $p = 0.2$.

Figure 4

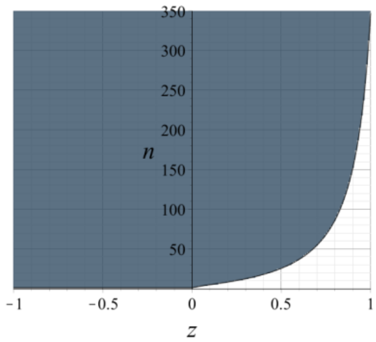


Table 2 summarizes this section.

Table 2: Expected Revenues across Sealed-Bid Auctions*

z	(1) **Regular Auction No cooling-off**	(2) **Relative Magnitude**	(3) **Asker's Auction**	(4) **Relative Magnitude**	(5) **Talmudic Auction**
$z < 0$	$pz\left[1-\left(\frac{-pz}{1-p}\right)^{n}\right]+$ $(1-p)\left(\frac{n-1}{n+1}\right)\left[1-\left(\frac{-pz}{1-p}\right)^{n+1}\right]$ $+(1-p)\left(\frac{-pz}{1-p}\right)\left(1+\frac{pz}{1-p}\right)$	$<$	$(1-p)\left(\frac{n-1}{n+1}\right)$	$<$	$\frac{n^2(n-1)(1-p)}{2(n+1)[p-(1-p)n^2]^2}\left[p(n+1)^2H_0 + (2-3p)n^2+p(2n+3)\right]^{***}$
$z = 0$	$(1-p)\left(\frac{n-1}{n+1}\right)$	$=$	$(1-p)\left(\frac{n-1}{n+1}\right)$	$<$	$\frac{n^2(n-1)(1-p)}{2(n+1)[p-(1-p)n^2]^2}\left[p(n+1)^2H_0 + (2-3p)n^2+p(2n+3)\right]^{***}$
$z \in (0,1]$	$(1-p)\left(\frac{n-1}{n+1}\right)+pz$	$>$	$pz +$ $(1-p)(1-z)\left(\frac{n-1}{n+1}+z^n\right)$	$\leq^{**}$	$\frac{n^2(n-1)(1-p)}{2(n+1)[p-(1-p)n^2]^2}\left[p(n+1)^2H_0 + (2-3p)n^2+p(2n+3)\right]^{***}$
$1 < z$	$(1-p)\left(\frac{n-1}{n+1}\right)+pz$	$>$	pz	$\leq^{****}$	$\frac{n^2(n-1)(1-p)}{2(n+1)[p-(1-p)n^2]^2}\left[p(n+1)^2H_0 + (2-3p)n^2+p(2n+3)\right]^{***}$

Notes: *Columns (1)-(3) were taken from Asker (2000). **For $n > \underline{n}$. ***$H_0 = \text{Hypergeom}\left(\left[1, \frac{2}{n-1}\right], \left[\frac{n+1}{n-1}\right], \frac{-(1-p)n^2+p}{p(n+1)}\right)$. ****Similar result was obtained regarding the comparison of Talmudic auction with regular auction with no cooling-off right (see Proposition 6).

7. SPURIOUS AND SHILL BIDDING ROBUSTNESS OF TALMUDIC AUCTIONS

Although our analysis is confined to symmetric equilibria, Asker (2000) indicated that in addition to (13), in some cases (particularly when $v_i < z \ \forall i \in N$) there exists an additional asymmetric spurious pure Nash equilibrium in his model of the form,

(28) $$b_l^{SA} \in \begin{cases} [z,\infty) & l \in \{i,j\} \\ 0 \cup [z,\infty) & l \in N \setminus \{i,j\}. \end{cases}$$

Asker raised several arguments against the plausibility of spurious equilibria. In a nutshell, Asker argued that rational agents would not spend time and exert fruitless efforts to submit spurious hopeless bids ensuring that no transaction will take place, and conjectured that non-spurious equilibria

> … might form the limit of the case where a fee is attached to the exercise of the cooling-off right … If we consider an epsilon (small) fee attached to the cooling-off right and then reduce this fee toward zero, the hedging behavior of the bidder against this fee will become insignificant and strategies will approach the non-spurious equilibria. However, such a fee will always remove any spurious bids from the best response set.

Nevertheless, Asker honestly reported that "spurious bidding behavior was observed in some of the auctions with cooling-off" [p. 599], but emphasized that "out of 180 bids collected from auctions with cooling-off, only five were spurious in the sense of inviting automatic cooling-off regardless of the state of the world that arose". Asker indicated that similar results have been observed in other experimental studies and argued that "[T]hese spurious bids can be seen as positive signals that people were aware of the structure of the bidding problem and had a feel for the equilibrium strategies and consequent payoffs" and that these spurious bids conform "exactly to asymmetric Bayesian Nash equilibrium in pure strategies".

Proposition 8:

Spurious bidding is incompatible with subgame-perfect Nash equilibrium in the Talmudic sealed-bid auction.

When bidding functions monotonically increase in n the auctioneer may benefit from inflating n artificially using phony bidders, a technique known as *shill-bidding*. That is, the auctioneer reports that $\hat{n} = n + s$ bids have been submitted, while the true number of sincere bids is n only. Suppose that the cost of misreporting the true value of n is zero (as is, for example, in online auctions), and that the uninformed sincere bidder naively believes that $\hat{n} = n$. Denote the bidding function of the uninformed naïve bidder in auction of type α under shill-bidding by $\hat{b}_i^{\alpha}$. Inserting $\hat{n}$ into (5), (11) and (13), and taking their limits with respect to s yields,

(29)
$$\lim_{s\to\infty} b_i^{SB} = v_i$$
$$\lim_{s\to\infty} b_i^{II} = \begin{cases} (1-p)v_i + pz & z \geq 0 \\ undefined & z < 0 \text{ and } v_i \geq \frac{-pz}{1-p} \\ 0 & \text{otherwise} \end{cases}$$
$$\lim_{s\to\infty} b_i^{SA} = \begin{cases} v_i & z \leq 0 \\ z & 0 < z \leq v_i \\ undefined & \text{otherwise.} \end{cases}$$

That is, if shill-bidding is costless for the auctioneer, then in certain cases the auctioneer may benefit from inflating $\hat{n}$ artificially, driving $\hat{b}_i^{SB}$, $\hat{b}_i^{II}$ and $\hat{b}_i^{SA}$ up to their limit. Assuming that the real value of the auctioned item for a shill-bidder is 0 (otherwise he would submit a sincere bid), implies that a shill-bidder is expected to withdraw any positive bid if (say, by accident), it wins the auction. Therefore, every shill bid is a spurious bid by definition, implying that if withdrawals are costless, once the no-collusion assumption is relaxed Asker's theoretical speculations are insufficient to discard spurious equilibria. However, if shill-bidding involves payments to shill-bidders, driving $\hat{b}_i^{SB}$, $\hat{b}_i^{II}$ and $\hat{b}_i^{SA}$ up to their limit may be expensive because it requires masses of colluders $(s \to \infty)$.

Regarding Talmudic sealed-bid auction, inserting $\hat{n}$ into (20) and taking the limit of $\hat{b}_i^{TS}$ with respect to s yields

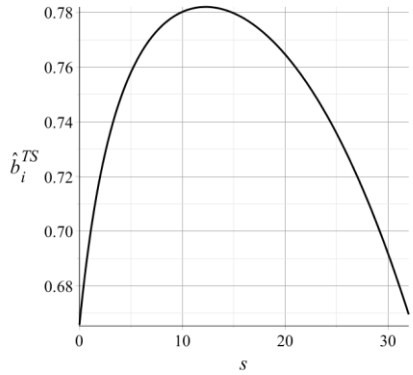
(30)
$$\lim_{s\to\infty} \hat{b}_i^{TS} = 0,\ \forall v_i,\ \forall p \in (0,1),$$

as can be seen in Figure 1. Nevertheless, (30) is insufficient to conclude that the Talmudic sealed-bid auction is totally shill-bidding robust. First, differentiating $\hat{b}_i^{TS}$ with respect to s yields,

(31)
$$\frac{\partial \hat{b}_i^{TS}}{\partial s} = \frac{(1-p)v_i^{n+s}\left(\begin{array}{r} p\left[\left((n+s)^2-1\right)(n+s)\ln(v_i)+(s+1)(2n+s)\right] \\ +(n+s)^2(1-p)v_i^{n+s-1}+n^2-1 \end{array} \right)}{\left[(n+s)^2(1-p)v_i^{n+s-1}+p(n+s+1)\right]^2}.$$

$\operatorname{sgn}\left(\partial \hat{b}_i^{TS} / \partial s\right)$ is generally indeterminate. Nevertheless, it can be verified that for sufficiently small p and n, and large v_i, $\left(\partial \hat{b}_i^{TS} / \partial s\right)_{s=0} > 0$. Figure 5 simulates $\hat{b}_i^{TS}$ calibrated for $n=5,\ p=0.2$ and $v_i = 0.9$ and shows that $\hat{b}_i^{TS}$ is concave. That is, the marginal benefit of shill-bidding is diminishing.

Figure 5



Second, shill-bidding is aimed at raising $E(R^{\alpha})$, not b_i^{α}. This distinction is crucially important in auctions with cooling-off right, because on the one hand shill-bidding may drive $\hat{b}_i^{\alpha}$ up, but on the other hand under shill-bidding more sincere bidders are incentivized to cool-off if state 2 eventuates.

The auctioneer's expected revenue in Asker's auction with shill bidding is,

$$(32)\ \hat{E}\left(R^{SA}\right)=\begin{cases}\dfrac{(1-p)n(n+s-1)}{(n+s)(n+1)} & z<0\\ \dfrac{n(z-1)(1-p)z^{n+s}}{(n+s)(s-1)}+\dfrac{1}{(n+s)(n+1)}\begin{bmatrix}n(2p-1)(2z-1)(n+s)\\ +n(p+z-1)+2pzs\end{bmatrix} & 0\le z\le 1\\ pz & z>1\end{cases}.$$

Define $\Delta_{\hat{R}^{SA},R^{SA}}=\hat{E}\left(\hat{R}^{SA}\right)-E\left(R^{SA}\right)$.

Proposition 9:

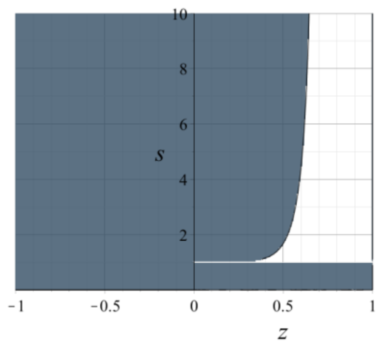
a. $\Delta_{\hat{R}^{SA},R^{SA}}\ge 0\ \forall z<0$ and $\forall z>1$.

b. $\exists z^{*}\in[0,1)$ *such that* $\Delta_{\hat{R}^{SA},R^{SA}}\ge 0\ \forall z\le z^{*}$.

c. $\lim_{s\to\infty}\Delta_{\hat{R}^{SA},R^{SA}}\ge 0\Rightarrow z^{*}=(n+1)^{-\frac{1}{n}}$.

The shaded area in Figure 6 contains combinations of z and s satisfying $\Delta_{\hat{R}^{SA},R^{SA}}\ge 0$ assuming $n=5$ and $p=0.2$.

Figure 6



Apparently, the expected revenue of the auctioneer in a Talmudic sealed-bid auction with shill-bidding is

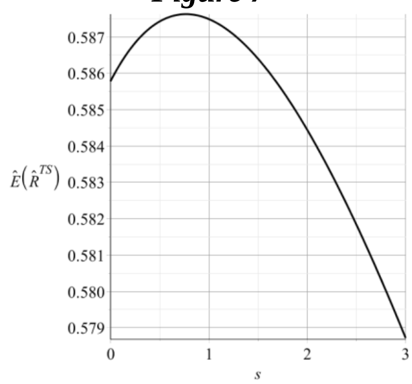
(33) $$E\left(\hat{R}^{TS}\right)=\sum_{i=1}^{n}\int_{0}^{1}\hat{b}_i^{TS}v_i^{n-1}dv_i = n\left(\frac{1-p}{p}\right)\left(\frac{n+s-1}{n+s+1}\right)\left(\frac{n+s}{2n+s}\right)H_0^s .$$

where $H_0^s = \text{Hypergeom}\left(\left[1,\frac{2n+s}{n+s-1}\right],\left[\frac{3n+2s-1}{n+s-1}\right],\frac{p-(1-p)(n+s)^2}{p(n+s+1)}\right)$. However, since under shill-bidding $\hat{E}\left[\hat{\varphi}_i\left(B^N\right)\right]=\frac{1}{n+s}\hat{E}\left(\hat{b}_n^{TS}\right)$ but only sincere withdrawers pay opting-out fee, the auctioneer's expected revenue in state 2 is $\left(\frac{n}{n+s}\right)E\left(\hat{R}^{TS}\right)$. It follows that the auctioneer's expected revenue in a sealed-bid Talmudic auction with shill-bidding is,

(34) $$\hat{E}\left(\hat{R}^{TS}\right)=\left[(1-p)+p\left(\frac{n}{n+s}\right)\right]E\left(\hat{R}^{TS}\right)=\frac{\left[n+s(1-p)\right]n(1-p)(n+s-1)H_0^s}{(2n+s)\,p\,(n+s+1)} .$$

Figure 7 presents a simulation of $\hat{E}\left(\hat{R}^{TS}\right)$ calibrated for $n=5$ and $p=0.2$, and demonstrates that $\hat{E}\left(\hat{R}^{TS}\right)$ is concave and under this calibration $s^*=1$ (recall that s must be an integer).

Figure 7



Define $\Delta_{\hat{R}^{TS},R^{TS}}=\hat{E}\left(\hat{R}^{TS}\right)-\hat{E}\left(R^{TS}\right)$.

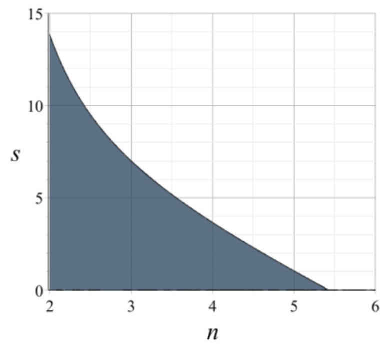
Proposition 10:

$\forall s\geq 0\ \exists \underline{n}$ *such that* $\Delta_{\hat{R}^{TS},R^{TS}}<0\ ,\forall n>\underline{n}$.

Proposition 10 establishes that if the auction is adequately competitive, (n is sufficiently large), the Talmudic sealed-bid auction is shill-bidding robust.

The shaded area in Figure 8 contains combinations of n and s satisfying $\Delta_{\hat{R}^{TS},R^{TS}}\geq 0$ assuming $p=0.2$.

Figure 8



8. SUMMARY AND DISCUSSION

In this article I analyzed an ancient auction model with uncertain posterior value and cooling-off right in return to endogenously determined opting-out fee, suggested in the Talmud. The Talmudic auction law introduced two novelties: (a) *Continuance commitment*. Bidder's liability does not expire at the submission of a higher bid, implying that a bid is a minimum price guarantee; (b) *Recursive winning procedure*. If the winner recants the auctioned item is offered to the subsequent bidder and the withdrawer pays an endogenously determined opting-out fee. The Talmudic law distinguished between *sequential* and *simultaneous* withdrawals, and ruled that in case of sequential cooling-off the withdrawer's opting-out fee equals to the difference between his bid and the subsequent bid, and in case of simultaneous withdrawals the seller's loss is distributed among all withdrawers. According to my (perhaps unorthodox) interpretation to this law, sequential withdrawals refer to open ascending auctions and simultaneous withdrawals refer to sealed-bid auctions.

In a regular English auction with uncertain posterior value and no cooling-off right, the entire risk is burdened on the winning bidder. The introduction of costless cooling-off right transfers the risk entirely to the seller. The Talmudic open auction mechanism removes uncertainty, redistributes the risk between the seller and the bidders and restores the equilibrium of English auction with certain value.

In a Talmudic sealed-bid auction, the loss of the seller in case of withdrawals is burdened on all effective withdrawers. The Talmudic law mentioned no specific sharing rule, and its classical commentators differed regarding the appropriate distribution scheme. However, the specification of the sharing rule has no effect on bidding and expected revenue. In a Talmudic sealed-bid auction equilibrium bids are lower compared with Asker's costless cooling-of right model equilibrium bids, but expected revenue is usually higher. Contrary to Asker's model, the Talmudic sealed-bid auction model is spurious bidding free, and if the auction is sufficiently competitive it is also shill-bidding robust.

The analysis was based on the prevailing assumption of risk-neutrality which is, of course, not very realistic but necessary to enable analytic solvability. While in many cases the direction and even the magnitude of the consequent bias can be estimated, this case, unfortunately, is different. The extent and the direction of risk-aversion influence on bidding are unknown. Apparently, a progress in our knowledge about the influence of risk-aversion on behavior in contests and auctions can be achieved through experiments only.

Another common assumption that I used is that the probability of state 2 is exogenously given, identical to all bidders and common knowledge. An interesting further research direction might be the influence informational asymmetry.

Last but not least, although the bidding strategy in Talmudic sealed-bid auction is indifferent to the sharing rule specification, from social and moral points of view the choice of a sharing rule is absolutely not an unimportant question. The normative characteristics of sharing rules were extensively studied in the economic literature[33], implying that the choice of a sharing rule reflects the norms and the prevailing social and legal philosophies of a society[34].

9. APPENDIX

Proof of Proposition 1:

From (1) and (7) we obtain that,

(A1) $$\Delta_{b_i^E, b_i^I} = \begin{cases} p(v_i - z) & 0 \le z \text{ or } z < 0 \text{ and } v_i \ge \frac{-pz}{1-p} \\ v_i & \text{otherwise.} \end{cases}$$

Part *a* of the proposition stems directly from (A1). Similarly, from (5) and (11) we obtain that,

(A2) $$\Delta_{b_i^{SB}, b_i^{II}} = \begin{cases} p\left[\left(\frac{n-1}{n}\right)v_i - z\right] & z \ge 0 \\ p\left[\left(\frac{n-1}{n}\right)v_i - z\right] - \left(\frac{1-p}{nv_i^{n-1}}\right)\left(\frac{-pz}{1-p}\right)^n & z < 0 \textit{ and } v_i \ge \frac{-pz}{1-p} \\ \left(\frac{n-1}{n}\right)v_i & \textit{otherwise} \end{cases}$$

implying that if $z \ge 0$ then $\Delta_{b_i^{SB}, b_i^{II}} \ge 0$ if and only if $v_i \ge \left(\frac{n}{n-1}\right)z$, and if $z < 0$ and

[33] For a survey see for example Thomson, (2003) and Moulin (2004).
[34] For an interdisciplinary analysis of sharing rule see Lipschütz & Schwarz (2015).

$v_i \geq \frac{-pz}{1-p}$ then $\Delta_{b_i^{SB},b_i^{II}} \geq 0$ if and only if $p \geq \frac{v_i\left[\frac{n(z-nb_i^{SB})}{z}\right]^{\frac{1}{n-1}}}{v_i\left[\frac{n(z-nb_i^{SB})}{z}\right]^{\frac{1}{n-1}} - z}$. Since $z > nb_i^{SB}$ is very unlikely, it follows that even if $z<0$ and $v_i \geq \frac{-pz}{1-p}$ usually $\Delta_{b_i^{SB},b_i^{II}} \geq 0$. ■

Proof of Proposition 2:

Bidder i's loss from buying the auctioned item in state 2 is $b_i^{TS}-z$, and $E\left[\varphi_i\left(B^K\right)\right]$ from cooling-off (conditional that $|K|=k$). Thus, if $z \leq b_i^{TS} - E\left[\varphi_i\left(B^K\right)\right], \forall i \in K, \ \forall K \subseteq N$, then in state 2 $C=K=N$, implying that $t=0$ and $E\left[\varphi_i\left(B^N\right)\right] = \frac{1}{n}E\left(b_n^{TS}\right)$.

$E\left(b_n^{TS}\right)$ is given by,

(A3) $$E\left(b_n^{TS}\right) = \Pr\left(b_i^{TS} = \max \mathbf{b}^{TS}\right)b_i^{TS} + \Pr\left(b_i^{TS} < \max \mathbf{b}^{TS}\right)E\left(b_n^{TS} \middle| b_i^{TS} < \max \mathbf{b}^{TS}\right).$$

Recall that $\Pr\left(b_i^{TS} = \max \mathbf{b}^{TS}\right) = \left[V\left(b_i^{TS}\right)\right]^{n-1}$. Using the "German Tank Problem" technique we estimate that $E\left(b_n^{TS} \middle| b_i^{TS} < \max \mathbf{b}^{TS}\right) = \left(\frac{n+1}{n}\right)b_i^{TS}$. It follows that,

(A4) $$E\left(b_n^{TS}\right) = \left(\frac{n+1-\left[V\left(b_i^{TS}\right)\right]^{n-1}}{n}\right)b_i^{TS}, \ \forall i \in N, \ \forall \varphi \in \mathbf{S}$$

implying that $E\left[\varphi_i\left(B^N\right)\right] = \frac{1}{n}E\left(b_n^{TS}\right) = \left(\frac{n+1-\left[V\left(b_i^{TS}\right)\right]^{n-1}}{n^2}\right)b_i^{TS}, \ \forall i \in N, \ \forall \varphi \in \mathbf{S}$. ■

Proof of Proposition 3:

Under both assumptions at the bidding stage bidder $i \in N$ knows only his own bid, b_i^{TS}, but neither δ_i nor L. Since by Proposition 2, the expected opting-out fee is indiferent to the sharing rule, it follows that both setting are strategically equivalent in subgame-perfect equilibrium. ■

Proof of Proposition 4:

From (20) and (13) we obtain,

(A5) $$\Delta_{b_i^{TS},b_i^{SA}} = \frac{(1-p)n(n-1)v_i^n}{\left[(1-p)n^2-p\right]v_i^{n-1}+p(n+1)} - \begin{cases} z & v_i \leq z \\ \left(\frac{n-1}{n}\right)v_i + \frac{z^n}{nv_i^{n-1}} & 0 \leq z < v \\ \left(\frac{n-1}{n}\right)v_i & z < 0 \end{cases}$$

And the proposition stems directly from (A5). ■

Proof of Proposition 5:

From (23) and (24) we obtain.

(A6) $$\Delta_{R^E,R^I} = \begin{cases} p(n-1)\left[\dfrac{n-(n+1)z}{n(n+1)}\right] & z \geq 0 \\ \dfrac{p}{n}\left(\dfrac{n-1}{n+1}\right)\left[(1-z)n - z\left(1-\left(\dfrac{-pz}{1-p}\right)^n\right)\right] & z < 0 \text{ and } \frac{-pz}{1-p} \leq 1 \\ \dfrac{n-1}{n+1} & \text{otherwise} \end{cases}.$$

and the proposition stems directly from (A6).■

Proof of Proposition 6:

From (25) and (27) we obtain by definition,

(A7) $$\Delta_{R^{TS},R^{II}} = \frac{n^2(n-1)(1-p)\left[p(n+1)^2 H_0 + (2-3p)n^2 - p(2n+3)\right]}{2(n+1)\left[p-(1-p)n^2\right]^2} - \begin{cases} pz\left[1-\left(\dfrac{-pz}{1-p}\right)^n\right] + (1-p)\left(\dfrac{n-1}{n+1}\right)\left[1-\left(\dfrac{-pz}{1-p}\right)^{n+1}\right] \\ \qquad + (1-p)\left(\dfrac{-pz}{1-p}\right)^n\left(1-\dfrac{pz}{1-p}\right) & z \leq 0 \\ pz + (1-p)(1-z)\left(\dfrac{n-1}{n+1} + z^n\right) & 0 < z \leq 1 \end{cases}$$

For $n \geq 1$ $\Delta_{R^{TS},R^{II}}$ is continuous with respect to n. Taking the limit of $\Delta_{R^{TS},R^{II}}$ with respect to n yields,

(A8) $$\lim_{n\to\infty} \Delta_{R^{TS},R^{II}} = \begin{cases} p(1-z) & z \leq 1 \\ 1-pz & z > 1 \end{cases}$$

$\lim_{n\to\infty} \Delta_{R^{TS},R^{II}} > 0 \ \forall z \leq p^{-1}$, thus by continuity $\forall z \leq p^{-1} \ \exists \underline{n} \Rightarrow \Delta_{R^{TS},R^{SA}} \geq 0 \ \forall n > \underline{n}$. ■

Proof of Proposition 7:

From (26) and (27) we obtain by definition,

(A9) $$\Delta_{R^{TS},R^{SA}} = \frac{n^2(n-1)(1-p)\left[p(n+1)^2 H_0 + (2-3p)n^2 - p(2n+3)\right]}{2(n+1)\left[p-(1-p)n^2\right]^2} - \begin{cases} (1-p)\left(\dfrac{n-1}{n+1}\right) & z \leq 0 \\ pz + (1-p)(1-z)\left(\dfrac{n-1}{n+1} + z^n\right) & 0 < z \leq 1 \\ pz & 1 < z. \end{cases}$$

For $n \geq 1$ $\Delta_{R^{TS},R^{SA}}$ is continuous with respect of n. Taking the limit of $\Delta_{R^{TS},R^{SA}}$ with respect to n yields,

(A10) $$\lim_{n\to\infty} \Delta_{R^{TS},R^{SA}} = \begin{cases} p & z < 0 \\ p + (1-2p)z & 0 \leq z \leq 1 \\ 1-pz & z > 1 \end{cases}$$

By (A10) $\lim_{n\to\infty}\Delta_{R^{TS},R^{SA}} > 0 \ \forall z<0$ and $\forall z\in\left[1,\frac{1}{p}\right]$ and $\begin{cases}\frac{p}{2p-1}\le z\le 1 & p\le\frac{1}{2}\\ 0\le z\le\frac{p}{2p-1} & p>\frac{1}{2}\end{cases}$, implying by continuity that for these values of z, $\exists\underline{n}\Rightarrow\Delta_{R^{TS},R^{SA}}\ge 0\ \forall n>\underline{n}$. ■

Proof of Proposition 8:

By definition, spurious bids are expected to be withdrawn in any state. Obviously, if state 1 eventuates the compensation is burdened on spurious bidders only (sincere bidders have no reason to cool-off in state 1). However, this is true also in state 2 because under spurious bidding $E\left[\varphi_i\left(B^K\right)\right]>b_i^{TS}-z,\ \forall i\in K,\ \ \forall K\subseteq N$, implying that in this case a sincere bidder will not cool-off and the seller's compensation is burdened on spurious bidders only. It follows that in the Talmudic sealed-bid auction, spurious bidding is incompatible with subgame-perfect Nash equilibrium. ■

Proof of Proposition 9:

Subtracting (26) from (32) yields,

$$\text{(A11)}\ \Delta_{\hat{R}^{SA},R^{SA}}=\frac{1-p}{(n+1)(n+s)}\begin{cases}s & z<0\\ \dfrac{(1-z)\left[(s-1)\left(s-(n+s)(n+1)z^n\right)-n(n+1)z^{n+s}\right]}{(s-1)} & 0\le z\le 1\\ 0 & z>1\end{cases}$$

Clearly, $\Delta_{\hat{R}^{SA},R^{SA}}\ge 0$ for all $z<0$ and $z>1$, and it also can be verified that $\exists z^*\in[0,1)$ such that $\left.\Delta_{\hat{R}^{SA},R^{SA}}\right|_{z\le z^*}\ge 0$.

Taking the limit of (A11) with respect to s yields,

$$\text{(A12)}\qquad \lim_{s\to\infty}\Delta_{\hat{R}^{SA},R^{SA}}=\left(\frac{1-p}{n+1}\right)\begin{cases}1 & z<0\\ (z-1)\left[z^n(n+1)-1\right] & 0\le z\le 1\\ 0 & z>1\end{cases}$$

implying that $\lim_{s\to\infty}\Delta_{\hat{R}^{SA},R^{SA}}\ge 0\Rightarrow z^*=(n+1)^{-\frac{1}{n}}$.■

Proof of Proposition 10:

Taking the limits of H_0^s and H_0 with respect to n yields $\lim_{n\to\infty}H_0^s=0$ and $\lim_{n\to\infty}H_0=1$, implying that $\lim_{n\to\infty}\Delta_{\hat{R}^{TS},R^{TS}}=-1$. Thus, by the continuity of $\Delta_{\hat{R}^{TS},R^{TS}}$ it follows that $\forall s\ge 0\ \exists\underline{n}$ such that $\Delta_{\hat{R}^{TS},R^{TS}}<0\ ,\forall n>\underline{n}$. ■

11. REFERENCES

Asker, J. (2000). Bidding Up, Buying Out and Cooling-Off: An Examination of Auctions with Withdrawal Rights. *Economic Theory*, *16*, 585-611.

Aumann, R. J., & Maschler, M. (1985). Game Theoretic Analysis of a Bankruptcy Problem from the Talmud. *Journal of Economic Theory*, *36*(2), 195-213.

Bazerman, M. H., & Samuelson, W. F. (1983). I Won the Auction but Don't want the Prize. *Journal of Conflict Resulution, 27*(4), 618-634.

Borad, S. (2007). Bidding into the Real: A Model of Post-Auction Bankruptcy. *Journal of Finance, 62*, 2695-2723.

Bulow, J., & Klemperer, P. (2002). Prices and the Winner's Curse. *The RAND Journal of Economics, 33*(1), 1-21.

Calveras, A. J., Ganuza, J., & Hauk, E. (2004). Wild Bids. Gambling for Resurrection in Procurement Contracts. *Journal of Regulatory Economics, 26*, 41-68.

Case, J. H. (1979). *Economics and the Competitive Process.* New York: New York University Press.

Chillemi, O., & Mezzetti, C. (2014). Optimal Procurement Mechanism: Bidding on Price and Damages for Breach. *Economic Theory, 55*, 335-355.

Cornes, R., & Hartley, R. (2003). Risk Aversion, Heterogeneity and Contests. *Public Choice, 117*, 1-25.

Cornes, R., & Hartley, R. (2010). Risk Aversion in Symmetric and Asymmetric Contests. *Economic Theory*, DOI: 10.1007/s00199-009-0490-9.

Eső, P., & WhiteSource, L. (2004). Precautionary Bidding in Auctions. *Econometrica, 72*(1), 77-92.

Goldwurm, H. (1982). *History of the Jewish People: The Second Temple Era.* Brooklyn, New York: Mesorah Publications.

Harstad, R. M., & Rothkopf, M. H. (1995). Withdrawable Bids as Winner's Curse Insurance. *Operations Research, 43*(6), 983-994.

Hillman, A. L., & Katz, E. (1984). Risk-Averse Rent-Seekers and the Social Cost of Monopoly Power. *Economic Journal, 94*, 104-110.

Hillman, A. L., & Samet, D. (1987). Dissipation of Rents and Revenues in Small Number Contests. *Public Choice, 54*, 63-82.

Kagel, J. H., & Levin, D. (1986). The Winner's Curse and Public Information in Common Value Auctions. *American Economic Review, 76*, 894-920.

Klemperer, P. (2002). What Really Matters in Auction Desigh. *Journal of Economic Perspectives, 16*, 169-190.

Konrad, K. A., & Schlesinger, H. (1997). Risk Aversion in Rent-Seeking and Rent-Augmenting Games. *The Economic Journal, 107*, 1671-1683.

Krishna, V. (2009). *Auction Theory.* Academic Press.

Lamping, J. (2007). The Value of Information in Auctions with Default Risk. *available at http://mpra.ub.uni-muenchen.de/24375/1/default.pdf*.

Lind, B., & Plott, C. R. (1991). The Winner's Curse: Experiments with Buyers and Sellers. *American Economic Review, 81*, 335-346.

Lipschütz, I., & Schwarz, M. E. (2015). Greek Versus Talmudic Fairness And Economic Approaches Towards Priority-Less Bankruptcies: An Interdisciplinary Analysis. *(unpublished)*.

Littlechild, S. C., & Owen, G. (1973). A Simple Expression For The Shapley Value In A Special Case. *Management Sciece, 20*(3), 370-372.

Moulin, H. J. (2004). *Fair Division and Collective Welfare.* Cambridge, Massachusetts: MIT Press.

O'Neill, B. (1982). A Problem of Rights Arbitration from the Talmud. *Mathematical Social Science, 2*(4), 345-71.

Oren, M. E., & Williams, A. (1975). On Competitive Bidding. *Operations Research, 23*, 1072-1079.

Riley, J. G. (1988). Ex Post Information in Auctions. *Review of Economic Studies, 55*(3), 409-429.

Roberts, J. W. (2011). Can Warranties Substitute for Reputations? *American Economic Journal: Microeconomics, 3*, 69-85.

Rothkopf, M. H. (1980). On Multiplicative Bidding Strategies. *Operations Research, 28*, 570-575.

Rothkopf, M. H. (1991). On Auctions With Withdrawable Winning Bids. *Marketing Science, 10*, 40-57.

Shapley, L. S. (1953). A Value For n Person Games. (H. W. Kuhn, & A. W. Tucker, Eds.) *Contributions To The Theory of Games (Annals of Mathematical Studies), 5*(28), pp. 307-319.

Skaperdas, S., & Gan, L. (1995). Risk Aversion in Contests. *The Economic Journal, 105*, 951-962.

Spulber, D. F. (1990). Auction and Contract Enforcement. *Journal of Law, Economics, & Organization, 6*, 325-344.

Thaler, R. H. (1988). Anomalies, The Winner's Curse. *Journal of Economic Perspectives, 2*(1), 191-202.

Thomson, W. (2003). Axiomatic and Game-Theoretic Analysis of Bankruptcy and Taxation Problems: A Survey. *Mathematical Social Science, 45*, 249-297.

Van Long, N. (2013). The Theory of Contests: A Unified Model and Review of the Literature. *European Journal of Political Economy, 32*, 161-181.

Van Long, N., & Vousden, N. (1987). Risk-averse Rent Seeking with Shared Rents. *The Economic Journal* , 971-985.

Vickery, W. (1961). Counterspeculation, Auctions and Competitive Sealed Tenders. *Journal of Finance, 16*(1), 8-37.

von Ungern-Sternberg, T. (1991). Swiss Auctions. *Economics, 58*, 341-357.

Waehrer, K. (1995). A Model of Auction Contracts with Liquidated Damages. *Journal of Economic Theory, 67*, 531-555.

Winkler, R. I., & Brooks, D. G. (1980). Competitive Bidding with Dependent Value Estimates. *Operations Research, 28*, 603-613.

Zheng, C. (2001). High Bids and Broke Winners. *Journal of Economic Theory, 100*, 129-171.