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Journal of Applied Logic 6 (2008) 24–46

JOURNAL OF APPLIED LOGIC

www.elsevier.com/locate/jal

Axiomatizing geometric constructions

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Received 1 March 2005; received in revised form 14 February 2007; accepted 15 February 2007

Available online 24 February 2007

Abstract

In this survey paper, we present several results linking quantifier-free axiomatizations of various Euclidean and hyperbolic geometries in languages without relation symbols to geometric constructibility theorems. Several fragments of Euclidean and hyperbolic geometries turn out to be *naturally occurring* only when we ask for the universal theory of the standard plane (Euclidean or hyperbolic), that can be expressed in a certain language containing only operation symbols standing for certain geometric constructions.

Keywords: Geometric constructions; Quantifier-free axiomatizations; Euclidean geometry; Absolute geometry; Hyperbolic geometry; Metric planes; Metric-Euclidean planes; Rectangular planes; *Treffgeradenebenen*

1. Introduction

The first modern axiomatizations of geometry, by Pasch, Peano, Pieri, and Hilbert, were expressed in languages which contained, in stark contrast to the axiomatizations of arithmetic or of algebraic theories, only relation (predicate) symbols, but no operation symbol.

On the other hand, geometric constructions have played an important role in geometry from the very beginning. It is quite surprising that it is only in 1968 that geometric constructions became part of the axiomatization of geometry.

Two papers broke the ice: Moler and Suppes [65] and Engeler [25]. In [65] we have the first axiomatization of geometry (to be precise of plane Euclidean geometry over Pythagorean ordered fields) in terms of two operations, by means of a quantifier-free axiom system. The first-order language in which it is expressed has one sort of variables, standing for *points*, and three individual constants a_0, a_1, a_2, as well as two quaternary operation symbols S and I as primitive notions. The primitive notions a_0, a_1, a_2, S and I have the following intuitive meanings: a_0, a_1, a_2 are three non-collinear points, $S(xyuv)$ is a point as distant from u on the ray $\overrightarrow{uv}$ as y is from x, provided that $u \neq v \vee (u = v \wedge x = y)$, an arbitrary point, otherwise, and $I(xyuv)$ is the point of intersection of the lines xy and uv, provided that $x \neq y$, $u \neq v$, the lines xy and uv are distinct and do intersect, an arbitrary point, otherwise.

Ten years later Seeland [115] rephrased the axiom system in [65] and also gave a quantifier-free axiom system for plane Euclidean geometry over Euclidean fields, in a language enlarged with a third quaternary symbol C, having

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1570-8683/$ – see front matter
doi:10.1016/j.jal.2007.02.001

the intuitive meaning: $C(xyuv)$ is the point of intersection of the circle centred at x and passing through y with the segment uv, provided that $x \neq y$, u lies inside and v lies outside the circle, an arbitrary point, otherwise.

The difference between the Euclidean geometries over Pythagorean and Euclidean ordered fields is that the circle axiom is not assumed in the former, i.e. one does not know whether a circle and a line passing through an inner point of the circle intersect or not, whereas the latter satisfies it.

Engeler's motivation, as he states it in [26], was that, as a student of P. Bernays, he got interested in the foundations of geometry, and "rereading Hilbert's *Grundlagen der Geometrie* [41] was struck by the fact that of all the topics of that book the one on geometric constructions was the least "modern", i.e. axiomatic".[1]

He thus devised a meaningful logic in which to address constructibility problems that may require a finite, but not a priori bounded, number of constructions. In this logic, we are, for example, able to determine constructively that two given segments 'behave Archimedeanly' (i.e. that an integer multiple of the length of either of them exceeds the length of the other), by laying off, in increasing order, integer multiples of one on the other from one of the latter's endpoints. If we get past the endpoint of the 'longer' segment, we stop, if not, we continue. If the logic allows us to state that such constructions terminate after finitely many steps, then we are able to express the Archimedeanity of the coordinate field. It turns out that a quantifier-free logic, containing only Boolean combinations of halting-formulas for flow-charts (that may contain loops but not recursive calls) is all one needs. This logic was introduced by E. Engeler [21] under the name of *algorithmic logic* and its relevance to geometry was studied in [22–25,80,115]. It is presented in the Appendix.

Such universal axiomatizations in languages without relation symbols capture the essentially constructive nature of geometry, that was the trademark of Greek geometry.[2]

For Proclus, who relates a view held by Geminus, "a postulate prescribes that we construct or provide some simple or easily grasped object for the exhibition of a character, while an axiom asserts some inherent attribute that is known at once to one's auditors" [99, p. 142 (181 in the Friedlein edition)]. And "just as a problem differs from a theorem, so a postulate differs from an axiom, even though both of them are undemonstrated; the one is assumed because it is easy to construct, the other accepted because it is easy to know" [99, p. 142 (182 in the Friedlein edition)].

That is, *postulates* ask for the production, the $\pi o \acute{\iota} \eta \sigma \iota \varsigma$ of something not yet given, of a $\tau \iota$, whereas *axioms* refer to the $\gamma \nu \tilde{\omega} \sigma \iota \varsigma$ of a given, to insight into the validity of certain relationships that hold between given notions (cf. [30,77,137]). In traditional axiomatizations, that contain relation symbols, and where axioms are not universal statements, such as Hilbert's, this ancient distinction no longer exists. The constructive axiomatics preserves this ancient distinction, as the ancient postulates are the primitive notions of the language, namely the individual constants and the geometric operation symbols themselves (in the Moler-Suppes case a_0, a_1, a_2, S, I), whereas what Geminus would refer to as "axioms" are precisely the axioms of the constructive axiom system.

In the present survey, which is meant to be a guide to the relevant literature, we shall present the results obtained so far in providing quantifier-free axiomatizations in languages without relation symbols for absolute and for several Euclidean and hyperbolic two-dimensional geometries, point out the connection with classical geometric construction theorems, such as the Mohr-Mascheroni theorem (see G.E. Martin [61], L. Bieberbach [13], A. Adler [1], or Gy. Szőkefalvi-Nagy [129] for a non-axiomatic treatment of Euclidean geometric constructions), and discuss the relevance of these axiomatizations.

Languages which contain only individual constants and operation symbols as primitive notions, as well as quantifier-free axiomatizations in such languages will be called *constructive*.

2. Euclidean geometries

Following the instruction of Voltaire's [140] geometer, "Je vous conseille de douter de tout, excepté que les trois angles d'un triangle sont égaux à deux droits", we will see what a progressive doubting of features of Euclidean geometry that are not related to its Euclidean metric (and thus to the sum of angles in a triangle) can lead us to.

[1] A more modern treatment can be found in [49], but it is not one in which constructions become part of the language of an axiom system. A further development in the direction of more logical look at geometric constructions, somewhat similar to that in [49], can be found in [109,110].

[2] Zeuthen [146] went so far as claiming that the geometric construction was the only means of establishing the existence of a geometric object, a claim refuted in this strong form by Knorr [52] (cf. also [53]).

A first step consists in doubting continuity, i.e. the need for $\mathbb{R}$ as the coordinate field of the Cartesian plane over the reals, as which standard Euclidean geometry is represented (to be referred as "the standard Euclidean plane"). This step leads to Tarski's [114,130,132] first-order theory of the standard Euclidean plane, which turns out to be Euclidean geometry over arbitrary real-closed fields. Real-closed fields are defined as ordered fields in which (i) every positive element has a square root, and (ii) every polynomial of odd degree has a zero. Since a geometric construction instrument can be expected to provide only zeros for polynomials with degrees bounded from above, we cannot conceive of this geometry as one of Euclidean constructions with finitely many instruments.

If we doubt even the Tarskian elementary form of continuity, and retain from the two conditions for real-closed fields only (i) and weaken (ii) to (ii)$'$ every polynomial of degree 3 has a zero (such fields are called in [61] *Vietan fields*, having been of interest to Viète) then the resulting geometry *is* one of geometric constructions. The instrument involved is the marked ruler (or twice-notched straightedge), which, in addition to being a ruler, allows the operation of *verging* or *insertion* or *neusis* (the marked ruler is a ruler with two marks (notches) on it, and verging allows, given any two intersecting lines g and h, for the positioning of the marked ruler such that one of its marks lands on g and the other lands on h). As shown in [3], [61, Th. 10.14], this is also the Euclidean geometry of origami constructions, when folding is allowed according to certain precise rules (see also [105]). The fact that the coordinate field is Vietan has been established in [15, Th. 10.3.4].

Doubting the amount of continuity still present in the geometry of the marked ruler, and insisting only on the amount of continuity required for the existence of the intersection point of a circle and a line passing through an inner point of that circle, i.e. dropping the requirement (ii) altogether from the coordinate field, we obtain Cartesian planes over Euclidean ordered fields, i.e. fields satisfying (i). These correspond to ruler and compass constructions.

A further step in doubting leads us to doubt this amount of continuity as well, i.e. (i), and to ask of our geometry only free mobility, i.e. weakening (i) to the requirement that the sum of two squares should always be a square, giving us ordered Pythagorean fields as coordinate fields. These Cartesian planes are those in which ruler and gauge (or segment-transporter) constructions can be performed.

Doubting even the need for free mobility, i.e. dropping the requirement of Pythagoreanness as well, we are left with Cartesian planes coordinatized by arbitrary ordered fields. These structures, which have been first axiomatized by H.N. Gupta (see [114]) cannot be characterized in terms of a set of geometric constructions that can be performed in it.

Doubting the need for order and free mobility as well, we get to structures in which, beside the Euclidean parallel postulate, all universal elementary Euclidean theorems involving only metric concepts, i.e. expressible in terms of the relation of segment congruence, hold. These structures go back to F. Schur [113], Bachmann and Reidemeister [9], and Baer [10]. The models of this geometry, which will be referred to as *Euclidean planes*, can be described quite simply as the structures $\mathfrak{D}_2(F,k) := \langle F \times F, \equiv_{(F,k)}\rangle$, where F is a field of characteristic $\neq 2$, $k \in F$, such that $-k \notin F^2$, and $\mathbf{ab} \equiv_{(F,k)} \mathbf{cd}$ iff $\|\mathbf{a}-\mathbf{b}\| = \|\mathbf{c}-\mathbf{d}\|$, where $\|(x_1,x_2)\| = x_1^2 + kx_2^2$, for $x_1, x_2 \in F$. The concept of line orthogonality may be defined as usual in Euclidean planes by means of the notion of congruence, the lines $ux + vy + w = 0$ and $u'x + v'y + w' = 0$ being orthogonal if and only if $kuu' + vv' = 0$. One can also give an alternate description of Euclidean planes, to be referred to as *Gaußian planes*. Let L/K be a quadratic extension of a field K of characteristic $\neq 2$ and let $\{1,\sigma\}$ be its Galois group. The *Gaußian plane over* (L,K) is the structure $\mathfrak{G}(L,K) := \langle L, \equiv\rangle$, with $xy \equiv uv$ iff $\|x-y\| = \|u-v\|$, with $\|x\| = x\sigma(x)$, for $x, y, u, v \in L$. It generalizes the classical Gaußian plane of complex numbers, $\mathfrak{G}(\mathbb{C},\mathbb{R})$, and we have $\mathfrak{G}(K(\sqrt{-k}),K) \simeq \mathfrak{D}_2(K,k)$, which allows us to treat the Euclidean plane over (K,k) and the Gaußian plane over (L,K), with $L = K(\sqrt{-k})$ as synonyms. Unlike the connection between Euclidean and Pythagorean fields with geometric constructions, which can be said to have been established as soon as the concept of field appeared in mathematics, the realization that Euclidean planes can be characterized in terms of geometric constructions is of a more recent date and provided a new elementary geometric justification for considering these structures.

The Euclideanity of a Euclidean plane may be considered as being determined by its affine structure (i.e. by the fact that an Euclidean plane is an affine plane), or as being determined by its Euclidean metric. Taking the second approach (given that we are not allowed to doubt the fact that the sum of the angles of a triangle is equal to two right ones), one may ask what the most general 'planes' with a Euclidean metric are, and whether having a Euclidean metric implies the affine structure (i.e. the intersection of non-parallel lines). It was shown by M. Dehn [19] that the latter is not the case, i.e. that there are planes with a Euclidean metric, to be called *metric-Euclidean planes*, that are not Euclidean planes (i.e. where the parallel axiom does not hold). Such planes must be non-Archimedean.

Metric-Euclidean planes were introduced by F. Bachmann [4,5,8], as a plane geometry with Euclidean metric, without order or free mobility, where the Euclidean parallel axiom need not hold. In addition to our previous doubts, we are thus also doubting the fact that non-parallel lines ought to intersect.

The point-set of a metric-Euclidean plane of characteristic $\neq 2$ is a subset E of $L = K(\sqrt{-k})$, where K is a field of characteristic $\neq 2$, $k \in K$, $-k \notin K^2$, which satisfies

(i) $(E, +)$ is a subgroup of $(L, +)$ and $1 \in E$,
(ii) $(\forall s \in L)\|s\| = 1 \Rightarrow s \cdot E \subseteq E$,
(iii) $(\forall s \in L)(\forall x \in E)\|s\| = 1 \Rightarrow \frac{1}{2}(x + sx) \in E$.

It inherits the collinearity and congruence relations from the Euclidean plane over (K, k) or, equivalently, of the Gaußian plane over (L, K). This means, geometrically speaking, that this point-set is a subset of the Gaußian plane over (L, K), contains 0, 1, is closed under translations and rotations around 0, and contains the midpoints of any point-pair consisting of an arbitrary point and its image under a rotation around 0.

Had we not doubted the order and the free mobility of Euclidean geometry, an just doubted the fact that non-parallel lines must intersect, then we would have obtained a special class of metric-Euclidean planes, namely that of *metric-Euclidean geometry with free mobility and order*. The point-sets of these planes are (cf. [8, §19,3, Satz 7]) $E_{F,M} = \{(x, y)\colon\ x, y \in M\}$, where F is a Pythagorean ordered field, $R = \{x \in F\colon\ (\exists n \in \mathbb{N})\, |x| \leqslant n$ for every ordering $\leqslant$ of $F\}$, $M \subset F$ and R-module with $0, 1 \in M$.

In metric-Euclidean planes, there is one class of pairs of distinguished non-parallel lines that we know must always intersect: that of orthogonal lines. We are thus not doubting that orthogonal lines intersect. If we start doubting this fact as well, we stumble upon structures called *rectangular planes*. A first example of a rectangular plane is mentioned in F. Bachmann [5, §4]. Rectangular planes were introduced in Karzel and Stanik [47] by a mixed geometric-group-theoretic axiom system, as a generalization of metric-Euclidean planes, where perpendicular lines need not intersect. A purely geometric axiomatization for rectangular planes, or *Rechtseitebenen*, of characteristic $\neq 2$, in a language without operation symbols, was provided by R. Stanik [126], where there are further references to planes with a Euclidean metric.

The point-set of a rectangular plane of characteristic $\neq 2$ is isomorphic to a subset E of $L = K(\sqrt{-k})$, where K is a field of characteristic $\neq 2$, $-k \in K$, $-k \notin K^2$, which satisfies

(i) $(E, +)$ is a subgroup of $(L, +)$ and $1 \in E$,
(ii) $(\forall s \in L)\|s\| = 1 \Rightarrow s \cdot E \subseteq E$.

It inherits the collinearity, congruence and parallelism relations from the Gaußian plane over (L, K). This means, geometrically speaking, that its point-set is a subset of the Gaußian plane over (L, K) which contains 0, 1 and is closed under translations and rotations around 0.

Both rectangular and metric-Euclidean planes can be understood as determined by the types of geometric constructions one may perform in them, and become, in that light, naturally occurring structures rather than exotic counterexamples.

In what follows we survey the constructive axiomatizability of these theories. There are two types of results that we will encounter. On the one hand, we have actual axiom systems, that are particularly elegant and particularly simple according to such simplicity criteria as the number of variables occurring in each axiom (which does not exceed 4 in these simplest possible axiomatizations) and the fact that the operation symbols are at most ternary (and thus have the smallest possible arity). On the other hand, we have results that show that a certain theory can be constructively axiomatized in a certain language, but the axiom systems that we would obtain by actually writing out the axioms would be far from elegant or simple. To some, to put forward an axiomatization is an excusable pastime *only* if the axiom system is particularly simple and appealing. To us, showing that a certain theory can be axiomatized in a certain manner (for example, by means of quantifier-free axioms) in a particular language has epistemological import, whereas finding a simple axiom system has significant aesthetic value, and both are worth pursuing separately. In those instances in which we know an axiom system that is particularly simple, we will list the axioms. In cases in which the axiom system is not worth displaying, we shall mention only the language in which the constructive axiomatization can be carried out.

2.1. Euclidean planes

Classical axiomatizations of Euclidean planes (in a language with points as variables, and with the single quaternary relation $\equiv$, standing for *equidistance*, with $ab \equiv cd$ to be read as 'a is as distant from b as c is from d' (or 'the segment ab is congruent to the segment cd' (here 'segments' are figures of speech))) were provided in [107,108], and [36]. Euclidean planes may also be constructively axiomatized in several constructive languages.

The first is a bi-sorted language containing only individual constants and binary operation symbols, with individual variables for both *points* (upper-case) and *lines* (lower-case), $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \gamma)$, where A_0, A_1, A_2 stand, as always, for three non-collinear points, φ has points as arguments and a line as value, with $\varphi(A, B)$ representing the line that passes through A and B if $A \neq B$, an arbitrary line, otherwise, ι has lines as arguments and a point as value, with $\iota(g, h)$ representing the point of intersection of g and h, provided that it exists and is unique, an arbitrary point, otherwise, and γ has a point and a line as arguments, and a line as value, with $\gamma(P, l)$ representing the perpendicular raised in P on l, whenever P is a point on l, an arbitrary line, otherwise. Such an axiomatization was carried out, based on Rautenberg and Quaisser [101], in Pambuccian [84], where it was also shown that γ may be replaced by γ', where $\gamma'(P, l)$ represents the perpendicular dropped from P to l, whenever P is a point that is not on l, an arbitrary line, otherwise. Thus Euclidean planes correspond to the geometry of ruler and set square constructions, where the ruler may be used both to join two points by a line and to intersect two given non-parallel lines, and the set square may only be used to raise perpendiculars to lines, or only to drop perpendiculars to lines.

It follows from [3, Cor. 2.2] that γ may also be replaced by μ, where $\mu(A, B)$ represents the perpendicular bisector of the segment AB, whenever $A \neq B$, an arbitrary line in case $A = B$.

A fourth constructive axiomatization of Euclidean planes, based on Schaeffer [106], was provided in Pambuccian [84], corresponding to the geometric constructions that can be performed with ruler and angle transporter.

Further constructive axiomatizations for Euclidean planes are possible, as shown in [86], in the bi-sorted languages $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \kappa)$ or $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \varrho)$. Here A_0, A_1, A_2, φ and ι are as above, and κ is a ternary operation symbol with points as arguments and the point $\kappa(A, B, C)$, the second intersection of the circle with centre C and radius CA with the line $\varphi(A, B)$, as value (whenever $A \neq B$ and $A \neq C$, an arbitrary point, otherwise). The ternary operation ϱ has points as arguments, and $\varrho(A, B, C)$ should be read as 'the second intersection point of the circle with centre A and radius AC with the circle with centre B and radius BC (provided that A, B, C are three non-collinear points, an arbitrary point, otherwise)'. These results reprove a strengthened version of a theorem proved by Tietze [133–135], which states that the geometry of ruler and restricted compass constructions coincides with the geometry of ruler and set square constructions. The restricted compass may be used only to draw uniquely determined points of intersection of either circles and lines (an operation formalized by κ) or circles and circles (an operation formalized by ϱ). In other words, one cannot use the compass to determine the two intersection points (if they exist) of a line determined by two points A and B with a circle π, with neither A nor B lying on π. The motivation behind this interdiction is: (i) one does not know whether π will actually intersect the line joining A with B, as this depends on whether the distance from the centre of π to the line joining A with B is less than or equal to the radius of π; (ii) even if one knew that they do intersect, in case there are two distinct intersection points, one would be unable to separate them by metric properties alone, without taking recourse to betweenness considerations, so one cannot consider any of the points as determined inside a Euclidean plane. For the same reasons one does not consider the points of intersection of two circles to be constructed, unless one of them is an already constructed point (in which case the second point is uniquely determined even if it coincides with the first one).

A seventh constructive language in which it has received two different axiomatizations, in [81] and [83], has *points* as variables, and the two ternary operations R and U, with $R(abc)$ and $U(abc)$ to be read as 'the reflection of c in line ab, provided that $a \neq b$, an arbitrary point, otherwise', and 'the centre of the circumcircle of triangle abc, provided that a, b, c are three non-collinear points, an arbitrary point, otherwise'. It can be obtained from the axiom system for metric-Euclidean planes below, by adding to that axiom system the axiom stating that every triangle has a circumcentre, after having defined the operations used in the metric-Euclidean case, namely F and P, in a constructive manner (without quantifiers) in terms of R and U. The axiom system thus obtained in this language, i.e. only by means of a_0, a_1, a_2, R and U, although consisting of axioms containing each no more than 4 variables, cannot be said to be elegant. It is not known whether R is actually needed for a constructive axiomatization, i.e. whether a synonymous theory could be constructively axiomatized by means of a_0, a_1, a_2, and U alone. The answer is very likely negative, given that collinearity can very likely not be defined in terms of a_0, a_1, a_2, and U in a quantifier-free manner (probably not in first-order logic either).

A constructive axiomatization of Euclidean planes in which all angles are bisectable (Euclidean geometry over Pythagorean fields) can be found in [85], the operations corresponding to construction with a double-edged ruler, which can be used as a ruler and to draw a parallel at a fixed distance from a given line. Another one may be rephrased from [3, Th. 3.3] as stating that Euclidean planes with bisectable angles can be constructively axiomatized in $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \mu, \xi)$, where $A_0, A_1, A_2, \varphi, \iota, \mu$ are as above, and $\xi(g, h)$ is one of the two angle bisectors of the angle $\angle(g, h)$, whenever g and h are different intersecting lines, an arbitrary line, otherwise (and it would be worth knowing whether μ is needed in this constructive axiomatization, i.e. whether the perpendicular bisector operation is needed in the presence of the angle bisector operation).

A whole book [111] is devoted to the elementary geometry of Euclidean planes, which is surprisingly rich (cf. [81, Th. 3] for an exact theorem regarding what one can prove in algorithmic logic inside the theory of Euclidean planes).

2.2. *Ordered metric-Euclidean planes with free mobility*

It was shown in [83,87] that one can axiomatize metric-Euclidean geometry with free mobility and order constructively in a language with *points* as variables, with three individual constants (standing for three non-collinear points), and with the operations T and M only, where M is the binary midpoint operation, $M(ab)$ being the midpoint of ab if $a \neq b$, and a itself if $a = b$, and where T is interpreted as 'the point $T(abc)$ is as distant from a on the ray $\overrightarrow{ac}$ as b is from a, provided that $a \neq c \vee (a = c \wedge a = b)$, arbitrary, otherwise'.

The resulting theory is synonymous (or logically equivalent, in the sense of [18,97,104,131], or [98]) with one first presented by Bachmann [6], who investigated it as the Euclidean geometry of ruler, set square, and segment-transporter constructions, in which the ruler can be used only to join two points by a straight line, but not to find the point of intersection of two lines (unless the lines are perpendicular, in which case the set square provides the intersection point), and the set square to raise and drop perpendiculars from points to lines. This type of ruler will be referred to as *restricted ruler*.

It is not known whether M is needed, i.e. whether M can be defined without quantifiers in terms of T, or, in other words, if a synonymous theory could be constructively axiomatized by means of a_0, a_1, a_2 and T alone.

2.3. *Metric-Euclidean planes*

Metric-Euclidean planes can be constructively axiomatized in the language $\mathsf{L}(a_0, a_1, a_2, P, F)$, where a_0, a_1, a_2 are individual constants (standing for three non-collinear points), P and F ternary operations, with $P(abc)$ representing the image of c under the translation that maps a onto b, and $F(abc) = d$ standing for 'd is the foot of the perpendicular from c to the line ab (if $a \neq b$; a itself in the degenerate case $a = b$)'.

Given that this particular axiom system is remarkably simple, and assuming that the reader is by now curious to see how a constructive axiom system looks like, we will list the axioms of this particular axiom system, first presented in [82].

In order to formulate the axioms in a more readable way, we shall use the abbreviations $\sigma(ba) := P(abb)$, $L(abc) :\Leftrightarrow F(abc) = c \vee a = b$, $V(abc) :\Leftrightarrow \sigma(F(cba)b) = c$, $ab \equiv cd :\Leftrightarrow V(cP(abc)d)$, and, for $a \neq b$, $R(abc) := \sigma(F(abc)c)$.

These may be read as '$\sigma(ba)$ is the point obtained by reflecting a in b'; '$R(abc)$ is the point obtained by reflecting c in ab' if $a \neq b$, an arbitrary point, otherwise; $L(abc)$ as 'a, b, c are collinear'; $V(abc)$ as 'ab is congruent to ac', and $ab \equiv cd$ as 'ab is congruent to cd'.

The axiom system for metric-Euclidean planes consists of the following axioms:

ME1 $L(aba)$,
ME2 $P(abc) = P(acb)$,
ME3 $P(abc) = c \rightarrow a = b$,
ME4 $\sigma(ax) = \sigma(bx) \rightarrow a = b$,
ME5 $a \neq c \wedge a \neq b \wedge F(abc) = c \rightarrow F(abx) = F(acx)$,
ME6 $F(abx) = F(bax)$,

ME7 $\neg L(abx) \wedge x \neq x' \wedge V(axx') \wedge V(bxx') \rightarrow x' = R(abx)$,
ME8 $L(ab\sigma(ab))$,
ME9 $a \neq b \rightarrow xy \equiv R(abx)R(aby)$,
ME10 $P(abd) = P(cP(abc)d)$,
ME11 $V(oab) \wedge V(obc) \rightarrow V(oac)$,
ME12 $V(oab) \rightarrow V(o'P(oao')P(obo'))$,
ME13 $\neg L(a_0a_1a_2)$.

Notice that ME10 is the minor Desargues axiom and that ME11 is what Bachmann [8, §4,1] calls a *Mittelsenkrechtensatz*, for it states that if in a triangle two of the perpendicular bisectors meet, then the third one is concurrent with the first two.

All models of this axiom system are isomorphic to models of metric-Euclidean geometry, in which the constants a_0, a_1, a_2, as well as the operations F and P have the intended interpretations.

It is not known whether the operation P is needed, i.e. whether a synonymous theory could be axiomatized by means of a_0, a_1, a_2 and F alone.

Metric-Euclidean geometry can thus be understood as the Euclidean geometry of an instrument allowing the translation of a point c in the direction $\overrightarrow{ab}$, and of a restricted kind of set square, with which we can only drop a perpendicular from a point to a line and draw its foot.

2.4. Rectangular planes

As shown in [82], rectangular planes can be constructively axiomatized in the language $\mathsf{L}(a_0, a_1, a_2, P, R)$, where the primitive notions have the usual interpretations. We can define σ in terms of P as above. With L and V defined now by means of R by $L(abc) :\Leftrightarrow R(abc) = c \vee a = b$ and $V(abc) :\Leftrightarrow c = b \vee \sigma(ab) = c \vee (R(cba) \neq a \wedge R(aR(cba)b) = c)$, the axioms for rectangular planes are ME1–ME4, ME7–ME13, together with ME5′ and ME6′, where ME5′ and ME6′ stand for ME5 and ME6 in which we have replaced all occurrences of F by R.

It is not known whether the operation P is needed, i.e. whether a synonymous theory could be axiomatized by means of a_0, a_1, a_2 and R alone.

In [82] it is not only shown that Euclidean planes, metric-Euclidean planes, and rectangular planes can be constructively axiomatized in the above-mentioned constructive languages, but also that the universal $\mathsf{L}(a_0, a_1, a_2, R', U)$-theory of the standard Euclidean plane is precisely the theory of Euclidean planes in which fields are formally real. In this sense, if we regard the fact that the field is formally real as of little geometric relevance, we notice the naturalness of the notion of Euclidean plane. For, if we wanted, with our language restricted to a_0, a_1, a_2, R and U, and not allowed to use quantifiers, to describe the geometry of the standard Euclidean plane, with a_0, a_1, a_2 being interpreted in the standard Euclidean plane as three points 'in general position', then all we would ever say would be theorems valid in all Euclidean planes, together with sentences stating, in the algebra of the coordinate field, that the sum of any number of squares of non-zero elements is never zero. Since it is also possible to define P, F, and R in a quantifier-free manner in terms of R and U, we can ask: What are the universal $\mathsf{L}(a_0, a_1, a_2, P, F)$- and $\mathsf{L}(a_0, a_1, a_2, P, R)$-reducts of the theory of Euclidean planes, enlarged with definitions for P, F, and R. It turns out (see [82]) that these universal theories are precisely the theories of metric-Euclidean planes and of rectangular planes. Metric-Euclidean and rectangular planes are thus shown to be very natural fragments of Euclidean planes, for if we were to restrict our language and were to narrate in a quantifier-free manner what we notice to be true in all Euclidean planes in terms of the notions in our language, then we would be uttering precisely the theorems of metric-Euclidean or those of rectangular planes, depending on the language.

2.5. Richer Euclidean structures

The classical elementary geometries are the geometries of ruler and compass, and that of ruler and segment transporter (or gauge), an instrument with which one can obtain circle-line intersections only for lines passing through the centre of the circle (which is of fixed radius in the case of the gauge), which correspond to the Euclidean planes over Pythagorean ordered and Euclidean ordered fields. These were first constructively axiomatized, as mentioned in the Introduction, in [65] and [115].

It is, however, possible to axiomatize them in languages with ternary operations only, as shown in [83]. Although all axioms of the axiom systems proposed therein have as their starting point the axiom system for metric-Euclidean planes presented above, and each axiom contains no more than 4 variables, no elegant versions of the axiom systems for ruler-and-compass or for ruler-and-gauge geometry are known in the very sparse minimalist languages listed below. If we were allowed to keep F and P, then adding just a few axioms describing what the added operation symbols actually do would suffice. Let B stand for the ternary betweenness relation, i.e. $B(abc)$ iff 'b lies between a and c'. The ruler and compass geometry can be constructively axiomatized in $\mathsf{L}(a_0, a_1, a_2, T, U, H)$, where the operation H has the following intended interpretation:

> $H(xyz) = t$ if and only if t is the vertex of $\triangle xzt$, right-angled at t, with y the foot of the altitude, and such that $(\overrightarrow{yx}, \overrightarrow{yt})$ has the same orientation as $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2})$, provided that x, y, z are three different points such that $B(xyz)$, an arbitrary point, otherwise.

The ruler and segment-transporter geometry can be axiomatized in $\mathsf{L}(a_0, a_1, a_2, T, U)$. That by paper-folding, using the rules proposed in [3, §3], one can carry out the same constructions that one with ruler and gauge, was shown in [25, §4] (cf. also [59]).

It follows from [3, Th. 4.2] that ruler and compass geometry can also be constructively axiomatized in $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \mu, \omega_1, \omega_2)$, where $\{\omega_1(P, Q, l), \omega_2(P, Q, l)\}$ are the lines through Q, which reflect P onto l (i.e. such that the reflection of P in $\omega_i(P, Q, l)$ belongs to l, for $i = 1, 2$), whenever P, Q, l are such that at least one such line exists, two arbitrary lines, otherwise.

The argument for naturalness made earlier for Euclidean planes can be made without the caveat "if we regard the fact that the field is formally real as of little geometric relevance" for Euclidean geometry over Euclidean and Pythagorean fields, as well as for the geometry of metric-Euclidean ordered planes with free mobility. These are precisely the universal $\mathsf{L}(a_0, a_1, a_2, T, U, H)$-, $\mathsf{L}(a_0, a_1, a_2, T, U)$-, and $\mathsf{L}(a_0, a_1, a_2, T, M)$-theories of the standard Euclidean plane, with a_0, a_1, and a_2 interpreted as three points in general position. Here, however, the fact that these three geometries are natural is not at all surprising or new, for we knew that these were the geometries of ruler-and-compass, ruler-and-gauge, and restricted ruler, set square, and gauge constructions. That these instruments can be reduced to $\{T, U, H\}$, $\{T, U\}$, and $\{T, M\}$ may contain an element of surprise.

Steiner's (cf. [1]) theorem, amounting to the sufficiency of a one-time use of the compass to produce all the ruler-and-compass constructible points, could be rephrased as stating that ruler and compass Euclidean geometry can be axiomatized in a one-sorted language with *points* as variables, by means of three individual constants a_0, a_1, a_2, standing for three non-collinear points, and two quaternary operations, $Q(uvxy)$, standing for the intersection of the circle with u as centre and uv as radius with the line xy, which is rightmost in the order on xy in which x precedes y, provided that $x \neq y$, $u = a_0$, $v = a_1$, and the line xy does intersect the circle with centre a_0 and radius a_0a_1, an arbitrary point, otherwise, and I, the operation defined in the Introduction.

Steiner's theorem has been generalized to absolute geometry with the circle axiom in [48] and [125] (see also [16]), and that variant should allow for a similar rephrasing as an axiomatizability theorem. It has also been generalized in the Euclidean setting along several lines in [12,14,45,54,55,67–69,78,79,116,117,128,143–145].

The even richer Euclidean geometry of marked ruler constructions (see [61, Ch. 9]). can be constructively axiomatized (see [3, Th. 5.3], [20,32]) in the bi-sorted language $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \mu, \omega_1, \omega_2, \theta_1, \theta_2, \theta_3)$, where $\{\theta_1(P, Q, l, m), \theta_2(P, Q, l, m), \theta_3(P, Q, l, m)\}$ is the set of lines which simultaneously reflect P onto l and Q onto m, whenever P, Q, m, l are such that at least one such line exists, three arbitrary lines, otherwise (it is not known whether both operation symbols μ and ω are needed in the presence of θ).

In algorithmic logic, in which one can axiomatize the Archimedean Euclidean geometry of ruler and compass (in fact, the only additional property, besides the first-order quantifier-free fragment, of the standard Euclidean plane, which algorithmic logic captures, is precisely the Archimedean axiom, as shown by Engeler [21–24], i.e. the algorithmic theory of the standard Euclidean plane in a language with, say S, I, and C from the introduction, can be axiomatized by the universal axioms of Euclidean geometry of ruler-and-compass constructions to which we add the Archimedean axiom as a halting formula), we can drop U from the language, provided that we change T to T', with $T'(abc) = d$ if 'd is as distant from a on the ray $\overrightarrow{ca}$ as b is from a, provided that $a \neq c \vee (a = c \wedge a = b)$, and arbitrary, otherwise'. Such an axiomatization in algorithmic logic, containing only a_0, a_1, a_2, T', and H as primitive symbols, was presented in [80]. It says, in geometric constructions parlance, that ruler and compass in Archimedean planes over Euclidean ordered fields may be replaced by the segment transporter and an instrument which allows the construction of the intersection points of a line perpendicular to a given diameter of a circle with that circle.

The oldest theorem in the theory of geometric constructions in the presence of the Archimedean axiom, the Mohr-Mascheroni theorem, was rephrased in [90], as the axiomatizability of the geometry of Euclidean planes over Archimedean ordered Euclidean fields in algorithmic logic in a language with *points* as individual variables and a quaternary operation symbol, υ, with $\upsilon(abcd)$ and $\upsilon(cdab)$ denoting—in arbitrary order—the intersection points of the circles with centres a and c and radii ab and cd, provided that they exist, and arbitrary points, otherwise. The resulting axiom system axiomatizes the algorithmic theory of the standard Euclidean plane.

An even stronger form of the Mohr-Mascheroni theorem, which states that a rusty compass (e.g. one of unit radius) is all one needs, and was proved in [147] (after preliminary forgotten work done in [44]), can be rephrased as the axiomatizability of Euclidean planes over Archimedean ordered Euclidean fields inside algorithmic logic, with variables to be interpreted as *points*, with two individual constants a_0 and a_1, standing for two different points at a distance $\leqslant 2$, and with one binary operation ϵ, where $\epsilon(a,b)$ and $\epsilon(b,a)$ represent the two intersection points of the circles with centres a and b and radius one, whenever $a \neq b$ and the two circles intersect, and $\epsilon(a,b) = a$ otherwise.

Analogous rephrasings of the Mohr-Mascheroni theorem ought to be possible for hyperbolic geometry, where it is also valid, as shown by Strommer [123,124] and Martynenko [64].

3. Absolute geometry

By plane absolute geometry we understand the theory axiomatized by the plane axioms of the first three groups of axioms from Hilbert's *Grundlagen der Geometrie*. Its models, also called H-panes, were described algebraically by W. Pejas [96].

Based on the fact, proved in Gupta [38], that the inner Pasch axiom implies the full Pasch axiom, a constructive axiomatization for H-planes was provided in Pambuccian [88]. It is expressed in $\mathsf{L}(a_0, a_1, a_2, T', J)$, where the a_i stand for three non-collinear points, T' is the segment transport operation defined earlier, and J is a quaternary segment-intersection predicate, $J(abcd)$ being interpreted as the point of intersection of the segments ab and cd, provided that a and b are two distinct points that lie on different sides of the line cd, and c and d are two distinct points that lie on different sides of the line ab, and arbitrary otherwise. This shows the remarkable fact that all ruler, set square, and segment-transporter constructions in plane absolute geometry can be carried out by means of two geometric instruments: segment-transporter and segment-intersector. The fact that *ruler* and segment transporter are sufficient for all constructions in absolute geometry has been pointed out repeatedly, by Hjelmslev [43], Forder [27–29], Guber [37], Szász [127], and is implicit in Gupta [38] and Rigby [102,103].

If we enlarge the language by adding a ternary operation A—with $A(abc)$ representing the point on the ray $\overrightarrow{ac}$, whose distance from the line ab is congruent to the segment ab, provided that a, b, c are three non-collinear points, and an arbitrary point, otherwise—we can, as shown in [88], axiomatize constructively both Euclidean planes over Pythagorean ordered fields and Klein models of hyperbolic geometry over Euclidean ordered fields. The two geometries can thus be axiomatized in a constructive language with operation symbols for geometric operations which are absolute, i.e. have the same meaning in both Euclidean and hyperbolic geometry.

4. Metric planes

The theory of metric planes is that common substratum of Euclidean and non-Euclidean plane geometries that can be expressed in terms of incidence and orthogonality, where order, free mobility, and the intersection of non-orthogonal lines is ignored.

The concept of a *metric plane*, one of the most remarkable concepts in the modern foundations of geometry, grew out of the work of Hessenberg, Hjelmslev, Reidemeister and A. Schmidt, and was provided with a simple group-theoretic axiomatics by F. Bachmann [8, p. 33], which was made first-order and expressed outside group theory in [95]. He also described them in a bi-sorted language with *points* and *lines* as individual variables, and point-line incidence, line orthogonality, and line-reflections, which are defined as bijections of the collection of all points and lines, which preserve incidence and orthogonality, are involutory, and fix all the points of a line. The axiom system for non-elliptic metric planes (i.e. metric planes that satisfy axiom non-P, which states that the composition of three reflections in lines is never the identity, which is equivalent to the requirement of uniqueness of the perpendicular from a point outside of a line to that line (cf. Bachmann [8, §3,4, Satz 5])) states that (the words 'intersect', 'through', 'perpendicular', 'have in common' are the usual paraphrases): (i) There are at least two points; (ii) For every two different points there is exactly one line incident with those points; (iii) If line a is orthogonal to line b, then b is orthogonal to a; (iv) Orthogonal lines intersect; (v) Through every point there is to every line a unique perpendicular; (vi) To every line there is at least a reflection in that line; (vii) The composition of reflections in three lines a, b, c which have a point or a perpendicular in common is a reflection in a line d.

Non-elliptic metric planes have been constructively axiomatized in [92] in the language $\mathsf{L}(a_0, a_1, a_2, F, \pi)$ with individual variables to be interpreted as *points*, with a_0, a_1, a_2, F having the usual interpretations, and with π a ternary operation symbol, $\pi(abc)$ being interpreted as the fourth reflection point whenever a, b, c are collinear points with $a \neq b$ and $b \neq c$, an arbitrary point, otherwise. By *fourth reflection point* we mean the following: if we designate by σ_x the mapping defined by $\sigma_x(y) = \sigma(xy)$, i.e. the reflection of y in the point x, then, if a, b, c are three collinear points, by Bachmann [8, §3,9, Satz 24b], the composition (product) $\sigma_c\sigma_b\sigma_a$, is the reflection in a point, which lies on the same line as a, b, c. That point is designated by $\pi(abc)$. It is not known whether π is actually needed, or whether it could be replaced by the point-reflection operation σ, defined by $\sigma(ab) := \pi(aba)$, i.e. whether a synonymous theory could be constructively axiomatized in $\mathsf{L}(a_0, a_1, a_2, F)$, or, if that turns out to not be the case, then in $\mathsf{L}(a_0, a_1, a_2, F, \sigma)$.

In order to describe the models of non-elliptic metric planes with a non-Euclidean metric, those with a Euclidean metric being metric-Euclidean planes, we need to introduce the concept of an ordinary projective-metric plane. Good references for this concept are [8,112].

By an *ordinary metric-projective* plane $\mathfrak{P}(K, \mathfrak{f})$ over a field K of characteristic $\neq 2$, with $\mathfrak{f}$ a symmetric bilinear form, which may be chosen to be defined by $\mathfrak{f}(\mathbf{x}, \mathbf{y}) = \alpha x_1 y_1 + \beta x_2 y_2 + \gamma x_3 y_3$, with $\alpha\beta\gamma \neq 0$, for $\mathbf{x}, \mathbf{y} \in K^3$, we understand a set of points and lines, the former to be denoted by (x, y, z) the latter by $[u, v, w]$ (determined up to multiplication by a non-zero scalar, not all coordinates being allowed to be 0), endowed with a notion of incidence, point (x, y, z) being incident with line $[u, v, w]$ if and only if $xu + yv + zw = 0$, an orthogonality of lines defined by $\mathfrak{f}$, under which lines $\mathbf{g}$ and $\mathbf{g}'$ are orthogonal if and only if $\mathfrak{f}(\mathbf{g}, \mathbf{g}') = 0$, and a segment congruence relation defined by $\frac{F(\mathbf{a},\mathbf{b})^2}{Q(\mathbf{a})Q(\mathbf{b})} = \frac{F(\mathbf{c},\mathbf{d})^2}{Q(\mathbf{c})Q(\mathbf{d})}$, where $F(\mathbf{x}, \mathbf{y}) = \beta\gamma x_1 y_1 + \alpha\gamma x_2 y_2 + \alpha\beta x_3 y_3$, $Q(\mathbf{x}) = F(\mathbf{x}, \mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, for points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ for which $Q(\mathbf{a})$, $Q(\mathbf{b})$, $Q(\mathbf{c})$, $Q(\mathbf{d})$ are all $\neq 0$. An ordinary projective metric plane is called *hyperbolic* if $F(\mathbf{a}, \mathbf{a}) = 0$ has non-zero ($\mathbf{a} \neq \mathbf{0}$) solutions, in which case the set of solutions forms a conic section, the *absolute* of that projective-metric plane.

The algebraic characterization of non-elliptic metric planes is given by the following representation theorem of Bachmann [8]:

The models of non-elliptic metric planes are either metric-Euclidean planes, or else they can be represented as embedded[3] subplanes that contain the point $(0, 0, 1)$ of a projective-metric plane $\mathfrak{P}(K, \mathfrak{f})$ over a field K of characteristic $\neq 2$, in which no point lies on the line $[0, 0, 1]$, from which it inherits the collinearity and segment congruence relations.

5. Hyperbolic geometry

5.1. Elementary hyperbolic geometry

Elementary hyperbolic geometry was born in 1903 when Hilbert [42] provided, using the end-calculus to introduce coordinates (see also [40]), a first-order axiomatization for it by adding to the axioms for plane absolute geometry (the plane axioms contained in groups I, II, III) a *hyperbolic parallel axiom* stating that "Through any point P not lying on a line l there are two rays r_1 and r_2, not belonging to the same line, which do not intersect l, and such that every ray through P contained in the angle formed by r_1 and r_2 does intersect l". It turned out that this was precisely 'ruler-and-compass' hyperbolic geometry, in the sense that it is all one can say about the standard Klein model of hyperbolic geometry (over the reals) if one is equipped only with ruler and compass and is allowed to make only first-order, or elementary, statements. That one does not get to construct more by using a variety of instruments

[3] A metric subplane of a projective-metric plane is *embedded* in it, if it contains with every point all the lines of the projective-metric plane that are incident with it.

specific to hyperbolic geometry than by means of ruler and compass, follows from an important characterization result of Mordoukhay-Boltovskoy [66]. The theory of hyperbolic geometric constructions was almost entirely developed by Russian-writing mathematicians ([17] is one of the few English-language contributions, where the theorem in [66] is reproved). They can be found in [31,33–35,66,72–75,118,138], several of which were collected in the textbooks [119] and [76].

Inspired by Handest [39], Strommer [120], Semënovic [118], and Rautenberg [100], Pambuccian [89] has shown that plane hyperbolic geometry can be axiomatized by universal axioms in a bi-sorted first-order language $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \pi_1, \pi_2)$, with variables for both *points* and *lines*, where A_0, A_1, A_2, φ, ι have the usual interpretations, and $\pi_1(P,l) = g_1$, $\pi_2(P,l) = g_2$ may be read as 'g_1 and g_2 are the two limiting parallel lines from P to l' (provided that P is not on l, arbitrary lines, otherwise).

Pambuccian [94] has also provided another constructive axiom system for plane hyperbolic geometry, expressed in the language $\mathsf{L}(a_0, a_1, T', C_1, C_2, K_1, K_2, P', A', H_1, H_2)$, with *points* as variables, which contains only ternary operation symbols having the following intended interpretations (here $ab \equiv cd$ should be read as 'ab is congruent to cd', $B(abc)$ as 'b lies between a and c', $ab \perp bc$ as 'line ab is perpendicular to bc'): $T'(abc) = d$ if d is as distant from a on the ray $\overrightarrow{ca}$ as b is from a, provided that $a \neq c \vee (a = c \wedge a = b)$, and arbitrary, otherwise; $C_i(abc)$, for $i = 1, 2$, stand for the two points d for which $da \equiv db$ and $da \equiv ac$, provided that $a \neq b$ and b lies between a and c, arbitrary points, otherwise; $K_i(abc)$, for $i = 1, 2$, stand for the two points d for which $ad \equiv ab$ and $bd \equiv bc$, provided that c lies strictly between a and the reflection of b in a, two arbitrary points, otherwise; $P'(abc)$ stands for the point d on the side ac or bc of triangle abc, for which $da \equiv db$ and $B(adc) \vee B(bdc)$, provided that a, b, c are three non-collinear points, an arbitrary point, otherwise; $A'(abc)$ stands for the point d on the ray $\overrightarrow{ac}$ for which $dd' \equiv ab$, where d' is the reflection of d in the line ab, provided that a, b, c are three non-collinear points, arbitrary, otherwise; $H_i(abc)$, for $i = 1, 2$, stand for the two points d for which $db \perp ba$ and $ad \perp dc$, provided that a, b, c are three different points with $B(abc)$, arbitrary, otherwise. All the operations used are absolute, and by replacing one axiom in it with its negation we obtain an axiom system for the Euclidean geometry of ruler and compass constructions.

Given its simplicity—it is stated using only ternary operations, and each axiom is a statement containing at most 4 variables—we will present it here as a second example of explicit constructive axiomatization.

To shorten and improve the readability of the axioms, we introduce the following abbreviations:

$$\sigma(ab) := T'(abb), \qquad T(abc) := T'(ab\sigma(ac)), \qquad \alpha(ab) := C_1(\sigma(ab)bb),$$
$$M(ab) := P(\alpha(ab)\sigma(a\alpha(ab))\alpha(ba)), \qquad \beta(abc) := \sigma(T'(acb)\alpha(T'(acb)a)),$$
$$\eta(abc) := P(\alpha(T'(acb)a)\beta(abc)c), \qquad \tau_1(abc) := T'(\eta(abc)c\alpha(T'(acb)a)),$$
$$\tau_2(abc) := T'(\eta(abc)c\beta(abc)), \qquad \mu(abc) := M(\tau_1(abc)\tau_2(abc)), \qquad R(abc) := \sigma(\mu(abc)c),$$
$$ba \perp ac :\Leftrightarrow a \neq b \wedge b \neq c \wedge c \neq a \wedge T(c\sigma(ab)b) = b,$$
$$B(abc) :\Leftrightarrow T'(bac) = a \vee b = c, \qquad L(abc) :\Leftrightarrow B(abc) \vee B(bca) \vee B(cab),$$
$$Z(abc) :\Leftrightarrow a \neq b \wedge b \neq c \wedge T'(bac) = a,$$
$$ab \equiv cd :\Leftrightarrow (a = b \wedge c = d) \vee (a \neq b \wedge ((a \neq c \wedge T(a\sigma(M(ac)d)b) = b) \vee (a = c \wedge T(adb) = b))),$$

σ being defined for all values of the arguments, α and M being defined whenever $a \neq b$, T whenever $a \neq c$, and the remaining operations whenever a, b, c are not collinear.

The intuitive meaning of some of these abbreviations are: $T(abc)$ is the point d on the ray $\overrightarrow{ac}$ for which $ad \equiv ab$, provided that $a \neq c$, an arbitrary point, otherwise; $M(ab)$ stands for the midpoint of ab, provided that $a \neq b$; $R(abc)$ stands for the reflection of c in ab, to be used only when a, b, c are not collinear; $Z(abc)$ stands for 'b lies between a and c, being different from a and different from c'; $L(abc)$ stands for 'a, b, c are collinear'.

The axioms are

H1 $a \neq b \wedge ((B(abc) \wedge B(abd)) \vee (B(abc) \wedge B(dab)) \vee (B(bca) \wedge B(bda))) \rightarrow L(acd)$,
H2 $T'(aab) = a$,
H3 $b \neq a \wedge c \neq a \rightarrow T'(abT'(acb)) = b$,
H4 $a \neq c \wedge a \neq b \wedge T(a\sigma(M(ac)d)b) = b \rightarrow T(c\sigma(M(ca)b)d) = d$,
H5 $a \neq c \rightarrow \sigma(M(ac)c) = a$,

H6 $a \neq b \to M(ab) = M(ba)$,
H7 $a \neq d \wedge T(abd) = d \wedge T(acd) = d \to T(abc) = c$,
H8 $T'(baa) = a \to a = b$,
H9 $b \neq d \wedge c \neq d \wedge T'(bad) = a \wedge T'(cbd) = b \to T'(bac) = a \vee b = c$,
H10 $a \neq b \wedge b \neq c \wedge c \neq a \wedge c \neq d \wedge T(adc) = c \wedge T(bdc) = c \to d = R(abc)$,
H11 $\neg L(abd) \wedge \neg L(acd) \wedge R(abd) = R(acd) \to L(abc)$,
H12 $L(abc) \wedge a \neq b \wedge a \neq c \wedge b \neq c \wedge T(adc) = c \wedge T(bdc) = c \to d = c$,
H13 $\neg L(abc) \wedge B(abd) \to T(dR(abc)c) = c$,
H14 $a \neq c \wedge B(abc) \wedge T(adc) = c \to B(aT(abd)d) \wedge bc \equiv T(abd)d$,
H15 $a \neq b \wedge B(abc) \wedge T(bda) = a \to ac \equiv dT'(bcd)$,
H16 $a \neq b \wedge B(abc) \wedge T(adb) = b \to dc \equiv bT(acd)$,
H17 $a \neq b \wedge B(bac) \wedge T(adb) = b \to dc \equiv bT'(acd)$,
H18 $a \neq b \to B(aM(ab)b) \wedge M(ab)a \equiv M(ab)b$,
H19 $c \neq a \to B(cT(cbT'(abc))T'(abc))$,
H20 $c \neq a \to T(aT'(abc)b) = b$,
H21 $c \neq a \wedge T'(abc) = a \to a = b$,
H22 $\neg L(abc) \to P'(abc) \neq a \wedge T(P'(abc)ba) = a \wedge (B(aP'(abc)c) \vee B(bP'(abc)c))$,
H23 $\neg L(abc) \to \neg L(M(ab)M(bc)M(ca))$,
H24 (i) $Z(bc\sigma(ab)) \to T(aK_j(abc)b) = b \wedge T(bK_j(abc)c) = c$,
(ii) $B(abc) \wedge a \neq b \to T(C_j(abc)ba) = a \wedge T(aC_j(abc)c) = c$,
H25 (i) $Z(bc\sigma(ab)) \to R(abK_1(abc)) = K_2(abc)$,
(ii) $B(abc) \wedge a \neq b \to R(abC_1(abc)) = C_2(abc)$.
H26 $Z(abc) \to ab \perp bH_j(abc) \wedge aH_j(abc) \perp H_j(abc)c$,
H27 $Z(abc) \to \sigma(bH_1(abc)) = H_2(abc)$,
H28 $\neg L(abc) \to A'(abc) \neq a \wedge (B(aA'(abc)c) \vee B(acA'(abc))) \wedge ab \equiv A'(abc)R(abA'(abc))$,
H29 $\neg ab \perp bc \vee \neg bc \perp cd \vee \neg cd \perp da \vee \neg da \perp ab$,
H30 $a_0 \neq a_1$.

As shown in [94], all models of this axiom system are isomorphic to the Klein model of hyperbolic geometry over some Euclidean ordered field, and the operation symbols have the desired interpretation. It is not known whether all the operations in $\mathsf{L}(a_0, a_1, T', C_1, C_2, K_1, K_2, P', A', H_1, H_2)$ are needed for a constructive axiomatization.

Another constructive axiomatization with *points* as variables, in $\mathsf{L}(a_0, a_1, a_2, I, \epsilon_1, \epsilon_2)$, where a_0, a_1, a_2 stand for three non-collinear points, with $\Pi(a_0a_1) = \pi/3$ ($\Pi(xy)$ standing here for the Lobachevsky function associating the angle of parallelism to the segment xy), I as in the Introduction, and the ternary operation symbols, ε_1 and ε_2, with $\varepsilon_i(abc) = d_i$ (for $i = 1, 2$) to be interpreted as 'd_1 and d_2 are two distinct points on line $\overline{ac}$ such that $ad_1 \equiv ad_2 \equiv ab$, provided that $a \neq c$, an arbitrary point, otherwise', was provided by Klawitter [50]. The axiomatization is inspired by the work of Strommer [121,122] and Gafurov [31], who have shown that, given a few points in convenient positions, one can perform all ruler and compass constructions with ruler and segment transporter.

5.2. *Klingenberg's generalized hyperbolic geometry*

Klingenberg [51] axiomatized in the group-theoretical style of [8] a theory whose models are isomorphic to the *generalized Kleinian models* over arbitrary ordered fields K. Their point-set consists of the points of a hyperbolic projective-metric plane over K that lie inside the absolute, the lines being all the lines of the hyperbolic projective-metric plane that pass through points that are interior to the absolute. The idea is thus to keep all the properties of the Kleinian inner-disc model, but to drop the Euclideanity requirement for the ordered coordinate field.

Klingenberg's generalized hyperbolic geometry can be axiomatized in the bi-sorted first-order language $\mathsf{L}(A_0, A_1, A_2, A_3, \varphi, F, \pi, \iota, \zeta)$, with *points* and *lines* as individual variables, where A_0, A_1, A_2, A_3 stand for four points such that the lines $\varphi(A_0, A_1)$ and $\varphi(A_2, A_3)$ have neither a point nor a perpendicular in common, and the operation symbols have the same interpretation as earlier, with ζ a binary operation with lines as arguments and a line as value, with $\zeta(g, h)$ to be interpreted as 'the common perpendicular to g and h, provided that $g \neq h$ and that the common perpendicular exists, an arbitrary line, otherwise'.

Klingenberg's generalization of hyperbolic geometry turns out to be precisely the universal $\mathsf{L}(A_0, A_1, A_2, A_3, \varphi, F, \pi, \iota, \zeta)$-theory of the standard hyperbolic plane. The constructive setting allows us thus to show that this generalization is, from a geometric point of view, a natural one. When presented by its models, it may seem to be a generalization motivated by algebraic concerns, by the desire to remove the condition of Euclideanity of the ordered coordinate field.

The difference between generalized Kleinian models and Kleinian models over Euclidean ordered fields is that in the former neither midpoints of segments nor hyperbolic (limiting) parallels from a point to a line (in other words intersection points of lines with the absolute) need exist. In fact, if we add to an axiom system for the former an axiom stating the existence of the midpoint of every segment (in the constructive setting this amounts to enlarging the language with the binary operation M and an axiom stating that M is the midpoint operation), then we obtain an axiom system for elementary hyperbolic geometry.

5.3. Hyperbolic geometry of restricted ruler, set square, and segment-transporter

In [93] we asked hyperbolic geometry the same question that led Bachmann [6] to the discovery of ordered metric-Euclidean geometry with free mobility, namely: What is the hyperbolic geometry of restricted ruler, set square and segment-transporter, in other words, what are all the universal sentences that one can express in the language of these construction instruments that are true in the standard hyperbolic plane? We found a constructive axiom system for this natural fragment of hyperbolic geometry, stated in the one-sorted language with *points* as variables, $\mathsf{L}(a_0, a_1, a_2, F, \tau)$, where τ is a segment transport operation similar to Klawitter's ϵ_i, which transports segments on lines, not on rays, not distinguishing between the two possible ways to transport them—with $\tau(abcd)$, $\tau(bacd)$ standing for the two points x on the line cd for which cx is congruent to ab, if $c \neq d$, arbitrary points otherwise—all of whose models are planes like those described in Bachmann's representation theorem, with K a Pythagorean field, $\alpha = \beta = 1$ and $\gamma \notin K^2$. That axiom system thus axiomatizes the universal $\mathsf{L}(a_0, a_1, a_2, F, \tau)$-theory of the standard hyperbolic plane.

5.4. Treffgeradenebenen

Trying to understand the geometrical significance of the *Treffgeradenebenen* introduced by Bachmann [8, §18,6], [7] (whose *lines* are all the *Treffgeraden*, i.e. those lines in the two-dimensional Cartesian plane over K, with K a Pythagorean field, which intersect the unit circle (the set of all points (x, y) with $x^2 + y^2 = 1$) in two points, and whose *points* are all points for which all lines of the plane that pass through them are *Treffgeraden*) we have also provided in Pambuccian [93] a constructive axiom system for them expressed in the bi-sorted language $\mathsf{L}(A_0, A_1, A_2, \varphi, \perp, \tau', \lambda_1, \lambda_2)$, where A_0, A_1, A_2 stand again for three non-collinear points, φ has the same interpretation as earlier, $\perp$ is a binary operation whose first argument is a point and whose second argument is a line variable, $\perp(P, g)$ standing for the foot of the perpendicular from P to g, τ' is a quaternary operation whose first three arguments are points, and whose fourth argument is a line, $\{\tau'(A, B, C, g), \tau'(B, A, C, g)\}$ standing for the two points P on the line g, for which the segments CP and AB are congruent, provided that C lies on g, two arbitrary points, otherwise, and λ_i is a ternary operation, whose arguments are point variables, $\{\lambda_1(A, B, C), \lambda_2(A, B, C)\}$ standing for the two lines that are hyperbolically parallel to $\varphi(A, B)$ and perpendicular to $\varphi(A, C)$, provided that A, B, C are three non-collinear points, and the lines $\varphi(A, B)$ and $\varphi(A, C)$ are not orthogonal, two arbitrary lines, otherwise. The operation λ_i corresponds to a geometric instrument whose constructive strength was investigated in Al-Dhahir [2], where it is called a *hyperbolic ruler*. It was shown that our axiom system for *Treffgeradenebenen* is an axiom system for the universal $\mathsf{L}(A_0, A_1, A_2, \varphi, \perp, \tau', \lambda_1, \lambda_2)$-theory of the standard hyperbolic plane. Again, the constructive axiomatization shows that these structures are natural, and not just counterexamples or gratuitously exotic planes.

6. Intuitionistic constructive geometry

Intuitionistic axiomatizations of geometry go back to Heyting's thesis in 1925 (published in 1927) on the axiomatic foundation of projective geometry. Further work on affine and projective geometry was done by Heyting and van Dalen, an account of which can be found in [136]. The intuitionistic counterpart to Tarski's first-order axiomatization of Euclidean geometry was obtained in [58]. All of these use quantifiers, and thus do not qualify as *constructive* in the sense used in this paper. Nor would we expect quantifier-free intuitionistic axiomatization to succeed in the absence of any relation symbol besides equality, for we know that there are some "essentially negative" relations in intuitionistic mathematics, that cannot be reduced to negated equality. We should thus allow in intuitionistic constructive axiomatizations the use of some relations beyond equality. Constructive axiomatizations in this sense have been put forward by J. von Plato [141,142], and the axiom system of [141] has been simplified in [56,57]. The geometries axiomatized therein, apartness, affine, ordered affine, and orthogonal geometry, are very weak and their models do not seem to correspond to some models admitting an algebraization. The aim there is rather to show the advantages of P. Martin-Löf's intuitionistic type theory [62,63], in which the axiomatizations can be expressed, as pointed out in an earlier paper [60]. A short comparison between the constructive approach in the classical and intuitionistic setting can be found in [139].

7. Conclusions and open problems

Several of our constructive axiomatizations have produced the simplest possible axiom systems, in the precise sense of having the least possible number of variables in each of their axioms among all possible axiom system whose individual variables have the same interpretation. Thus constructive axiomatizations allowed us to prove in [82] and [94] that all fragments of the Euclidean geometry of ruler and compass admit axiomatizations, all of whose axioms are statement about at most 4 points, and the same holds for elementary hyperbolic geometry.

Aside from this aesthetic value of constructive axiomatization, and perhaps much more important, is the fact that it has showed that several weak geometries that had appeared in various contexts in the literature turn out to be *natural*, when viewed from the point of view of a restrictive constructive language.

Perhaps even more important is the fact that it showed us how little we know about geometries we thought were relegated to the dustbin of history. The most pressing open problems are those regarding what one may call the constructive independence of geometric constructive operations, such as: Do we need P to constructively axiomatize metric-Euclidean planes, or does a_0, a_1, a_2 and F suffice? Do we need π to constructively axiomatize non-elliptic planes, or do a_0, a_1, a_2, F, and σ suffice?

The process of translating a result on equivalences of constructibilities, as they are customarily phrased (such as Steiner's theorem or Mohr-Mascheroni), into one of constructive axiomatizability is by no means straightforward. When one proves that with instruments X and Y one can construct in some geometry all the points that can be constructed with instruments U and V one assumes the underlying geometry and all the properties of that geometry in the proof of the equivalence of the two sets of instruments. Moreover, very often one chooses convenient points, which ensure that certain lines actually do intersect (in hyperbolic geometry this is often the case), and it is not at all clear that these choices can be made by means of the available instruments. Moreover, in a constructive axiomatization one needs to define the underlying geometry while one is trying to define the modus operandi of the operation standing for the geometric construction instrument. Our statement, that there ought to be a constructive axiomatization as a counterpart of the hyperbolic Mohr-Mascheroni theorem or to the absolute Steiner theorem of [16] does not mean that we know how to obtain one, so these can be considered to be open problems.

Somewhat unsurprisingly, we know much less about constructions in the hyperbolic plane. It is not known what the ruler and set square hyperbolic geometry is, i.e. what the universal $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \perp)$-theory of the standard hyperbolic plane is. Nor is it known what one can achieve with the marked ruler, or what the hyperbolic geometry of origami constructions would be, i.e. the $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \mu, \omega_1, \omega_2)$-theory or the $\mathsf{L}(A_0, A_1, A_2, \varphi, \iota, \mu, \omega_1, \omega_2, \theta_1, \theta_2, \theta_3)$-theory of the standard hyperbolic plane.

It is also not known what the precise algebraic counterpart, that is, what types of fields serve as coordinate fields for the geometry of marked ruler and compass, where the neusis process can be carried out not only between two lines, but also between a line and a circle. A partial result in this direction can be found in [11]. The same question for hyperbolic geometry has not been investigated so far.

Another, hitherto unexploited, advantage of the constructive axiomatization is a proof-theoretical one. When axioms are rephrased as rules of inference, as proposed in [70] and [71], and as carried out for affine geometry in [91, p. 370–373], all formulas appearing in the sequents involved will be equalities. Besides being logic-free (as there are no logical connectives or quantifiers in any of the rules of inference replacing the axioms), the rules are also predicate-free (since the language contains no predicates). This could lead to improved means of proving the independence of a certain statement in a constructive geometry, by providing syntactic arguments for the non-derivability of the sequent corresponding to that statement.

8. Appendix

8.1. Algorithmic logic

We present the definition of Engeler's *algorithmic logic*, following closely [115]. We begin with a formal description of flow-charts, which may be thought of as trees consisting, besides exactly one root and at least one leaf, of two kinds of nodes: nodes at which there is one input and one output, and nodes at which there is one input and two outputs. At the first kind of nodes a variable is assigned a certain value, at the second kind of nodes the question whether the input is equal to a certain value is asked and the outcome (yes, no) determines which path is to be followed.

A *directed graph* is a relational structure $\mathbf{G} = \langle V_{\mathbf{G}}, E_{\mathbf{G}}\rangle$, where $E_{\mathbf{G}} \subseteq V_{\mathbf{G}} \times V_{\mathbf{G}}$ (the elements of $V_{\mathbf{G}}$ will be called *vertices* and $e = (v, v')$ with $e \in E_{\mathbf{G}}$ (which will be denoted by $E_{\mathbf{G}}(v, v')$) will be called an *edge*). For $v \in V_{\mathbf{G}}$ let $dg^+(v) = |\{v' \in V_{\mathbf{G}} \mid E_{\mathbf{G}}(v, v')\}|$ and $dg^-(v) = |\{v' \in V_{\mathbf{G}} \mid E_{\mathbf{G}}(v', v)\}|$. A sequence of vertices $p = (v_1, \dots, v_n)$ with $v_i \in V_{\mathbf{G}}$ $(i = 1, \dots, n)$ will be called a *path* in $\mathbf{G}$ if $E_{\mathbf{G}}(v_i, v_{i+1})$ for $i = 1, \dots, n-1$. The vertices $v \in V_{\mathbf{G}}$ with $dg^-(v) = 0$ (respectively $dg^+(v) = 0$) will be denoted by 'S' for 'start' (respectively 'H' for 'halt').

A finite directed graph (i.e. $V_{\mathbf{G}}$ is a finite set) will be called a *flow-chart* if:

(i) there is exactly one $v \in V_{\mathbf{G}}$ with $dg^-(v) = 0$;
(ii) there is at least one $v \in V_{\mathbf{G}}$ with $dg^+(v) = 0$;
(iii) for all $v \in V_{\mathbf{G}}$ with $dg^+(v) \neq 0$ either $dg^+(v) = 1$ or $dg^+(v) = 2$.

Let $\mathrm{L} = \mathrm{L}(\mathbf{F}, r)$ be a first-order language, where $\mathbf{F}$ is a finite set of operation symbols, and $r : \mathbf{F} \to \mathbb{N}$ is a function assigning to each $f \in \mathbf{F}$ its arity $r(f)$. Let $\mathbf{V}$ be the set of variables for L. Let $\Sigma_{\mathrm{L}}^{\mathbf{F}}$ be the set of assignments of the form $x \leftarrow \tau$ where $\tau = x_1$, or $\tau = f(x_1 \dots x_{r(f)})$ with $x_1, \dots, x_{r(f)}, x \in \mathbf{V}$, and $f \in \mathbf{F}$; Σ_{L}^{f} be the set of quantifier-free formulas $\varphi(x_1 \dots x_n)$ with free variables $x_1 \dots x_n$ $(n \in \mathbf{N})$ and $x_1 \dots x_n \in \mathbf{V}$[4]; Σ_{L} be $\Sigma_{\mathrm{L}}^{\mathbf{F}} \cup \Sigma_{\mathrm{L}}^{f}$.

For any sequence of symbols α (formula or assignment) let $V(\alpha)$ be the set of all variables in α. Let $\mathbf{G}$ be a flow-chart, $j : E_{\mathbf{G}} \to \Sigma_{\mathrm{L}}$ a map and $V_n = \{x_1, \dots, x_n\}$ with $x_1, \dots, x_n \in \mathbf{V}$. $\Pi_{\mathrm{L}}(x_1, \dots, x_n) = (\mathbf{G}_\Pi, j_\Pi, V_\Pi) = (\mathbf{G}, j, V_n)$ will be called a *program* over Σ_{L} in the variables $x_1, \dots, x_n$ if

(i) for $v \in V_{\mathbf{G}}$ with $dg^+(v) = 1$ there is an $v' \in V_{\mathbf{G}}$ and $\alpha \in \Sigma_{\mathrm{L}}^{\mathbf{F}}$ with $E_{\mathbf{G}}(v, v')$, $j(v, v') = \alpha$ and $V(\alpha) \subseteq V_\Pi$;
(ii) for $v \in V_{\mathbf{G}}$ with $dg^+(v) = 2$ there are $v', v'' \in V_{\mathbf{G}}$ and $\alpha \in \Sigma_{\mathrm{L}}^{f}$ with $E_{\mathbf{G}}(v, v'), E_{\mathbf{G}}(v, v'')$, $j(v, v') = \alpha$, $j(v, v'') = \neg\alpha$ and $V(\alpha) \subseteq V_\Pi$.

The map j_Π thus establishes a correspondence between the paths p in $\mathbf{G}_\Pi$ and sequences $w_\Pi(p)$ of elements from $\Sigma_\Pi \overset{\text{def}}{=} \{j(v, v') \in \Sigma_{\mathrm{L}} \mid (v, v') \in E_{\mathbf{G}_\Pi}\}$, which we interpret as words over Σ_Π. Let $W(\Pi) \overset{\text{def}}{=} \{w_\Pi(p) \mid p$ a path in $\mathbf{G}_\Pi\}$ and $W_{SH}(\Pi) \overset{\text{def}}{=} \{w_\Pi(p) \mid p$ is a path from S to H in $\mathbf{G}_\Pi\}$.

For a given program $\Pi_{\mathrm{L}}(x_1, \dots, x_n)$, we define a quantifier-free formula ϕ_w (describing under what conditions on x the program $\Pi_{\mathrm{L}}(x_1, \dots, x_n)$ will follow the path w) with $V(\phi_w) \subseteq V_\Pi$ inductively over $w \in W(\Pi)$.

(i) If $\lambda \in W(\Pi)$ is the empty word, then $\phi_\lambda(x_1, \dots, x_n) = (x_1 = x_1 \wedge \dots \wedge x_n = x_n)$;
(ii) if $\alpha \cdot w \in W(\Pi)$ with $\alpha \in \Sigma_{\mathrm{L}}^{\mathbf{F}}$, $\alpha = x \leftarrow \tau$ and $V(\alpha) \subseteq V(\Pi)$, then $\phi_{\alpha \cdot w}(x_1, \dots, x_n)$ is the formula obtained by substituting τ for x in $\phi_w(x_1, \dots, x_n)$;
(iii) if $\alpha \cdot w \in W(\Pi)$ with $\alpha \in \Sigma_{\mathrm{L}}^{f}$, and $V(\alpha) \subseteq V(\Pi)$, then $\phi_{\alpha \cdot w}(x_1, \dots, x_n) = \alpha \wedge \phi_w(x_1, \dots, x_n)$.

ϕ_Π will be called a *halting formula* for a program $\Pi_{\mathrm{L}}(x_1, \dots, x_n)$ if

$$\phi_\Pi(x_1, \dots, x_n) = \bigvee_{w \in W_{SH}(\Pi)} \phi_w(x_1, \dots, x_n).$$

[4] We could have taken Σ_{L}^{f} to be the set of formulas $x_i = x_j$ and $\neg(x_i = x_j)$ with $x_i, x_j \in \mathbf{V}$. The drawback would have been that programs would have become longer.

ϕ_Π will in general no longer be a first-order formula, but one in $L_{\omega_1\omega}$. The *algorithmic language* $aL(\mathrm{L})$ is the least language over L for which

(i) for every program Π over Σ_{L}, $\phi_\Pi \in aL(\mathrm{L})$;
(ii) if $\alpha \in aL(\mathrm{L})$, then $\neg\alpha \in aL(\mathrm{L})$;
(iii) if $\alpha, \beta \in aL(\mathrm{L})$, then $\alpha \vee \beta \in aL(\mathrm{L})$.

$aL(\mathrm{L})$ is a sublanguage of $L_{\omega_1\omega}$. Let $\mathfrak{M}$ be a structure for L and let Σ denote a set of sentences in $aL(\mathrm{L})$. We denote by $aTh(\mathfrak{M}) \stackrel{\text{def}}{=} \{\alpha \in aL(\mathrm{L}) \mid \mathfrak{M} \models \alpha\}$ the *algorithmic theory of* $\mathfrak{M}$ and by $aCn(\Sigma) \stackrel{\text{def}}{=} \{\alpha \in aL(\mathrm{L}) \mid \Sigma \models \alpha\}$ the set of *algorithmic consequences* of Σ.

8.2. Construction instruments

The *compass*. In its most generous form, the compass can be used to draw circles, as well as to construct the intersection points of two circles, or the intersection points of a circle and an already constructed line. We denote by $\mathcal{C}(a, bc)$ the circle with centre a and radius bc. Aspects of this operation can be found among several of our construction operations. In all our construction operations that reflect a compass use, the compass is *collapsible*, that is, cannot take a segment ab as its radius and draw it with a centre that is neither a nor b.

(1) Seeland's [115] quaternary operation C—with $C(xyuv)$ being the point of intersection of the circle $\mathcal{C}(x, xy)$ with the segment uv, provided that $x \neq y$, u lies inside and v lies outside $\mathcal{C}(x, xy)$, an arbitrary point, otherwise—stands for the segment and circle intersection aspect of the compass.
(2) The operation H—with $H(xyz) = t$ if and only if t is the vertex of $\triangle xzt$, right-angled at t, with y the foot of the altitude, and such that $(\overrightarrow{yx}, \overrightarrow{yt})$ has the same orientation as $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2})$, provided that x, y, z are three different points such that $B(xyz)$, an arbitrary point, otherwise—stands for the circle and ray intersection aspect of the compass, for $H(xyz)$ provides one of the intersection point of the circle with diameter xz with the perpendicular in y to the diameter xz. The operations $H_1(xyz)$ and $H_2(xyz)$ provide the pair of intersection points from which $H(xyz)$ was chosen (we thus known only what the set $\{H_1(xyz), H_2(xyz)\}$ consists of, and not what the individual points $H_1(xyz)$ and $H_2(xyz)$ are).
(3) The operations C_1 and C_2—the set $\{C_1(abc), C_2(abc)\}$ consists of the points of intersection of $\mathcal{C}(a, ac)$ with the perpendicular bisector of the segment ab, provided that $a \neq b$ and b lies between a and c, two arbitrary points, otherwise—stand for the circle and line intersection aspect of the compass.
(4) The operations K_1 and K_2—the set $\{K_1(abc), K_2(abc)\}$ consists of the points of $\mathcal{C}(a, ab)$ and $\mathcal{C}(b, bc)$, provided that c lies strictly between a and the reflection of b in a, two arbitrary points, otherwise—stand for the circle and circle intersection aspect of the compass.
(5) The quaternary operation υ—the set $\{\upsilon(abcd), \upsilon(cdab)\}$ consists of the points of intersection of $\mathcal{C}(a, ab)$ and $\mathcal{C}(c, cd)$, provided that the two circles have at least one point in common, two arbitrary points, otherwise.

According to the Poncelet–Steiner theorem, all constructions that can be performed with ruler and compass can be performed with ruler and a one-time use of the compass. The operation reflecting that single-use-compass is:

(6) $Q(uvxy)$—where u must be a_0 and v must be a_1 (i.e. these are the only values that appear as the first two arguments of Q in the axiom system). It stands for that intersection point of the circle $\mathcal{C}(u, uv)$ with the line xy, which is rightmost in the order on xy in which x precedes y, provided that $x \neq y$.

According to the Mohr-Mascheroni theorem, all constructions that can be performed with ruler and compass can be performed with compass alone.

The *rusty compass*. Moreover, as shown in [147], all ruler and compass constructions can be performed with a rusty compass, which is a compass with a *fixed* opening. Our construction operation corresponding to the rusty compass is the binary operation ϵ, where $\epsilon(a, b)$ and $\epsilon(b, a)$ represent the two intersection points of the circles with centres a and b and radius one, whenever $a \neq b$ and the two circles intersect, and $\epsilon(a, b) = a$ otherwise.

The *segment transporter*. As shown by Hilbert [41], one cannot perform in Euclidean geometry all ruler and compass constructions with the ruler and the segment transporter alone. As shown by Gafurov [31] and Strommer [121,122], the ruler and the segment transporter *are* equivalent to ruler and compass in hyperbolic geometry, if two points at a special distance are given. The segment transporter allows the transport of segment ab on ray $\overrightarrow{cd}$. If only one length (a unit length) can be transported, the instrument is called a *gauge*. Kürschák (see [41]) showed that the gauge is equivalent to the segment transporter in Euclidean geometry. It can be considered to be a very restricted version of the compass, in the sense that it is a compass which delivers intersection points only with its own diameters. It appears in our constructive setting under several guises.

(1) $S(xyuv)$—the point of intersection of $\mathcal{C}(u, xy)$. with the ray $\overrightarrow{uv}$, provided that $u \neq v \vee (u = v \wedge x = y)$, an arbitrary point, otherwise—reflects the result of using a movable segment transporter, i.e. one than can transport the segment xy to a different location.
(2) $\tau(abcd)$, $\tau(bacd)$—standing for the two intersection of $\mathcal{C}(c, ab)$ with the line cd—reflects a segment transporter which is not capable of making order-related decisions. In the bi-sorted logic with points and lines as variables, τ becomes
(3) a quaternary operation τ' whose first three arguments are points, and whose fourth argument is a line—$\{\tau'(A, B, C, g), \tau'(B, A, C, g)\}$ stands for the two points of intersection of $\mathcal{C}(C, AB)$ and the line g, provided that C lies on g, two arbitrary points, otherwise.
(4) $\varepsilon_1(abc)$ and $\varepsilon_2(abc)$—the points of intersection of $\mathcal{C}(a, ab)$ with the line $\overline{ac}$ provided that $a \neq c$, arbitrary points, otherwise—is similar to τ and τ' in the sense that it reflects the operation of an order-free segment transporter, but it also corresponds to a *restricted* form of that instrument, which we may call *collapsible segment transporter*, an instrument which can transport a segment ab only on a ray passing through a.
(5) $T(abc)$ (or $T'(abc)$)—standing for the intersection of $\mathcal{C}(a, ab)$ with the ray $\overrightarrow{ac}$ (or $\overrightarrow{ac}$), provided that $a \neq c \vee (a = c \wedge a = b)$, arbitrary, otherwise—reflect the application of an order-discernible collapsible segment transporter.

The *set square* is an instrument which allows both the raising and the dropping of perpendiculars from points to lines. Only our first operation $\perp$ reflects both of these capabilities.

(1) $\perp$, a binary operation, whose first argument is a point and whose second argument is a line variable, $\perp(P, g)$ standing for the foot of the perpendicular from P to g.
(2) F, a ternary operation with points as variables, $F(abc)$ standing for the foot of the perpendicular from c to the line ab (if $a \neq b$; a itself in the degenerate case $a = b$).
(3) γ, a binary operation whose first argument is a point and whose second argument is a line variable, $\gamma(P, l)$ standing for the perpendicular raised in P on l, whenever P is a point on l, an arbitrary line, otherwise.
(4) γ', a binary operation whose first argument is a point and whose second argument is a line variable, $\gamma'(P, l)$ standing for the perpendicular dropped from P on l, whenever P is a point which is not on l, an arbitrary line, otherwise.
(5) μ, a binary operation with points as variables—where $\mu(A, B)$ stands for the perpendicular bisector to segment AB in case $A \neq B$, an arbitrary line, otherwise—reflects an iterated use of the set square (given that the midpoint of AB has to be first constructed)
(6) P', a ternary relation with points as variables—where $P'(abc)$ stands for the intersection point of the perpendicular bisector of segment ab with one of the segments ac or bc, provided that a, b, c are three non-collinear points, an arbitrary point, otherwise—combines the raising a perpendicular property with the line-segment intersection property and with the constructing a midpoint property, all properties that may be construed as belonging to the set square.
(7) U, a ternary operation with points as variables—where $U(abc)$ stands for the intersection of the perpendicular bisectors of segments ab and ac, provided that a, b, c are three non-collinear points, an arbitrary point, otherwise—combines the raising a perpendicular operation with the constructing a midpoint and the line-intersection operations.

The next three operations can be seen as reflecting both the outcome of Euclidean constructions with a set square, or the outcome of Euclidean constructions with a *restricted compass*, an instrument with which one can draw circles, but one can only obtain those points of intersection between lines and circles or between circles and circles whose choice is unambiguous, i.e. for which we do not have to choose one among a set of two intersection points, given that one of the two points is already known.

(8) R, a ternary operation with points as variables—$R(abc)$ being the point obtained by reflecting c in ab, if $a \neq b$, an arbitrary point, otherwise—also reflects both an iterated use of the set square (requiring the construction of the intersection point of the parallels from a to bc, from b to ac, of the parallel from that intersection to ab, followed by the intersection of the latter with the perpendicular from c to ab) and a decision capability imbedded in the ability to produce c itself in case c lies on the line ab.
(9) ϱ, a slight variation of R, a ternary operation with points as variables (in a bi-sorted language), with $\varrho(A, B, C)$ standing for the point obtained by reflecting C in AB if A, B, C are not collinear, an arbitrary point, otherwise.
(10) κ, a ternary operation with points as variables (in a bi-sorted language), with $\kappa(A, B, C)$ standing for the reflection of A in the perpendicular from C to AB, if $A \neq B$ and $A \neq C$, an arbitrary point, otherwise.

The *ruler* is an instrument that allows us to construct the line joining two different points, as well as to find the point of intersection of two intersecting lines. These two properties will be treated separately by our operations. An instrument with which we can only join points by a line is referred to as a *restricted ruler*. It is represented by our operation

(1) φ, a binary operation with points as variables, with $\varphi(A, B)$ standing for the line joining A and B, provided that $A \neq B$, an arbitrary line, otherwise.
The line-intersection operation is incorporated in the next two operations:
(2) ι, a binary operation with lines as variables, $\iota(g, h)$ standing for the intersection point of g and h, provided that $g \neq h$, and that g and h have a common point.
(3) I, a quaternary operation with points as variables, where $I(xyuv)$ is the point of intersection of the lines xy and uv, provided that $x \neq y$, $u \neq v$, the lines xy and uv are distinct and do intersect, an arbitrary point, otherwise.
(4) J, a quaternary operation with points as variables—$J(abcd)$ being the point of intersection of the *segments ab* and cd, provided that $a \neq b$, $c \neq d$, and that the two segments intersect—reflects the capabilities of a *bounded ruler*, an instrument with which one can only draw the segment joining two points and find the intersection point of segments.

The *parallel-ruler* is an instrument which one can use both as a ruler and as an instrument with which one can draw a parallel line from a given point to a given line. Aspects of this instrument are reflected in

(1) the affine operation P—$P(abc)$ being the image of c under the translation that maps a onto b,
(2) the absolute operation M—$M(ab)$ being the midpoint of the segment ab.

A special kind of parallel-ruler is the *double-edged ruler*, which can be used both as a ruler and to draw a parallel to a given line at an a priori *fixed* distance. It was investigated as a geometric construction instrument in [46], and a constructive axiomatization based on its modus operandi was presented in [85].

The angle bisector is an instrument that enables the construction of the angle bisectors of a given angle. It is represented by our operation ξ, a binary operation with lines as variables, $\xi(g, h)$ being one of the two angle bisectors of the angle $\angle(g, h)$, whenever g and h are different intersecting lines, an arbitrary line, otherwise.

We have also used, in [84], an operation corresponding to the action of an *angle transporter*, an instrument which is capable of transporting an angle such that its vertex lands on a given point P and one its legs lands on a given line passing through P.

An absolute operation that does not correspond to a traditional construction instrument (given that, in the context in which it is used, non-elliptic metric planes, midpoints need not exist) is π, with $\pi(abc)$ standing for the *fourth reflection point* of three collinear points a, b, c, i.e. the point with the property that the composition (in this order) of the reflections a, in b, and in c is the reflection in $\pi(abc)$.

So far our instruments were either Euclidean or absolute. We now turn to specifically hyperbolic construction instruments. The *hyperbolic parallel-ruler* is an instrument which allows the construction of the two limiting parallel lines from a point to a line. Its operation is reflected by π_1 and π_2, two binary operations (the first argument a point, the second a line), with $\pi_1(P,l)$ and $\pi_2(P,l)$ standing for the two limiting parallel lines from P to l (provided that P is not on l, arbitrary lines, otherwise).

The *hyperbolic ruler* (going back to Al-Dhahir [2]) allows, given two rays r_1 and r_2 that are not perpendicular, the construction of that limiting parallel to r_1 which is perpendicular to r_2. It is reflected by λ_1 and λ_2, two ternary operations, whose arguments are point variables, $\{\lambda_1(A,B,C), \lambda_2(A,B,C)\}$ standing for the two lines that are hyperbolically parallel to $\varphi(A,B)$ and perpendicular to $\varphi(A,C)$, provided that A,B,C are three non-collinear points, and the lines $\varphi(A,B)$ and $\varphi(A,C)$ are not orthogonal, two arbitrary lines, otherwise.

One can also image the *common-perpendicular ruler*, an instrument with which one can construct the common perpendicular to two hyperparallel (non-intersecting and non-hyperbolically parallel lines) lines. It is reflected by our ζ, with lines as variables, $\zeta(g,h)$ being the common perpendicular to g and h, provided that $g \neq h$ and that the common perpendicular exists, an arbitrary line, otherwise.

Two closely related ternary operations, A and A'—where $A(abc)$ (or $A'(abc)$) stands for the point of intersection of the ray $\overrightarrow{ac}$ with the equidistant curve (a line in the Euclidean case, a hypercycle in the hyperbolic case) whose distance from line ab is the segment ab (or half the segment ab), provided that a,b,c are three non-collinear points, and an arbitrary point, otherwise—correspond to different geometric constructions in Euclidean and hyperbolic geometry, although they have the same description in terms of betweenness and congruence (B and $\equiv$).

Finally, we have specifically *origami* geometric constructions, reflected in our setting by the operations ω_1, ω_2—where $\{\omega_1(P,Q,l), \omega_2(P,Q,l)\}$ are the lines through Q, which reflect P onto l (i.e. such that the reflection of P in $\omega_i(P,Q,l)$ belongs to l, for $i = 1,2$), whenever P,Q,l are such that at least one such line exists, two arbitrary lines, otherwise—and the operation θ_i with $i = 1,2,3$, that allows for constructions of the kind possible only with the marked ruler, but not with ruler and compass, where $\{\theta_1(P,Q,l,m), \theta_2(P,Q,l,m), \theta_3(P,Q,l,m)\}$ is the set of lines which simultaneously reflect P onto l and Q onto m, whenever P,Q,m,l are such that at least one such line exists, three arbitrary lines, otherwise.

8.3. Fields

There are several classes of (commutative) fields that appeared as coordinate fields of our geometries:

(1) non-quadratically closed fields are fields that have an element which is not a square.
(2) formally real fields are fields for which no sum of squares is equal to -1. Formally real fields are known to be orderable (if one assumes AC).
(3) Pythagorean ordered fields are ordered fields in which every sum of two squares is a square.
(4) Euclidean ordered fields are fields in which every positive element is a square.
(5) Vietan fields are Euclidean fields in which every cubic polynomial has a zero.
(6) real closed fields are Euclidean fields in which every polynomial of odd degree has a zero.

Acknowledgement

The author thanks two anonymous referees for suggestions that significantly improved the presentation of this survey.

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