



## On generalized weakly concircularly symmetric manifolds

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**Abstract.** The object of the present paper is to study generalized weakly concircularly symmetric manifold and study its geometric properties. We also study conformally flat generalized weakly concircularly symmetric manifolds. Existence of such space is also ensured by a non-trivial example.

**2000 Mathematics Subject Classification,** 53C15, 53C25.

**Key words and phrase:** weakly symmetric manifold, generalized weakly symmetric manifold, generalized weakly concircularly symmetric manifolds.

**Introduction:** Let  $M$  be a  $n(> 3)$ -dimensional manifold equipped with Riemannian metric  $g$ , Levi-Civita connection  $\nabla$ , Riemannian curvature  $R$ , Ricci tensor  $S$  and scalar curvature  $r$ . Extending Chaki pseudo symmetric Tamássy and Binh [15] introduced the notion of weakly symmetric manifold  $(W.S)_n$ .

A manifold is said to be  $(W.S)_n$  if it satisfies the relation

$$\nabla R = A_1 \otimes R + B_1 \otimes R + B_1 \otimes R + D_1 \otimes R + D_1 \otimes R.$$

For details we refer to see the paper of A. A. Shaikh and his co-authors ([4],[5], [6],[7],[8],[10], [11], [12], [13], [14], etc) and also the reference there in.

Generalizing  $(W.S)_n$  recently Baishya [1] introduced the notion of generalized weakly symmetric space (briefly  $(G.W.S)_n$ ) defined by the following equations

$$(1.1) \quad \nabla R = A_2 \otimes R + B_2 \otimes R + B_2 \otimes R + D_2 \otimes R + D_2 \otimes R \\ + \alpha_2 \otimes G + \beta_2 \otimes G + \beta_2 \otimes G + \gamma_2 \otimes G + \gamma_2 \otimes G.$$

Where  $G$  is Gaussian curvature tensor and  $A_2, B_2, D_2, \alpha_2, \beta_2$  and  $\gamma_2$  are non-zero 1-forms.

Generalizing the notion of generalized weakly symmetric manifold, in this paper we introduce the notion of generalized weakly concircularly symmetric manifold. A Riemannian manifold  $M, n > 2$ , is said to be generalized weakly concircularly symmetric manifold if its concircular curvature tensor  $\tilde{C}$  satisfies the following relation:

$$(1.2) \quad (\nabla_X \tilde{C})(Y, U, V, W) = \bar{A}(X) \tilde{C}(Y, U, V, W) + \bar{B}(Y) \tilde{C}(X, U, V, W) \\ + \bar{B}(U) \tilde{C}(Y, X, V, W) + \bar{D}(V) \tilde{C}(Y, U, X, W) + \bar{D}(W) \tilde{C}(Y, U, V, X)$$

$$+\bar{\alpha}(X) G(Y, U, V, W) + \bar{\beta}(Y) G(X, U, V, W) + \bar{\beta}(U) G(Y, X, V, W) \\ +\bar{\gamma}(V) G(Y, U, X, W) + \bar{\gamma}(W) G(Y, U, V, X).$$

Where  $\bar{A}, \bar{B}, \bar{D}, \bar{\alpha}, \bar{\beta}$  and  $\bar{\gamma}$  are non-zero 1-forms which are defined as  $\bar{A}(X) = g(X, \theta_1)$ ,  $\bar{B}(X) = g(X, \phi_1)$ ,  $\bar{D}(X) = g(X, \pi_1)$ ,  $\bar{\alpha}(X) = g(X, \theta_2)$ ,  $\bar{\beta}(X) = g(X, \phi_2)$  and  $\bar{\gamma}(X) = g(X, \pi_2)$ . Such an n-dimensional manifold is denoted by  $(G.W.\tilde{C}.S)_n$ . The local expression of (1.2) is

$$(1.3) \quad \tilde{C}_{mnpq,k} = \bar{A}_k \tilde{C}_{mnpq} + \bar{B}_m \tilde{C}_{knpq} + \bar{B}_n \tilde{C}_{mkpq} + \bar{D}_p \tilde{C}_{mnkq} + \bar{D}_q \tilde{C}_{mnpk} \\ + \bar{\alpha}_k G_{mnpq} + \bar{\beta}_m G_{knpq} + \bar{\beta}_n G_{mkpq} + \bar{\gamma}_p G_{mnkq} + \bar{\gamma}_q G_{mnpk}.$$

Where  $\bar{A}_i, \bar{B}_i, \bar{D}_i, \bar{\alpha}_i, \bar{\beta}_i$  and  $\bar{\gamma}_i$  are non-zero co-vectors. If  $r = 0$  then  $(G.W.\tilde{C}.S)_n$  reduces to  $(G.W.S)_n$ . In this paper we show that every  $(G.W.\tilde{C}.S)_n$  is a  $(G.W.S)_n$ . So the main interest to study of such  $(G.W.\tilde{C}.S)_n$ -space is that as in [1], for suitable choice of 1-forms we get different space such as weakly concircularly symmetric manifolds [9], symmetric space [2], recurrent space [16], generalized recurrent space [3], weakly symmetric space [15] etc.

We organized this paper as follows; section 2 is concerned with some basic geometric properties of  $(G.W.\tilde{C}.S)_n$ . In section 3 we study conformally flat  $(G.W.\tilde{C}.S)_n$  and the form of scalar curvature is also found. Section 4 deals with examples of  $(G.W.\tilde{C}.S)_n$  to establish the existence of such space.

## 2. Some basic geometric properties of $(G.W.\tilde{C}.S)_n$ .

In this section, we consider a  $(G.W.\tilde{C}.S)_n$ . Now, contracting (1.2) we have

$$(2.1) \quad (\nabla_X S)(Y, W) - \frac{dr(X)}{n} g(Y, W) = \bar{A}(X)[S(Y, W) - \frac{r}{n} g(Y, W)] \\ + \bar{B}(Y)[S(X, W) - \frac{r}{n} g(X, W)] + [\bar{D}(R(X, W)Y) - \bar{B}(R(Y, X)W)] \\ - \frac{r}{n(n-1)} [\bar{B}(X)g(Y, W) - \bar{B}(Y)g(X, W) + \bar{D}(X)g(Y, W) - \bar{D}(W)g(Y, X)] \\ + \bar{D}(W)[S(Y, X) - \frac{r}{n} g(X, Y)] + (n-1)[\bar{\alpha}(X)g(Y, W) + \bar{\beta}(Y)g(X, W) + \bar{\gamma}(W)g(Y, X)] \\ - \bar{\beta}(Y)g(X, W) + [\bar{\beta}(X) + \bar{\gamma}(X)]g(Y, W) - \bar{\gamma}(W)g(Y, X).$$

Taking contraction over  $Y$  and  $W$  in (2.1) we obtain

$$(2.2) \quad 2 [\bar{B}(QX) - \frac{r}{n} \bar{B}(X)] + 2 [\bar{D}(QX) - \frac{r}{n} \bar{D}(X)] \\ + (n-1)[n \bar{\alpha}(X) + 2 \bar{\beta}(X) + 2 \bar{\gamma}(X)] = 0.$$

Where  $Q$  be the symmetric endomorphism of tangent bundle corresponding to the Ricci tensor  $S$ . This gives the following:

**Theorem 2.1.** For a  $(G.W.\tilde{C}.S)_n$ , the 1-forms are related by the expression (2.2) .

Again contraction of (2.1) over  $X$  and  $W$  yields

$$(2.3) \quad \frac{n-2}{2n} dr(Y) = [\bar{A}(QY) - \frac{r}{n} \bar{A}(Y)] - [\bar{B}(QY) - \frac{r}{n} \bar{B}(Y)] \\ + [\bar{D}(QY) - \frac{r}{n} \bar{D}(Y)] + (n-1)[\bar{\alpha}(Y) + (n-1)\bar{\beta}(Y) + \bar{\gamma}(Y)].$$

Further contraction of (2.1) over  $X$  and  $Y$  gives

$$(2.4) \quad \frac{n-2}{2n} dr(W) = [\bar{A}(QW) - \frac{r}{n} \bar{A}(W)] + [\bar{B}(QW) - \frac{r}{n} \bar{B}(W)] \\ - [\bar{D}(QW) - \frac{r}{n} \bar{D}(W)] + (n-1)[\bar{\alpha}(W) + \bar{\beta}(W) + (n-1)\bar{\gamma}(W)].$$

If the scalar curvature is constant then  $dr(X) = 0$ , then from (2.3) and (2.4) we have

$$(2.5) \quad 0 = [\bar{A}(QY) - \frac{r}{n} \bar{A}(Y)] - [\bar{B}(QY) - \frac{r}{n} \bar{B}(Y)] \\ + [\bar{D}(QY) - \frac{r}{n} \bar{D}(Y)] + (n-1)[\bar{\alpha}(Y) + (n-1)\bar{\beta}(Y) + \bar{\gamma}(Y)].$$

$$(2.6) \quad 0 = [\bar{A}(QW) - \frac{r}{n} \bar{A}(W)] + [\bar{B}(QW) - \frac{r}{n} \bar{B}(W)] \\ - [\bar{D}(QW) - \frac{r}{n} \bar{D}(W)] + (n-1)[\bar{\alpha}(W) + \bar{\beta}(W) + (n-1)\bar{\gamma}(W)].$$

This gives the following:

**Theorem 2.2.** If the scalar curvature of a  $(G.W.\tilde{C}.S)_n$  is non-zero constant, then the 1-forms are related by the expression (2.5) and (2.6) .

Again from (2.3) and (2.4) we have

$$(2.7) \quad r = \frac{n}{2} \frac{[2\{\bar{B}(QX) - 2\bar{D}(QX)\} - (n-1)(n-2)(\bar{\beta}(X) - \bar{\gamma}(X))]}{[\bar{B}(X) - \bar{D}(X)]}.$$

This gives the following:

**Theorem 2.3.** For a  $(G.W.\tilde{C}.S)_n$  the scalar curvature exist and is given by (2.7) provided  $[\bar{B}(X) - \bar{D}(X)] \neq 0$ .

**Theorem 2.4.** In a  $(G.W.\tilde{C}.S)_n$ ,  $\frac{r}{n}$  is the eigenvalue of the Ricci tensor  $S$  corresponding to the eigenvector  $\bar{\mu}$  defined by  $g(X, \bar{\mu}) = H_1(X)$  where  $H_1(X) = \bar{B}(X) - \bar{D}(X)$  provided  $\bar{\beta} = \bar{\gamma}$ .

Again from (1.2) we have

$$(2.8) \quad (\nabla_X R)(Y, U, V, W) = \bar{A}(X) R(Y, U, V, W) + \bar{B}(Y) R(X, U, V, W) \\ + \bar{B}(U) R(Y, X, V, W) + \bar{D}(V) R(Y, U, X, W) + \bar{D}(W) R(Y, U, V, X) \\ + \left[ \frac{dr(X)}{n(n-1)} - \frac{r \bar{A}(X)}{n(n-1)} + \bar{\alpha}(X) \right] G(Y, U, V, W) + \left[ \bar{\beta}(Y) - \frac{r \bar{B}(Y)}{n(n-1)} \right] G(X, U, V, W) \\ + \left[ \bar{\beta}(U) - \frac{r \bar{B}(U)}{n(n-1)} \right] G(Y, X, V, W) + \left[ \bar{\gamma}(V) - \frac{r \bar{D}(V)}{n(n-1)} \right] G(Y, U, X, W) \\ + \left[ \bar{\gamma}(W) - \frac{r \bar{D}(W)}{n(n-1)} \right] G(Y, U, V, X).$$

This leads the following:

**Theorem 2.5.** A  $(G.W.\tilde{C}.S)_n$  is a  $(G.W.S)_n$ .

So by [1] we have the following:

**Theorem 2.6.** A  $(G.W.\tilde{C}.S)_n$  is a generalized weakly Ricci symmetric manifolds [1].

### 3. Conformally flat $(G.W.\tilde{C}.S)_n$

The Weyl conformal curvature tensor  $C$  of type (0,4) is given by

$$C = R - \frac{1}{n-2} g \wedge S + \frac{r}{(n-1)(n-2)} G.$$

Where  $\wedge$  is the tensor product of (0,2) tensors. A Riemannian manifold  $(M, g), n > 3$ , is said to be conformally flat if its conformal curvature tensor  $C(X, Y, Z, W) = 0$  where  $X, Y, Z, W \in \chi(M)$ . Then we have

$$(3.1) \quad R = \frac{1}{n-2} g \wedge S - \frac{r}{(n-1)(n-2)} G.$$

Using (3.1) in (1.2) we have

$$\begin{aligned} (3.2) \quad & (\nabla_X R)(Y, U, V, W) - \frac{dr(X)}{n(n-1)} G(Y, U, V, W) \\ & = \bar{A}(X) \left[ \frac{1}{n-2} (g \wedge S)(Y, U, V, W) - \frac{2r}{n(n-2)} G(Y, U, V, W) \right] \\ & \quad + \bar{B}(Y) \left[ \frac{1}{n-2} (g \wedge S)(X, U, V, W) - \frac{2r}{n(n-2)} G(X, U, V, W) \right] \\ & \quad + \bar{B}(U) \left[ \frac{1}{n-2} (g \wedge S)(Y, X, V, W) - \frac{2r}{n(n-2)} G(Y, X, V, W) \right] \\ & \quad + \bar{D}(V) \left[ \frac{1}{n-2} (g \wedge S)(Y, U, X, W) - \frac{2r}{n(n-2)} G(Y, U, X, W) \right] \\ & \quad \quad \quad \bar{D}(W) \left[ \frac{1}{n-2} (g \wedge S)(Y, U, V, X) - \frac{2r}{n(n-2)} G(Y, U, V, X) \right] \\ & \quad + \bar{\alpha}(X) G(Y, U, V, W) + \bar{\beta}(Y) G(X, U, V, W) + \bar{\beta}(U) G(Y, X, V, W) \\ & \quad \quad \quad + \bar{\gamma}(V) G(Y, U, X, W) + \bar{\gamma}(W) G(Y, U, V, X). \end{aligned}$$

Contracting (3.2) we have

$$\begin{aligned} (3.3) \quad & (\nabla_X S)(Y, W) = \left[ \bar{A}(X) + \frac{\bar{B}(X)}{n-2} + \frac{\bar{D}(X)}{n-2} \right] S(Y, W) \\ & \quad + \frac{n-3}{n-2} [\bar{B}(Y)S(X, W) + \bar{D}(W)S(Y, X)] + \frac{1}{(n-2)} [\bar{B}(QX)g(Y, W) - \bar{B}(QY)g(X, W)] \\ & \quad + \frac{1}{(n-2)} [\bar{D}(QX)g(Y, W) - \bar{D}(QW)g(Y, X)] - \frac{2r}{n(n-2)} [\bar{B}(X)g(Y, W) - \bar{B}(Y)g(X, W)] \\ & \quad - \frac{2r}{n(n-2)} [\bar{D}(X)g(Y, W) - \bar{D}(W)g(Y, X)] - \frac{2r(n-1)}{n(n-2)} \bar{B}(Y)g(X, W) - \frac{2r(n-1)}{n(n-2)} \bar{D}(W)g(Y, X) \\ & \quad \quad \quad + \frac{r}{(n-2)} [\bar{A}(X)g(Y, W) + \bar{B}(Y)g(X, W) + \bar{D}(W)g(Y, X)] \end{aligned}$$

$$+ \left[ \frac{dr(X)}{n(n-1)} - \frac{2r\bar{A}(X)}{n(n-2)} \right] (n-1)g(Y, W) + (n-1)[\bar{\alpha}(X)g(Y, W) + \bar{\beta}(Y)g(X, W) + \bar{\gamma}(W)g(Y, X)] - \bar{\beta}(Y)g(X, W) + [\bar{\beta}(X) + \bar{\gamma}(X)]g(Y, W) - \bar{\gamma}(W)g(Y, X).$$

Again contracting (3.3) we have

$$(3.4) \quad r = \frac{n}{2} \frac{[2\{\bar{B}(QX) + \bar{D}(QX)\} + (n-1)(n\bar{\alpha}(X) + 2\bar{\beta}(X) + 2\bar{\gamma}(X))]}{[\bar{B}(X) + \bar{D}(X)]}.$$

This gives the following:

**Theorem 3.1.** In a conformally flat  $(G.W.\tilde{C}.S)_n$  the scalar curvature is given by (3.4) provided  $[\bar{B}(X) + \bar{D}(X)] \neq 0$ .

Again contraction of (3.3) over  $X$  and  $Y$  yields

$$(3.5) \quad \frac{n-2}{2n} dr(X) = \bar{A}(QX) + \bar{B}(QX) - \bar{D}(QX) - \frac{r}{n} [\bar{A}(X) + \bar{B}(X) - \bar{D}(X)] + 2(n-1)[\bar{\alpha}(X) + \bar{\beta}(X) + (n-1)\bar{\gamma}(X)].$$

If we take the scalar curvature of a conformally flat  $(G.W.\tilde{C}.S)_n$ -space is non-zero constant, then we have  $dr(X) = 0$  and from (3.5) we have

$$(3.6) \quad 0 = \bar{A}(QX) + \bar{B}(QX) - \bar{D}(QX) - \frac{r}{n} [\bar{A}(X) + \bar{B}(X) - \bar{D}(X)] + 2(n-1)[\bar{\alpha}(X) + \bar{\beta}(X) + (n-1)\bar{\gamma}(X)].$$

This gives the following:

**Theorem 3.2.** If the scalar curvature of a conformally flat  $(G.W.\tilde{C}.S)_n$  is constant, then the 1-forms are related by the expression (3.6).

#### 4. Existence of a generalized weakly concircularly symmetric space

**Example 1.** Let  $(\mathbb{R}^4, g)$  be a 4-dimensional Riemannian space endowed with the Riemannian metric  $g$  given by

$$(4.1) \quad ds^2 = g_{ij}dx^i dx^j = e^{x^1+x^2+x^3} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2,$$

where  $(i, j = 1, 2, 3, 4)$ .

The non-zero components of concircular curvature tensor and scalar curvature are

$$\begin{aligned} \tilde{C}_{1212} = \tilde{C}_{1313} = \tilde{C}_{2323} &= -\frac{1}{8} e^{x^1+x^2+x^3}, \\ \tilde{C}_{1213} = -\tilde{C}_{1223} = \tilde{C}_{1323} &= \frac{1}{4} e^{x^1+x^2+x^3}, \\ \tilde{C}_{1414} = \tilde{C}_{2424} = \tilde{C}_{3434} &= \frac{1}{8}, \\ r &= \frac{3}{2} [\cosh(x^1 + x^2 + x^3) - \sinh(x^1 + x^2 + x^3)]. \end{aligned}$$

With the help of (4.1), we can find out

$$\begin{aligned} G_{1212} = G_{2323} = G_{1313} &= -e^{2(x^1+x^2+x^3)}, \\ G_{2424} = G_{1414} = G_{3434} &= -e^{(x^1+x^2+x^3)}. \end{aligned}$$

The non-vanishing component of covariant derivatives of concircular curvature tensors are

$$\begin{aligned} \tilde{C}_{1212,1} = \tilde{C}_{1212,2} = -\tilde{C}_{1212,3} = \tilde{C}_{1313,1} = -\tilde{C}_{1313,2} = \tilde{C}_{1313,3} &= \frac{3}{8} e^{x^1+x^2+x^3}, \\ -\tilde{C}_{2323,1} = \tilde{C}_{2323,2} = \tilde{C}_{2323,3} &= \frac{3}{8} e^{x^1+x^2+x^3}, \end{aligned}$$

$$\begin{aligned} \tilde{C}_{1213,1} &= -\tilde{C}_{1223,2} = \tilde{C}_{1323,3} = -\frac{1}{2}e^{x^1+x^2+x^3}, \\ \tilde{C}_{1213,2} &= \tilde{C}_{1213,3} = -\tilde{C}_{1223,1} = \tilde{C}_{1223,3} = \tilde{C}_{1323,1} = \tilde{C}_{1323,2} = -\frac{1}{8}e^{x^1+x^2+x^3}, \\ \tilde{C}_{1414,1} &= \tilde{C}_{1414,2} = \tilde{C}_{1414,3} = \tilde{C}_{2424,1} = \tilde{C}_{2424,2} = \tilde{C}_{2424,3} = \tilde{C}_{3434,1} = \tilde{C}_{3434,2} = \tilde{C}_{3434,3} \\ &= -\frac{1}{8}. \end{aligned}$$

We consider the 1-forms as follows:

$$\bar{A}(\partial_i) = \bar{A}_i = \begin{cases} 1 & \text{for } i = 1, \\ -2 & \text{for } i = 2, \\ -2 & \text{for } i = 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{B}(\partial_i) = \bar{B}_i = \begin{cases} -\frac{3}{2} & \text{for } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{D}(\partial_i) = \bar{D}_i = \begin{cases} -\frac{3}{2} & \text{for } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\alpha}(\partial_i) = \bar{\alpha}_i = \begin{cases} \frac{1}{4}e^{-(x^1+x^2+x^3)} & \text{for } i = 1, \\ -\frac{1}{8}e^{-(x^1+x^2+x^3)} & \text{for } i = 2, \\ -\frac{1}{8}e^{-(x^1+x^2+x^3)} & \text{for } i = 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\beta}(\partial_i) = \bar{\beta}_i = \begin{cases} -\frac{3}{16}e^{-(x^1+x^2+x^3)} & \text{for } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\gamma}(\partial_i) = \bar{\gamma}_i = \begin{cases} -\frac{3}{16}e^{-(x^1+x^2+x^3)} & \text{for } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

Where  $\partial_i = \frac{\partial}{\partial u^i}$ ,  $u^i$  being the local coordinates of  $R^4$ .

In our  $R^4$ , (4.1) becomes with these 1-forms to the following equations:

$$\begin{aligned} \tilde{C}_{1212,k} &= \bar{A}_k \tilde{C}_{1212} + \bar{B}_1 \tilde{C}_{k212} + \bar{B}_2 \tilde{C}_{1k12} + \bar{D}_1 \tilde{C}_{12k2} + \bar{D}_2 \tilde{C}_{121k} \\ &+ \bar{\alpha}_k G_{1212} + \bar{\beta}_1 G_{k212} + \bar{\beta}_2 G_{1k12} + \bar{\gamma}_1 G_{12k2} + \bar{\gamma}_2 G_{121k}. \end{aligned}$$

$$\begin{aligned} \tilde{C}_{2424,k} &= \bar{A}_k \tilde{C}_{2424} + \bar{B}_2 \tilde{C}_{k424} + \bar{B}_4 \tilde{C}_{2k24} + \bar{D}_2 \tilde{C}_{24k4} + \bar{D}_4 \tilde{C}_{242k} \\ &+ \bar{\alpha}_k G_{2424} + \bar{\beta}_2 G_{k424} + \bar{\beta}_4 G_{2k24} + \bar{\gamma}_2 G_{24k4} + \bar{\gamma}_4 G_{242k}. \end{aligned}$$

$$\begin{aligned} \tilde{C}_{1414,k} &= \bar{A}_k \tilde{C}_{1414} + \bar{B}_1 \tilde{C}_{k414} + \bar{B}_4 \tilde{C}_{1k14} + \bar{D}_1 \tilde{C}_{14k4} + \bar{D}_4 \tilde{C}_{141k} \\ &+ \bar{\alpha}_k G_{1414} + \bar{\beta}_1 G_{k414} + \bar{\beta}_4 G_{1k14} + \bar{\gamma}_1 G_{14k4} + \bar{\gamma}_4 G_{141k}. \end{aligned}$$

$$\tilde{C}_{1213,k} = \bar{A}_k \tilde{C}_{1213} + \bar{B}_1 \tilde{C}_{k213} + \bar{B}_2 \tilde{C}_{1k13} + \bar{D}_1 \tilde{C}_{12k3} + \bar{D}_3 \tilde{C}_{121k}$$

$$\begin{aligned}
 & + \bar{\alpha}_k G_{1213} + \bar{\beta}_1 G_{k213} + \bar{\beta}_2 G_{1k13} + \bar{\gamma}_1 G_{12k3} + \bar{\gamma}_3 G_{121k} \cdot \\
 \tilde{C}_{1313,k} & = \bar{A}_k \tilde{C}_{1313} + \bar{B}_1 \tilde{C}_{k313} + \bar{B}_3 \tilde{C}_{1k13} + \bar{D}_1 \tilde{C}_{13k3} + \bar{D}_3 \tilde{C}_{131k} \\
 & + \bar{\alpha}_k G_{1313} + \bar{\beta}_1 G_{k313} + \bar{\beta}_3 G_{1k13} + \bar{\gamma}_1 G_{13k3} + \bar{\gamma}_3 G_{131k} \cdot \\
 \tilde{C}_{1323,k} & = \bar{A}_k \tilde{C}_{1323} + \bar{B}_1 \tilde{C}_{k323} + \bar{B}_3 \tilde{C}_{1k23} + \bar{D}_2 \tilde{C}_{13k3} + \bar{D}_3 \tilde{C}_{132k} \\
 & + \bar{\alpha}_k G_{1323} + \bar{\beta}_1 G_{k323} + \bar{\beta}_3 G_{1k23} + \bar{\gamma}_2 G_{13k3} + \bar{\gamma}_3 G_{132k} \cdot \\
 \tilde{C}_{3434,k} & = \bar{A}_k \tilde{C}_{3434} + \bar{B}_3 \tilde{C}_{k434} + \bar{B}_4 \tilde{C}_{3k34} + \bar{D}_3 \tilde{C}_{34k4} + \bar{D}_4 \tilde{C}_{343k} \\
 & + \bar{\alpha}_k G_{1323} + \bar{\beta}_3 G_{k434} + \bar{\beta}_4 G_{3k34} + \bar{\gamma}_3 G_{34k4} + \bar{\gamma}_4 G_{343k} \cdot \\
 \tilde{C}_{1223,k} & = \bar{A}_k \tilde{C}_{1223} + \bar{B}_1 \tilde{C}_{k223} + \bar{B}_2 \tilde{C}_{1k23} + \bar{D}_2 \tilde{C}_{12k3} + \bar{D}_3 \tilde{C}_{122k} \\
 & + \bar{\alpha}_k G_{1223} + \bar{\beta}_1 G_{k223} + \bar{\beta}_2 G_{1k23} + \bar{\gamma}_2 G_{12k3} + \bar{\gamma}_3 G_{122k} \cdot \\
 \tilde{C}_{2323,k} & = \bar{A}_k \tilde{C}_{2323} + \bar{B}_2 \tilde{C}_{k323} + \bar{B}_3 \tilde{C}_{2k23} + \bar{D}_2 \tilde{C}_{23k3} + \bar{D}_3 \tilde{C}_{232k} \\
 & + \bar{\alpha}_k G_{2323} + \bar{\beta}_2 G_{k323} + \bar{\beta}_3 G_{2k23} + \bar{\gamma}_2 G_{23k3} + \bar{\gamma}_3 G_{232k} \cdot
 \end{aligned}$$

Where,  $k = 1,2,3$ . As a result of the above we can state the following:

**Theorem 4.1.** Let  $(\mathbb{R}^4, g)$  be a Riemannian manifold equipped with the metric given by (4.1). Then  $(\mathbb{R}^4, g)$  is a  $(G.W.\tilde{C}.S)_4$  with non-vanishing and non-constant scalar curvature which is non-conformally flat.

**Example 2.** Let  $(\mathbb{R}^4, g)$  be a 4-dimensional Riemannian space endowed with the Riemannian metric  $g$  given by

$$(4.1) \quad ds^2 = g_{ij} dx^i dx^j = (x^4)^{\frac{4}{3}} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \quad \text{where } (i, j = 1,2,3,4) \text{ with } x^4 > 0.$$

The non-zero components of concircular curvature tensor and scalar curvature are

$$\begin{aligned}
 \tilde{C}_{1212} = \tilde{C}_{1313} = \tilde{C}_{2323} & = -\frac{5}{9(x^4)^{\frac{2}{3}}}, \\
 \tilde{C}_{1414} = \tilde{C}_{2424} = \tilde{C}_{3434} & = \frac{5}{9(x^4)^{\frac{2}{3}}}, \\
 r & = -\frac{4}{3(x^4)^{\frac{10}{3}}}.
 \end{aligned}$$

With the help of (4.2), we can find out

$$\begin{aligned}
 G_{1212} = G_{2323} = G_{2424} & = -(x^4)^{\frac{8}{3}}, \\
 G_{1313} = G_{1414} = G_{3434} & = -(x^4)^{\frac{8}{3}}.
 \end{aligned}$$

The non-vanishing component of covariant derivatives of concircular curvature tensors are

$$\tilde{C}_{1212,4} = \tilde{C}_{1313,4} = \tilde{C}_{2323,4} = -\tilde{C}_{1414,4} = -\tilde{C}_{2424,4} = -\tilde{C}_{3434,4} = \frac{50}{27(x^4)^{\frac{5}{3}}}$$

$$\tilde{C}_{1214,2} = \tilde{C}_{1314,3} = \tilde{C}_{2324,3} = -\tilde{C}_{1224,1} = -\tilde{C}_{1334,1} = -\tilde{C}_{2334,2} = \frac{20}{27(x^4)^{\frac{5}{3}}}$$

If we consider the 1-forms as follows:

$$\bar{A}(\partial_i) = \bar{A}_i = \begin{cases} -\frac{2}{x^4} & \text{for } i = 4, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{B}(\partial_i) = \bar{B}_i = \begin{cases} -\frac{4}{3x^4} - \frac{9}{5}(x^4)^{\frac{10}{3}} & \text{for } i = 4, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{D}(\partial_i) = \bar{D}_i = \begin{cases} -\frac{4}{3x^4} + \frac{9}{5}(x^4)^{\frac{10}{3}} & \text{for } i = 4, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\alpha}(\partial_i) = \bar{\alpha}_i = \begin{cases} -\frac{20}{27(x^4)^{\frac{13}{3}}} & \text{for } i = 4, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\beta}(\partial_i) = \bar{\beta}_i = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\gamma}(\partial_i) = \bar{\gamma}_i = \begin{cases} -1 & \text{for } i = 4, \\ 0 & \text{otherwise,} \end{cases}$$

At any point  $x \in M$ , then proceeding similarly as in Example 1, it can be easily seen that the space under consideration is a  $(G.W.\tilde{C}.S)_4$  which is conformally flat. Hence we can state the following:

**Theorem 4.2.** Let  $(\mathbb{R}^4, g)$  be a Riemannian manifold equipped with the metric given by (4.2). Then  $(\mathbb{R}^4, g)$  is a  $(G.W.\tilde{C}.S)_4$  with non-vanishing and non-constant scalar curvature which is conformally flat.

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