

# Applications of Gröbner Bases in Synthesis of Multidimensional Control Systems

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## §1 Introduction

- *Stabilization, asymptotic tracking and disturbance rejection or regulation* are basic and important requirements for feedback control system synthesis.
- The purpose of this presentation is to show the possibility on applications of Gröbner bases in synthesis of multidimensional ( $nD$ ) feedback control systems. In particular, the following topics will be explored and discussed.
  - Definitions and conditions for left, right and duoble co-primeness and skew primeness of  $nD$  matrices over the ring of structurally stable  $nD$  rational functions;

- Formulations and solvability conditions for stabilization problem using 1DOF (degree-of-freedom) or 2DOF controller, asymptotic tracking problem and disturbance rejection problem, which lead to the conclusion that these problems can be essentially reduced to the solvability problems of coprime matrix equation and skew prime matrix equation;
- How to solve these two kind of matrix equations by utilizing Gröbner basis approach;
- Open problems.

## §2 Coprimeness and Skew Primeness of $n$ D Matrices

Let  $\mathbf{R}[z]$ : the ring of real polynomials in  $z = (z_1, \dots, z_n)$ ,  
 $\bar{U}^n = \{z \in \mathbf{C}^n \mid |z| \leq 1\}$ , and

$$\mathbf{G} = \{n/d \mid n, d \in \mathbf{R}[z], d(0, \dots, 0) \neq 0\},$$

$$\mathbf{H} = \{n/d \in \mathbf{G} \mid d(z) \neq 0, \forall z \in \bar{U}^n\}$$

$$\mathbf{I} = \{h \in \mathbf{H} \mid h^{-1} \in \mathbf{G}\},$$

$$\mathbf{J} = \{h \in \mathbf{H} \mid h^{-1} \in \mathbf{H}\}.$$

Denote by  $\mathbf{M}(* )$  the set of matrices with entries in  $*$  (e.g.,  $\mathbf{G}$ ,  $\mathbf{H}$ ).

$A \in \mathbf{M}(\mathbf{H})$  is said to be  $\mathbf{G}$ -unimodular (respectively  $\mathbf{H}$ -unimodular) iff it is square and  $\det A \in \mathbf{I}$  ( $\mathbf{J}$ ).

$D^{-1}N$ , with  $D, N \in \mathbf{M}(\mathbf{H})$  and  $D$   $\mathbf{G}$ -unimodular, is called a left matrix fractional description (MFD) (on  $\{\mathbf{G}, \mathbf{H}, \mathbf{I}, \mathbf{J}\}$ ).

Moreover,  $D, N$  are said to be left coprime, and correspondingly  $D^{-1}N$  left coprime MFD, iff there exist  $U, V \in \mathbf{M}(\mathbf{H})$  such that

$$DU + NV = I. \quad (1)$$

The dual definitions on right coprimeness can be given analogously.

$P \in \mathbf{M}(\mathbf{G})$  is said to admit a left (resp. right) coprime factorization if there exist  $D, N \in M(\mathbf{H})$  ( $\tilde{D}, \tilde{N} \in M(\mathbf{H})$ ) such that  $P = D^{-1}N$  ( $P = \tilde{N}\tilde{D}^{-1}$ ) and  $D, N$  ( $\tilde{D}, \tilde{N}$ ) are left (right) coprime.

Moreover,  $P$  is said to admit a doubly coprime factorization if it admits both left and right coprime factorizations, or equivalently, there exist  $D, N, \tilde{D}, \tilde{N}, X_1, Y_1, X_2, Y_2 \in M(\mathbf{H})$  such that  $P = D^{-1}N = \tilde{N}\tilde{D}^{-1}$  and the doubly coprime relation

$$\begin{bmatrix} X_2 & Y_2 \\ -N & D \end{bmatrix} \begin{bmatrix} \tilde{D} & -Y_1 \\ \tilde{N} & X_1 \end{bmatrix} = I \quad (2)$$

or equivalently,

$$\begin{bmatrix} \tilde{D} & -Y_1 \\ \tilde{N} & X_1 \end{bmatrix} \begin{bmatrix} X_2 & Y_2 \\ -N & D \end{bmatrix} = I \quad (3)$$

holds.

Further, two matrices  $D, N \in M(\mathbf{H})$  are said to be (externally) skew prime iff there exist  $U, V \in M(\mathbf{H})$  such that

$$DU + VN = I. \quad (4)$$

For  $P \in M(\mathbf{G})$ , it is always possible to find  $N, D \in M(\mathbf{H})$  such that  $P = D^{-1}N$ , but  $D, N$  are not in general left coprime even when they possess no left common factor. The following theorem gives the necessary and sufficient condition for the existence of coprime factorizations.

**Theorem 1.** [Quadrat 04, Quadrat 06]

1.  $P \in G^{q \times r}$  admits a left coprime factorization iff there exists  $D \in \mathbf{H}^{q \times q}$  such that  $\det D \neq 0$  and

$$[I_q \ - P]\mathbf{H}^{q+r} = D^{-1}\mathbf{H}^q, \quad (5)$$

*i.e.,  $[I_q \ - P]\mathbf{H}^{q+r}$  is a free lattice of  $\mathbf{G}^q$ , or equivalently,  $[I_q \ - P]\mathbf{H}^{q+r}$  is a free  $\mathbf{H}$ -submodule of  $\mathbf{G}^q$  of rank  $q$ . Then,  $P = D^{-1}N$ , where  $N = DP \in \mathbf{H}^{q \times r}$ , is a left coprime factorization of  $P$ .*



2.  $P \in G^{q \times r}$  admits a right coprime factorization iff there exists  $\tilde{D} \in \mathbf{H}^{r \times r}$  such that  $\det \tilde{D} \neq 0$  and

$$\mathbf{H}^{1 \times (q+r)} \begin{bmatrix} P \\ I_r \end{bmatrix} = \mathbf{H}^{1 \times r} \tilde{D}^{-1}, \quad (6)$$

i.e.,  $\mathbf{H}^{1 \times (q+r)} \begin{bmatrix} P \\ I_r \end{bmatrix}$  is a free lattice of  $\mathbf{G}^{1 \times r}$ , or equivalently,  $\mathbf{H}^{1 \times r} \tilde{D}^{-1}$  is a free  $\mathbf{H}$ -submodule of  $\mathbf{G}^{1 \times r}$  of rank  $r$ . Then,  $P = \tilde{N} \tilde{D}^{-1}$ , where  $\tilde{N} = P \tilde{D} \in \mathbf{H}^{q \times r}$ , is a right coprime factorization of  $P$ .

**Theorem 2.** *If  $P \in G^{q \times r}$  admits a left or right coprime factorization, then it admits a doubly coprime factorization.*

**Proof:** Clear from Corollary 3, Theorem 3 and Corollary 5 of [Quadrat 04].

**Theorem 3.**  *$D, N \in M(\mathbf{H})$  where  $\det D \neq 0$  are (externally) skew prime iff there exist  $\bar{N}, \bar{D} \in M(\mathbf{H})$  such that*

$$ND = \bar{D}\bar{N} \tag{7}$$

*with  $D$  and  $\bar{N}$  right coprime and  $N$  and  $\bar{D}$  left coprime.  
(Proof omitted.)*

The following lemma will be used later.

**Lemma 1.** *Suppose  $V, T, F \in \mathbf{M}(\mathbf{H})$ ,  $\det T \neq 0$ , and  $T, F$  are left coprime. Then  $VT^{-1}F \in \mathbf{M}(\mathbf{H})$  iff  $VT^{-1} \in \mathbf{M}(\mathbf{H})$ .*

**Proof.** The sufficiency is obvious and the necessity can be shown as the 1D case [Vidyasagar 85].

### §3 $n$ D Stabilization Problem

Consider the  $n$ D feedback system shown in Fig. 1 where  $P \in \mathbf{M}(\mathbf{G})$  is a linear  $n$ D plant and  $C \in \mathbf{M}(\mathbf{G})$  is a 1DOF controller.

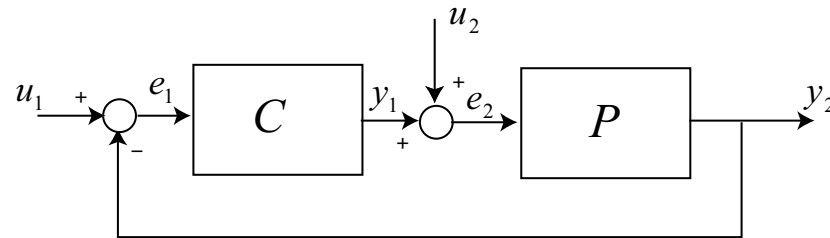


Fig.1 A Feedback System

It is easy to see that

$$e = T_{eu}u, \quad y = T_{yu}u \quad (8)$$

where

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$$\begin{aligned}
T_{eu} &= \begin{bmatrix} (I + PC)^{-1} & -P(I + PC)^{-1} \\ C(I + PC)^{-1} & (I + PC)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} (I + PC)^{-1} & -(I + PC)^{-1}P \\ C(I + PC)^{-1} & I - C(I + PC)^{-1}P \end{bmatrix}, \\
T_{yu} &= \begin{bmatrix} C(I + PC)^{-1} & -C(I + PC)^{-1}P \\ (I + PC)^{-1}PC & (I + PC)^{-1}P \end{bmatrix}.
\end{aligned}$$

Note that

$$T_{eu} = I - FT_{yu}, \quad T_{yu} = F(T_{eu} - I) \quad (9)$$

where  $F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ .

If there exists a controller  $C$  such that the closed-loop system is structurally stable, i.e.,  $T_{yu} \in \mathbf{M}(\mathbf{H})$ , or equivalently,  $T_{eu} \in \mathbf{M}(\mathbf{H})$ , then  $P$  is said to be internally stabilizable and  $C$  is called a stabilizing controller of  $P$ .

### Theorem 4. [Quadrat 04]

A given plant  $P \in \mathbf{G}^{q \times r}$  is internally stabilizable iff either of the following equivalent statements is true.

1. There exists  $S = [U \ T \ V^T]^T \in \mathbf{H}^{(q+r) \times q}$  such that  $\det U \neq 0$  and

$$(a) \quad SP = \begin{bmatrix} UP \\ VP \end{bmatrix} \in \mathbf{H}^{(q+r) \times r},$$

$$(b) \quad [I_q \ -P]S = U - PV = I_q.$$

Then,  $C = VU^{-1}$  internally stabilizes  $P$ , and

$$U = (I - PC)^{-1}, \quad V = C(I_q - PC)^{-1}. \quad (10)$$

2. There exists  $T = [-XY] \in \mathbf{H}^{r \times (q+r)}$  such that  $\det Y \neq 0$  and

$$(a) \quad PT = \begin{bmatrix} -PT & PY \end{bmatrix} \in \mathbf{H}^{q \times (q+r)},$$

$$(b) \quad T \begin{bmatrix} P \\ I_r \end{bmatrix} = -XP + Y = I_r.$$

Then,  $C = Y^{-1}X$  internally stabilizes  $P$ , and

$$Y = (I_r - CP)^{-1}, \quad X = (I_r - CP)^{-1}C. \quad (11)$$

**Corollary 1.** [Quadrat 04]

1. If  $P \in \mathbf{G}^{q \times r}$  admits the left coprime factorization

$$P = D^{-1}N, \quad DX - NY = I_q, \quad \det X \neq 0$$

with  $[X^T \ Y^T]^T \in \mathbf{H}^{(q+r) \times q}$ , then  $C = YX^{-1}$  is a stabilizing controller of  $P$ .

2. If  $P \in \mathbf{G}^{q \times r}$  admits the right coprime factorization

$$P = \tilde{N}\tilde{D}^{-1}, \quad \tilde{X}\tilde{D} - \tilde{Y}\tilde{N} = I_r, \quad \det \tilde{X} \neq 0$$

with  $[\tilde{Y} \ \tilde{X}] \in \mathbf{H}^{r \times (q+r)}$ , then  $C = \tilde{X}^{-1}\tilde{Y}$  is a stabilizing controller of  $P$ .



**Theorem 5.** [Quadrat 04, Quadrat 06]

*Every internally stabilizable  $nD$  system defined by a transfer matrix  $P \in \mathbf{M}(\mathbf{G})$  admits a doubly coprime factorization.*

**Theorem 6.**  *$P \in \mathbf{M}(\mathbf{G})$  is internally stabilizable iff it admits a doubly coprime factorization.*

**Proof.** It is clear from Theorem 5 and Corollary 1.

**Corollary 2.** [Vidyasagar 85, Quadrat 04, Quadrat 06]

If  $P \in \mathbf{M}(\mathbf{G})$  is internally stabilizable, i.e., there exists a doubly coprime factorization  $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ ,

$$\begin{aligned} & \begin{bmatrix} X_2 & Y_2 \\ -N & D \end{bmatrix} \begin{bmatrix} \tilde{D} & -Y_1 \\ \tilde{N} & X_1 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{D} & -Y_1 \\ \tilde{N} & X_1 \end{bmatrix} \begin{bmatrix} X_2 & Y_2 \\ -N & D \end{bmatrix} \\ &= I, \end{aligned} \tag{12}$$

then the class of all stabilizing controllers of  $P$  is given by

$$\begin{aligned} C &= (Y_1 + \tilde{D}Q)(X_1 - \tilde{N}Q)^{-1} \\ &= (X_2 - RN)^{-1}(Y_2 + RD) \end{aligned} \tag{13}$$

where  $Q, R \in \mathbf{M}(\mathbf{H})$  are arbitrary but  $\det(X_1 - \tilde{N}Q) \neq 0$ ,  $\det(X_2 - RN) \neq 0$ .

## §4 General $n$ D Stabilization Problem

Consider the  $n$ D general feedback system shown in Fig.2

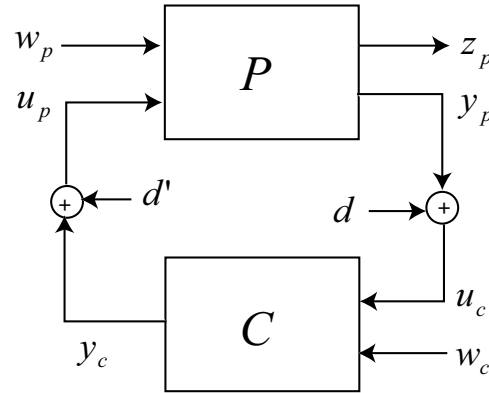


Fig.2 A General Feedback System

where  $P \in \mathbf{M}(\mathbf{G})$  is linear  $n$ D *generalized plant* given by

$$\begin{bmatrix} z_p \\ y_p \end{bmatrix} = P \begin{bmatrix} w_p \\ u_p \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad (14)$$

and  $C \in \mathbf{M}(\mathbf{G})$  is a linear  $n$ D controller described by

$$y_c = [C_1 \quad -C_2] \begin{bmatrix} w_c \\ u_c \end{bmatrix}; \quad (15)$$

$z_p$ : a vector containing the plant variables to be controlled;

$y_p, y_c$ : the utilized output vectors of the plant and controller, respectively;

$w_p, w_c$ : the vectors of all the exogenous inputs (such as disturbances, initial conditions, reference signals, etc.);

$u_p, u_c$ : the utilized inputs to the plant and controller, respectively;

$d, d'$ : the exogenous unmodeled signals (such as noise, interference, etc.).

The problem considered here is to find a controller  $C$  such that the closed-loop feedback system (16) is structurally stable, i.e.,  $\Phi \in \mathbf{M}(\mathbf{H})$ .

$$\begin{bmatrix} z_p \\ y_p \\ y_c \end{bmatrix} = \Phi \begin{bmatrix} w_p \\ d' \\ w_c \\ d \end{bmatrix}, \quad \begin{bmatrix} u_p \\ u_c \end{bmatrix} = \Psi \begin{bmatrix} w_p \\ d' \\ w_c \\ d \end{bmatrix} \quad (16)$$

where

$$\Phi = \begin{bmatrix} \Phi_{11} & P_{12}\tilde{\Delta} & P_{12}\tilde{\Delta}C_1 & -P_{12}C_2\Delta \\ \Delta P_{21} & P_{22}\tilde{\Delta} & P_{22}\tilde{\Delta}C_1 & -P_{22}C_2\Delta \\ -C_2\Delta P_{21} & -C_2P_{22}\tilde{\Delta} & \tilde{\Delta}C_1 & -C_2\Delta \end{bmatrix} \quad (17)$$

$$(18)$$

$$\Psi = \begin{bmatrix} 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \Phi + \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad (19)$$

with  $\Phi_{11} = P_{11} - P_{12}C_2\Delta P_{21}$ ,  $\Delta = (I + P_{22}C_2)^{-1}$  and  $\tilde{\Delta} = (I + C_2P_{22})^{-1}$ .

Let  $P_{22} \in \mathbf{M}(\mathbf{G})$  be internally stabilizable. Then, there exist a doubly coprime factorization  $P_{22} = N_{22}D_{22}^{-1} = \tilde{D}_{22}^{-1}\tilde{N}_{22}$  and the relation

$$\begin{bmatrix} X_1 & Y_1 \\ -\tilde{N}_{22} & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} D_{22} & -Y_2 \\ N_{22} & X_2 \end{bmatrix} = I \quad (20)$$

such that

$$C_2 = (Y_2 + D_{22}Q)(X_2 - N_{22}Q)^{-1} \triangleq N_{c2}D_c^{-1} \quad (21)$$

$$= (X_1 - R\tilde{N}_{22})^{-1}(Y_1 + R\tilde{D}_{22}) \triangleq \tilde{D}_c^{-1}\tilde{N}_{c2} \quad (22)$$

is the class of stabilizing controllers for  $P_{22}$ , where  $D_{22}, N_{22}, \tilde{D}_{22}, \tilde{N}_{22}, X_1, Y_1, X_2, Y_2, \in \mathbf{M}(\mathbf{H})$ , and  $Q, R \in \mathbf{M}(\mathbf{H})$  are arbitrary but  $\det(X_2 - N_{22}Q) \neq 0$ ,  $\det(X_1 - R\tilde{N}_{22}) \neq 0$ .

It should be clear that  $D_c$ ,  $N_{c2}$  and  $\tilde{D}_c$ ,  $\tilde{N}_{c2}$  are also right and left coprime, respectively.

Substituting the results of (20) – (22) into (40) yields

$$\Phi = \begin{bmatrix} \Phi_{11} & P_{12}D_{22}\tilde{D}_cP_{12}D_{22}\tilde{D}_cC_1 - P_{12}D_{22}\tilde{N}_{c2} \\ D_c\tilde{D}_{22}P_{21} & D_c\tilde{N}_{22} & N_{22}\tilde{D}_cC_1 & -N_{22}\tilde{N}_{c2} \\ -N_{c2}\tilde{D}_{22}P_{21} & -N_{c2}\tilde{N}_{22} & D_{22}\tilde{D}_cC_1 & -N_{c2}\tilde{D}_{22} \end{bmatrix} \quad (23)$$

with  $\Phi_{11} = P_{11} - P_{12}N_{c2}\tilde{D}_{22}P_{21}$ .

In view of the coprimeness of the pairs of  $(\tilde{D}_c, \tilde{N}_{c2})$ ,  $(D_c, N_{c2})$ ,  $(D_{22}, N_{22})$ , it is easy to see that  $\Phi \in \mathbf{M}(\mathbf{H})$  iff the conditions of (24)–(27) hold.

$$P_{11} + P_{12}N_{c2}\tilde{D}_{22}P_{21} \in \mathbf{M}(\mathbf{H}) \quad (24)$$

$$P_{12}D_{22}[\tilde{D}_c \ \tilde{N}_{c2}] \in \mathbf{M}(\mathbf{H}) \Leftrightarrow P_{12}D_{22} \in \mathbf{M}(\mathbf{H}) \quad (25)$$

$$\begin{aligned} \begin{bmatrix} D_c \\ N_{c2} \end{bmatrix} [D_c^T \ N_{c2}^T]^T \tilde{D}_{22}P_{21} \in \mathbf{M}(\mathbf{H}) \\ \Leftrightarrow \tilde{D}_{22}P_{21} \in \mathbf{M}(\mathbf{H}) \end{aligned} \quad (26)$$

$$\begin{aligned} \begin{bmatrix} D_{22} \\ N_{22} \end{bmatrix} [D_{22}^T \ N_{22}^Y]^T \tilde{D}_c C_1 \in \mathbf{M}(\mathbf{H}) \\ \Leftrightarrow \tilde{D}_c C_1 \in \mathbf{M}(\mathbf{H}) \end{aligned} \quad (27)$$



(24)–(26) reveal the conditions for the generalized plant  $P$  to admit a stabilizing controller when  $P_{22}$  is internally stabilizable.

On the other hand, (27) gives the admissible condition for the 2DOF (degree-of-freedom) controller  $C = [C_1 \ -C_2]$ . In the sequel, therefore, assume that

$$[C_1 \ C_2] = \tilde{D}_c^{-1}[\tilde{N}_{c1} \ \tilde{N}_{c2}],$$

$\tilde{D}_c, \tilde{N}_{c1}, \tilde{N}_{c2} \in \mathbf{M}(\mathbf{H})$ . For the special 1DOF case, i.e.,

$$C_1 = C_2 = \tilde{D}_c^{-1}\tilde{N}_{2c},$$

the condition (27) is always satisfied.

## §5 $n$ D Regulation and Tracking Problems

The configuration for  $n$ D *generalized* regulation and tracking problems shown in Fig.3 will be used.

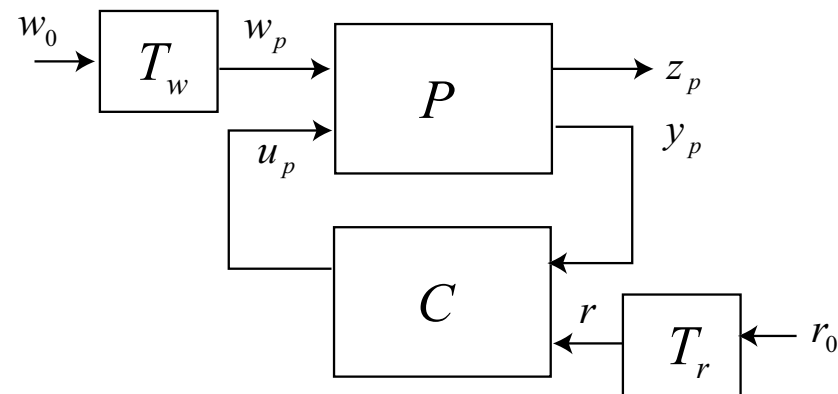


Fig.3 The  $n$ D Regulation and tracking Configuration

where  $T_r, T_w \in \mathbf{M}(\mathbf{G})$  are generators of the reference signal  $r$  and the disturbance signal  $w_p$ ;  $w_0, r_0$  correspond to the initial or boundary conditions of the generators, respectively.  $P$ ,  $C$  and the other variables are defined as in Fig.2.

The problems to be considered can be stated as follows:

given  $P$ ,  $T_w$  and  $T_r$ , to search a controller  $C$  for both the 1DOF ( $C_1 = C_2$ ) and 2DOF ( $C_1 \neq C_2$ ) cases such that  $C$  stabilizes  $P$  and further

**Regulation Problem (RP):**  $T_{z_p w_0} \in \mathbf{M}(\mathbf{H})$ , i.e., the transfer matrix from  $w_0$  to  $z_p$ , is structurally stable.

**Tracking Problem (TP):**  $T_{e r_0} \in \mathbf{M}(\mathbf{H})$ , i.e., the transfer matrix from  $r_0$  to  $e = r - z_p$  is structurally stable.

**Regulation and Tracking Problem (RTP):** the requirements of RP and TP are satisfied simultaneously.

Assume in the following that  $P_{22}$  is internally stabilizable, and  $T_w = \tilde{D}_w^{-1}\tilde{N}_w$ ,  $T_r = \tilde{D}_r^{-1}\tilde{N}_r$ , are both left coprime MFDs.

Define  $\bar{P}_1 = P_{12}D_{22}$ ,  $\bar{P}_2 = \tilde{D}_{22}P_{21}$ ,  $\bar{P}_3 = P_{11} - P_{12}Y_2\tilde{D}_{22}P_{21}$  and note that  $\bar{P}_1, \bar{P}_2, \bar{P}_3 \in \mathbf{M}(\mathbf{H})$  by (24)–(26).

Then we have:

**Theorem 7.** *For either the 1DOF or 2DOF C, RP is solvable iff there exist  $V, Q \in \mathbf{M}(\mathbf{H})$  such that, when  $z_p \neq y_p$*

$$\bar{P}_1 Q \bar{P}_2 + V \tilde{D}_w = \bar{P}_3; \quad (28)$$

when  $z_p = y_p$

$$N_{22} Q \bar{P}_2 + V \tilde{D}_w = X_2 \bar{P}_2. \quad (29)$$

**Theorem 8.** *TP is solvable iff*

(a) *for the 2DOF C, there exist  $W, \tilde{N}_{c1} \in \mathbf{M}(\mathbf{H})$  such that, when  $z_p \neq y_p$ ,*

$$\bar{P}_1 \tilde{N}_{c1} + W \tilde{D}_r = I, \quad (30)$$

*i.e.,  $\bar{P}_1$  and  $\tilde{D}_r$  are skew prime; when  $z_p = y_p$ ,*

$$N_{22} \tilde{N}_{c1} + W \tilde{D}_r = I, \quad (31)$$

*i.e.,  $N_{22}$  and  $\tilde{D}_r$  are skew prime.*

(b) *for the 1DOF C, there exist  $Q, W \in \mathbf{M}(\mathbf{H})$  such that, when  $z_p \neq y_p$ ,*

$$\bar{P}_1 Q \tilde{D}_{22} + W \tilde{D}_r = I - \bar{P}_1 Y_1; \quad (32)$$

*when  $z_p = y_p$ ,*

$$N_{22} Q \tilde{D}_{22} + W \tilde{D}_r = X_2 \tilde{D}_{22}. \quad (33)$$

**Theorem 9.** *RTP is solvable iff*

(a) *for the 2DOF C, RP and TP are independently solvable, i.e., when  $z_p \neq y_p$ , (28) and (30), when  $z_p = y_p$ , (29) and (31), are separately solvable.*

(b) *for the 1DOF C, when  $z_p \neq y_p$ , (28), (32) and (34), when  $z_p = y_p$ , (29), (33) and (35), are simultaneously satisfied, respectively.*

$$V\tilde{D}_w - W\tilde{D}_r P_{21} = P_{11} - P_{21} \quad (34)$$

$$V\tilde{D}_w = W\tilde{D}_r P_{21} \quad (35)$$

**Theorem 10.** *Suppose that  $\bar{P}_2\tilde{D}_w^{-1} = N_{2w}D_{2w}^{-1} = \tilde{D}_{2w}^{-1}\tilde{N}_{2w}$ ,  $\bar{P}_3\tilde{D}_w^{-1} = N_{3w}D_{3w}^{-1}$ , and  $N_{2w}D_{2w}^{-1}$ ,  $\tilde{D}_{2w}^{-1}\tilde{N}_{2w}$ ,  $N_{3w}D_{3w}^{-1}$  are coprime MFDs. Then (28) is solvable iff  $D_{1w} = D_{3w}^{-1}D_{2w} \in \mathbf{M}(\mathbf{H})$  and there exist  $Q, \bar{V} \in \mathbf{M}(\mathbf{H})$  such that*

$$\bar{P}_1Q + \bar{V}\tilde{D}_{2w} = \bar{N} \quad (36)$$

where  $\bar{N} \triangleq N_{3w}D_{1w}Y_{2w} \in \mathbf{M}(\mathbf{H})$ , and  $Y_{2w}$  is determined by the equation  $X_{2w}D_{2w} + Y_{2w}N_{2w} = I$ .

**Theorem 11.** *Let  $\tilde{D}_{2w}, \tilde{N}_{2w}, D_{2w}, N_{2w}$  as defined in Theorem 10. Then, (29) is solvable iff  $\tilde{N}_{22}$  and  $\tilde{D}_{2w}$  are skew prime.*



Further, the following theorem gives a necessary and sufficient condition for the solvability of (36).

**Theorem 12.** *If  $\bar{P}_1$  is square and nonsingular, and  $\bar{P}_1$  and  $\bar{N}$  are left coprime, then (36) is solvable iff  $\tilde{P}_1$  and  $\tilde{N}$  are skew prime where  $\tilde{P}_1$  is given by  $\bar{P}_1^{-1}\bar{N} = \tilde{N}\tilde{P}_1^{-1}$  with  $\tilde{N}, \tilde{P}_1$  right coprime. (Proof omitted.)*

Due to the results of Theorems 8, 10, 11, and 12, we see that the solvability problems of RP and TP have been essentially reduced to the skew primeness of certain matrices over  $\mathbf{H}$ .

## §6 Solution of Coprime Matrix Equation by Gröbner Basis Approach

It has been clarified that

- Synthesis of stabilizing controllers  $\Rightarrow$  find left or/and right coprime MFDs of  $P$ , and solve the corresponding coprime matrix equations;
- The solution of RP and TP  $\Rightarrow$  solve certain skew prime matrix equations.



By Theorem 3, the solution of a skew prime matrix equation can be essentially reduced to the solution problems of left and right coprime matrix equations.

Therefore, in the following, we focus ourselves on the problems:

- how to obtain a coprime MFD for a given  $P$ , say a left coprime MFD  $P = D^{-1}N$ , and
- how to solve the left coprime matrix equation

$$DX + NY = I. \quad (37)$$

As  $\mathbf{R}[z] \subset \mathbf{H}$ , we can only consider, without loss of generality, the left coprime MFD  $P = D^{-1}N$  with  $D, N \in M(\mathbf{R}[z])$ , and (37) can be equivalently transformed to

$$D\bar{X} + N\bar{Y} = V \quad (38)$$

where  $\bar{X}, \bar{Y}$  and  $V \in M(\mathbf{R}[z])$  and  $\det V(z) \neq 0$  on  $\bar{U}^n$ .

Further, applying Cauchy-Binet theorem to (38) yields that

$$\sum_{i=1}^{\beta} a_i(z)x_i(z) = \det V(z) \quad (39)$$

where  $a_i(z)$  are the maximal order minors of the matrix  $F = [D \ N]$  and  $x_i(z)$  are the maximal order minors of  $[\bar{X}^T \ \bar{Y}^T]^T$ .

Let  $\mathcal{I}$  be the ideal generated by  $a_i(z)$  ( $i = 1, \dots, \beta$ ) and  $\mathcal{V}(\mathcal{I})$  the variety of  $\mathcal{I}$ . A necessary condition for  $D$  and  $N$  to be left coprime is that  $\mathcal{V}(\mathcal{I}) \cap \bar{U}^n = \emptyset$ , i.e.,  $a_i(z)$  ( $i = 1, \dots, \beta$ ) possess no common zeros in  $\bar{U}^n$ .

For 2D case, it is well known that

- factor coprimeness  $\Leftrightarrow$  minor coprimeness [Youla 79],
- a factor/minor coprime MFD over  $\mathbf{R}([z_1, z_2])$  for a given  $P \in \mathbf{M}(\mathbf{G})$  (with  $n = 2$ ) can always be obtained [Guiver and Bose 82].

**Theorem 13.** [Guiver and Bose 85, Bisiacco 86]

*$P \in \mathbf{M}(\mathbf{G})$  ( $n = 2$ ) admits a left coprime factorization iff, for any left MFD  $P = D^{-1}N$  with  $D, N \in \mathbf{M}(\mathbf{R}[z_1, z_2])$  being left factor/minor coprime,  $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ .*

Theorem 13 implies that a left coprime MFD for a 2D causal plant  $P$  can be obtained, if it exists, by using any left factor coprime 2D polynomial MFD of  $P$ .

In the following, we show a test for the condition  $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$  by Gröbner basis approach [Xu et al. 94].

Consider  $P(z_1, z_2) = D^{-1}(z_1, w)N(z_1, z_2)$  where  $D(z_1, z_2) \in \mathbf{R}[z_1, z_2]^{m \times m}$  and  $N(z_1, z_2) \in \mathbf{R}[z_1, z_2]^{m \times l}$ , are left factor coprime.

By the results of [Morf et al., 77], then, we always have  $X_1, Y_1, X_2$  and  $Y_2 \in \mathbf{M}(\mathbf{R}[z_1, z_2])$  such that

$$D(z_1, z_2)X_1(z_1, z_2) + N(z_1, z_2)Y_1(z_1, z_2) = V_1(z_1) \quad (40)$$

$$D(z_1, z_2)X_2(z_1, z_2) + N(z_1, z_2)Y_2(z_1, z_2) = V_2(z_2) \quad (41)$$

where  $V_1(z_1), V_2(z_2)$  are diagonal 1D polynomial matrices with non-zero determinants.

Decompose  $V_1(z_1)$  and  $V_2(z_2)$  as

$$V_1(z_1) = V_{1u}(z_1)V_{1s}(z_1) \quad (42a)$$

$$V_2(z_2) = V_{2u}(z_2)V_{2s}(z_2) \quad (42b)$$

such that all the entries of  $V_{1s}(\xi)$  and  $V_{2s}(\xi)$  are 1D stable polynomials, while all the entries of  $V_{1u}(\xi)$  and  $V_{2u}(\xi)$  are 1D completely unstable polynomials, i.e., have only unstable zeros in  $\bar{U}$  [Xu et al., 94].

Define

$$\mathcal{V}(\det V_{1u}(z_1), \det V_{2u}(z_2)) = \{(z_1, z_2) \in \mathbf{C}^2 \mid \det V_{1u}(z_1) = 0, \det V_{2u}(z_2) = 0\}.$$

Let

$$F(z_1, z_2) = \begin{bmatrix} D(z_1, z_2) & N(z_1, z_2) \end{bmatrix} = [\vec{f}_1 \cdots \vec{f}_k]$$

where  $k = m + l$  and  $\vec{f}_i$  ( $i = 1, \dots, k$ ) are  $m \times 1$  2D polynomial vectors.

**Theorem 14.** [Xu et al., 94]

*The following statements are equivalent:*

(i)  $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ , or equivalently, (38) is solvable when  $n = 2$ ;

(ii) For any  $(z_{10}, z_{20}) \in \mathcal{V}(\det V_{1u}(z_1), \det V_{2u}(z_2))$ ,  $F(z_{10}, z_{20})$  is of full rank;

(iii) A non-zero constant is an element in the Gröbner basis of the ideal generated by  $\det V_{1u}(z_1)$ ,  $\det V_{2u}(z_2)$  and  $a_i(z_1, z_2)$ ,  $i = 1, \dots, \beta$ ;



(iv) For  $i = 1, \dots, m$ ,  $\vec{e}_i$  is an element of the Gröbner basis of the module generated by

$$\{\vec{f}_1, \dots, \vec{f}_k, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det V_{1u}(z_1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} *, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det V_{2u}(z_2) \\ 0 \\ \vdots \\ 0 \end{bmatrix} *\} \quad (43)$$

where  $\vec{e}_i$  denotes an  $m \times 1$  vector having 1 at the  $i$ th position and 0 at the other positions, and  $*$  denotes the  $i$ th position of the associated vectors.

Next, we show a procedure for constructing a solution to (38) by using Gröbner basis approach.

**Theorem 15.** [Xu et al., 94] *Suppose that  $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ . Then the polynomial  $s(z_1, z_2)$  defined as*

$$s(z_1, z_2) = \det V_{1s}(z_1) \det V_{2s}(z_2) \quad (44)$$

*vanishes on  $\mathcal{V}(\mathcal{I})$  and is stable, namely, devoid of zeros in  $\bar{U}^2$ .*

## 2D Solution Procedure

**Input:**  $F(z_1, z_2) = [D(z_1, z_2) \ N(z_1, z_2)] = [\vec{f}_1, \dots, \vec{f}_k]$ , a stable polynomial  $s(z_1, z_2) \in \mathbf{R}[z_1, z_2]$  vanishing over  $\mathcal{V}(\mathcal{I})$ .

**Output:**  $\bar{X}(z_1, z_2)$ ,  $\bar{Y}(z_1, z_2)$  and  $V(z_1, z_2)$  for (38).

**step 1.** Solve the following equation using the Gröbner basis approach for  $i = 1, \dots, m$ :

$$\bar{x}_{i,1}(z_1, z_2, t)\vec{f}_1(z_1, z_2) + \dots + \bar{x}_{i,k}(z_1, z_2, t)\vec{f}_k(z_1, z_2) + \tilde{x}_i(z_1, z_2, t) \begin{bmatrix} 0 \\ \vdots \\ 1 - ts(z_1, z_2) \\ \vdots \\ 0 \end{bmatrix} * = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} *$$

where  $t$  is a new indeterminate.

**step 2.** Substituting  $t = 1/s$  into the above equations and clearing out the denominators yields

$$F(z_1, z_2) \vec{f}'_i(z_1, z_2) = [0, \dots, s(z_1, z_2)^{r_i}, \dots, 0]^T$$

or equivalently,

$$F(z_1, z_2) [\vec{f}'_1, \dots, \vec{f}'_m] = \begin{bmatrix} s^{r_1} & & \\ & \dots & \\ & & s^{r_m} \end{bmatrix} \\ \triangleq V(z_1, z_2)$$

where  $\vec{f}'_i(z_1, z_2)$  ( $i = 1, \dots, m$ ) are  $k \times 1$  2D polynomial vectors and  $r_i$  ( $i = 1, \dots, m$ ) are non-negative integers.

**step 3.** Partitioning  $[\vec{f}_1'(z_1, z_2), \dots, \vec{f}_m'(z_1, z_2)]$  as  $[\bar{X}(z_1, z_2)^T \bar{Y}(z_1, z_2)^T]^T$ , we have that

$$D(z_1, z_2)\bar{X}(z_1, z_2) + N(z_1, z_2)\bar{Y}(z_1, z_2) = V(z_1, z_2)$$

where  $\det V(z_1, z_2) = s(z_1, z_2)^{r_1 + \dots + r_m}$  is obviously a stable 2D polynomial.

For a general  $n$ D ( $n \geq 3$ ) case, we encounter the difficulties that factor coprimeness of two  $n$ D polynomial matrices does not imply the minor coprimeness of them, and we do not know how to construct a minor coprime MFD for a given  $P \in \mathbf{M}(\mathbf{G})$ .

It is still an open problem to construct a coprime MFD for a given  $P \in \mathbf{M}(\mathbf{G})$ , though its existence condition has been shown recently by the result of Theorem 5 [Quadrat 04, Quadrat 06].

By introducing the concept of reduced minors of an  $nD$  polynomial matrix, it has been shown that the  $nD$  stabilization problem can be characterized by using polynomial MFDs of a given plant  $P \in \mathbf{M}(\mathbf{G})$  which are not necessarily minor coprime [Lin 98, Lin 01].

Let  $a_1(\mathbf{z}), \dots, a_\beta(\mathbf{z})$  denote the  $m \times m$  minors of the  $nD$  polynomial matrix  $F(\mathbf{z}) = [D(\mathbf{z}) \ N(\mathbf{z})]$ . Extracting a greatest common divisor  $d(\mathbf{z})$  of  $a_i(\mathbf{z})$  ( $i = 1, \dots, \beta$ ) yields

$$a_i(\mathbf{z}) = d(\mathbf{z})b_i(\mathbf{z}), \quad i = 1, \dots, \beta.$$

Then,  $b_i(\mathbf{z}), \dots, b_\beta(\mathbf{z})$  are called the reduced minors of  $F(\mathbf{z})$  [Sule 94, Lin 98, Lin 01].

It has been shown that  $P(\mathbf{z}) = D(\mathbf{z})^{-1}N(\mathbf{z})$  (not necessarily left minor coprime) is stabilizable iff  $b_1(\mathbf{z}), \dots, b_\beta(\mathbf{z})$  have no common zeros in  $\bar{U}^n$ , or equivalently, there exist  $x_1(\mathbf{z}), \dots, x_\beta(\mathbf{z})$  such that

$$\sum_{i=1}^{\beta} b_i(\mathbf{z})x_i(\mathbf{z}) = s(\mathbf{z}) \quad (45)$$

where  $s(\mathbf{z})$  is stable, i.e.,  $s(\mathbf{z}) \neq 0$  for any  $\mathbf{z} \in \bar{U}^n$ .

Further, it has been shown that if the solution  $x_i(\mathbf{z})$  ( $i = 1, \dots, \beta$ ) can be found, then a stabilizing controller  $C(\mathbf{z})$  can be constructively obtained [Lin 98, Lin 01].



## §7 Open Problems

Let  $\mathcal{I}$  be the ideal generated by  $b_1(z), \dots, b_\beta(z)$ , and  $\mathcal{V}(\mathcal{I})$  the variety of  $\mathcal{I}$ .

**Problem 1.** [Lin 01] find an efficient method to determine whether or not  $\mathcal{V}(\mathcal{I}) \cap \bar{U}^n = \emptyset$ .

**Problem 2.** [Lin 01, Xu et al. 04]

Suppose that  $\mathcal{V}(\mathcal{I}) \cap \bar{U}^n = \emptyset$ . Find a constructive method to obtain a stable polynomial  $\tilde{s}(z)$  such that  $\tilde{s}(z)$  vanishes on  $\mathcal{V}(\mathcal{I})$ .

For the case when  $\mathcal{I}$  is of zero dimension, i.e.,  $\mathcal{V}(\mathcal{I})$  consists of only a finite number of points, such a stable polynomial  $\tilde{s}(z)$  can be constructed by utilizing Gröbner bases [Xu et al. 98].

Note also that, once this problem is solved, it will be ready to construct a solution to (45) by Gröbner basis approach [Xu et al. 94, Xu et al. 98].

**Problem 3.** Find a constructive method to obtain a left or/and right coprime MFDs for a given stabilizable  $P \in \mathbf{M}(\mathbf{G})$ .

**Problem 4.** Find a constructive method to construct the matrices  $\bar{D}$  and  $\bar{N}$  defined in Theorem 3 when  $n \geq 3$ . (Solution for the 2D case can be found in [Xu et al. 90].)

## §8 Conclusions

- Necessary and sufficient solvability conditions have been given for  $n$ D stabilization, regulation and tracking problems under various configurations and/or cases, which show that these problems can be essentially reduced to the solvability problems of coprime and skew prime matrix equations.
- It has been shown that for the 2D case, these equations, and thus the synthesis problems of various kinds of 2D controllers can be constructively solved by utilizing Gröbner basis approach. However, for the general  $n$ D ( $n \geq 3$ ) cases, some substantial challenges still remain, which are summarized as several open problems.

Thanks for your attention!