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Almost Automorphic Groups and Semigroups in Fréchet Spaces

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Abstract

In this paper we characterize almost automorphic and asymptotically almost automorphic groups and semigroups in Fréchet spaces.

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1. Introduction

Harald Bohr's interest in which functions could be represented by a Dirichlet series, i.e. of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, where $a_n, z \in \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence of real numbers (series which play an important role in complex analysis and analytic number theory), led him to devise a theory of almost periodic real (and complex) functions, founding this theory between the years 1923 and 1926.

The theory of almost periodic functions was strongly extended to abstract spaces, see for example the monographs [9], [28], [20], [21] (for Banach space valued functions), and the works [7], [17], [18], [20] (for Fréchet space valued functions). Also, in the recent paper [2] (see also Chapter 3 in the book [21]), the theory of real-valued almost periodic functions has been extended to the case of fuzzy-number-valued functions.

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The concept of almost automorphy is a generalization of almost periodicity. It has been introduced in the literature by S. Bochner in relation to some aspects of differential geometry [3–6]. Important contributions to the theory of almost automorphic functions have been obtained, for example, by the papers [14], [16], [19], [22], [25–27], [29] and by the books [28], [20], [21] (concerning almost automorphic functions with values in Banach spaces), and by the paper [24] (concerning almost automorphy on groups). Also, the theory of almost automorphic functions with values in fuzzy-number-type spaces was developed in the papers [12], [10] (see also Chapter 4 in [21]).

Recently we developed the theory of almost automorphic functions with values in a locally convex space (Fréchet space) in our paper [11].

In the recent paper [8], the theory of almost automorphic and asymptotically almost automorphic semigroups of linear operators on Banach spaces is studied.

The main aim of this paper is to extend this theory on complete metrizable locally convex (Fréchet) spaces.

2. Almost Automorphic Functions in Fréchet spaces

In this section we recall some elements of the theory of almost automorphic functions with values in Fréchet spaces we need in Section 4. First we recall the following well-known concept.

Definition 2.1: A linear space $(X, +, \cdot)$ over $\mathbb{R}$ is called Fréchet space if X is a metrizable, complete, locally convex space.

Remark: It is a classical fact that the Fréchet spaces are characterized by the existence of a countable, sufficient and increasing family of semi-norms $(p_i)_{i\in\mathbb{N}}$ (that is $p_i(x) = 0, \forall i \in \mathbb{N}$ implies $x = 0_X$ and $p_i(x) \leq p_{i+1}(x), \forall x \in X, i \in \mathbb{N}$), which define the pseudo-norm

$$|x|_X = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{p_i(x)}{1 + p_i(x)}, x \in X,$$

and the metric $d(x, y) = |x - y|_X$ invariant with respect to translations, such that d generates a complete (by sequences) topology equivalent to that of locally convex space. That is, d has the properties: $d(x, y) = 0$ iff $x = y$, $d(x, y) = d(y, x)$, $d(x, y) \leq d(x, z) + d(z, y)$, $d(x + u, y + u) = d(x, y)$ for all $x, y, z \in X$. Also, notice that since $\frac{p_i(x)}{1+p_i(x)} \leq 1$ and $\sum_{i=0}^{\infty} \frac{1}{2^i} = 1$, it follows that $|x|_X \leq 1, \forall x \in X$.

Moreover, d has the properties given by the following.

Theorem 2.2 (see [11]):

(i) $d(cx, cy) \leq d(x, y)$ for $|c| \leq 1$;

(ii) $d(x + u, y + v) \leq d(x, y) + d(u, v)$;

(iii) $d(kx, ky) \leq d(rx, ry)$ if $k, r \in \mathbb{R}, 0 < k \leq r$;

(iv) $d(kx, ky) \leq kd(x, y), \forall k \in \mathbb{N}, k \geq 2$;

(v) $d(cx, cy) \leq (|c| + 1)d(x, y), \forall c \in \mathbb{R}$.

Everywhere in the rest of the paper, $(X, (p_i)_{i\in\mathbb{N}}, d)$ will be a Fréchet space with $(p_i)_{i\in\mathbb{N}}$ and d as in the Remark following Definition 2.1.

We start with the following Bochner-kind definition.

Definition 2.3 (see [11]): We say that a continuous function $f : \mathbb{R} \to X$, is almost automorphic, if every sequence of real numbers $(r_n)_n$, contains a subsequence $(s_n)_n$, such that for each $t \in \mathbb{R}$, there exists $g(t) \in X$ with the property

$$\lim_{n\to+\infty} d(g(t), f(t + s_n)) = \lim_{n\to+\infty} d(g(t - s_n), f(t)) = 0.$$

(The above convergence on $\mathbb{R}$ is pointwise).

Remark: Almost automorphy in Definition 2.3, is a more general concept than almost periodicity in Fréchet spaces, as it was defined in [20, p. 51]. Indeed, by the Bochner's criterion (see e.g. [20, p. 55, Theorem 3.1.8]), a function with values in a Fréchet space is almost periodic if and only if for every sequence of real numbers $(r_n)_n$, there exists a subsequence $(s_n)_n$, such that the sequence $(f(t+s_n))_n$ converges uniformly with respect to $t \in \mathbb{R}$, in the metric d. Obviously this is a stronger condition than the pointwise convergence in Definition 2.3.

Also, note that the limits with respect to d in Definition 2.3, are equivalent to the corresponding limits with respect to each seminorm p_j, that is to

$$\lim_{n\to+\infty} p_j(g(t) - f(t + s_n)) = \lim_{n\to+\infty} p_j(g(t - s_n) - f(t)) = 0, \forall j \in \mathbb{N}.$$

The following elementary properties hold.

Theorem 2.4 (see [11]): Let $(X, (p_i)_{i\in\mathbb{N}}, d)$ be a Fréchet space. If $f, f_1, f_2 : \mathbb{R} \to X$ are almost automorphic functions then we have:

(i) $f_1 + f_2$ is almost automorphic;

(ii) cf is almost automorphic for every scalar $c \in \mathbb{R}$;

(iii) $f_a(t) = f(t + a), \forall t \in \mathbb{R}$ is almost automorphic for each fixed $a \in \mathbb{R}$;

(iv) For all $i \in \mathbb{N}$, we have $\sup\{p_i[f(t)]; t \in \mathbb{R}\} < +\infty$ and $\sup\{p_i[g(t)]; t \in \mathbb{R}\} < +\infty$, where g is the function attached to f in Definition 2.3;

(v) The range $R_f = \{f(t); t \in \mathbb{R}\}$ is relatively compact in the complete metric space (X, d);

(vi) The function h defined by $h(t) = f(-t)$, $t \in \mathbb{R}$ is almost automorphic;

(vii) If $f(t) = 0_X$ for all $t > a$ for some real number a, then $f(t) = 0_X$ for all $t \in \mathbb{R}$;

(viii) If $A : X \to Y$ is continuous, where Y is another Fréchet space, then $A(f) : \mathbb{R} \to Y$ also is almost automorphic.

(ix) Let $h_n : \mathbb{R} \to X$, $n \in \mathbb{N}$ be a sequence of almost automorphic functions such that $h_n(t) \to h(t)$ when $n \to +\infty$, uniformly in $t \in \mathbb{R}$ with respect to the metric d. Then h is almost automorphic.

Similar to the case of Banach spaces (see e.g. [20, p. 37]), the concept in Definition 2.3 can be generalized as follows.

Definition 2.5 (see [11]): Let $(X, (p_i)_{i\in\mathbb{N}}, d)$ be a Fréchet space. A continuous function $f : \mathbb{R}_+ \to X$ is said to be asymptotically almost automorphic if it admits the decomposition $f(t) = g(t) + h(t)$, $t \in \mathbb{R}_+$, where $g : \mathbb{R} \to X$ is almost automorphic and $h : \mathbb{R}_+ \to X$ is a continuous function with $\lim_{t\to+\infty} |h(t)|_X = 0$. Here g and h are called the principal and the corrective terms of f, respectively.

Remark: Every almost automorphic function restricted to $\mathbb{R}_+$ is asymptotically almost automorphic, by taking $h(t) = 0_X$, $\forall t \in \mathbb{R}_+$.

Regarding this new concept, the following results similar to those in the case of Banach spaces hold.

Theorem 2.6 (see [11]): Let f, f_1, f_2 be asymptotically almost automorphic. Then we have:

(i) $f_1 + f_2$ and $c \cdot f$, $c \in \mathbb{R}$ are asymptotically almost automorphic;

(ii) For fixed $a \in \mathbb{R}_+$, the function $f_a(t) = f(t + a)$ is asymptotically almost automorphic;

(iii) For any $i \in \mathbb{N}$ we have $\sup\{p_i(f(t)); t \in \mathbb{R}_+\} < +\infty$.

(iv) Let $(X, (p_i)_{i\in\mathbb{N}}, d)$, $(Y, (q_j)_{j\in\mathbb{N}}, \rho)$ be two Fréchet spaces and $f : \mathbb{R}_+ \to X$ be an almost automorphic function, $f = g + h$. Let $\phi : X \to Y$ be continuous and assume there is a compact set B in (X, d) which contains the closures of $\{f(t); t \in \mathbb{R}_+\}$ and $\{g(t); t \in \mathbb{R}_+\}$. Then $\phi \circ f : \mathbb{R}_+ \to Y$ is asymptotically almost periodic;

(v) The decomposition of an asymptotically almost automorphic function is unique.

Theorem 2.7 (see [11]): If $(X, (p_i)_{i\in\mathbb{N}}, d)$ is a Fréchet space, then the space of almost automorphic X-valued functions $AA(X)$, is a Fréchet space with respect to the countable family of seminorms given by $q_i(f) = \sup\{p_i(f(t)); t \in \mathbb{R}\}$, $i \in \mathbb{N}$, which generates the metric D on $AA(X)$ defined by

$$D(f, g) = \sum_{i=0}^{+\infty} \frac{1}{2^i} \frac{q_i(f-g)}{1+q_i(f-g)}.$$

3. Semigroups of operators on Fréchet spaces

In this section we recall some elements of the semigroup theory on Fréchet spaces we need in the next section.

First let us recall the characterization of the continuity for linear operators between locally convex (Fréchet) spaces.

Theorem 3.1 (see e.g. [13, p. 128]): Let $(X, (p_i)_{i\in J_1})$, $(Y, (q_j)_{j\in J_2})$ be two locally convex spaces, where $(p_i)_i$ and $(q_j)_j$ are the corresponding families of semi-norms. A linear operator $A : X \to Y$ is continuous on X if and only if for any $j \in J_2$, there exists $i \in J_1$ and a constant $M_j > 0$, such that

$$q_j(A(x)) \le M_j p_i(x), \forall x \in x.$$

The space of all linear and continuous operators from X to Y is denoted by $B(X, Y)$. If $X = Y$, then $B(X, Y)$ will be denoted by $B(X)$.

Remark: For $A \in B(X)$, let us denote $||A||_{i,j} = \sup\{p_j(A(x)); x \in X, p_i(x) \le 1\}$. Then it is well-known that $A \in B(X)$ if and only if for every j there exists i (depending on j) such that $||A||_{i,j} < +\infty$.

Definition 3.2 (see e.g. [15],[23]): Let $(X, (p_j)_{j\in J})$ be a locally convex space. A family $T = (T(t))_{t\ge 0}$ with $T(t) \in B(X)$, $\forall t \ge 0$ is called C_0-semigroup on X if:

(i) $T(0) = I$ (the identity operator on X);

(ii) $T(t+s) = T(t)T(s)$, $\forall t, s \ge 0$ (here the product means composition);

(iii) For all $j \in J$, $x \in X$ and $t_0 \in \mathbb{R}_+$ we have $\lim_{t\to t_0} p_j[T(t)(x) - T(t_0)(x)] = 0$.

(iv) The operator A is called the (infinitesimal) generator of the C_0-semigroup T on X, if for every $j \in J$ we have

$$\lim_{t\to 0^+} p_j\left[A(x) - \frac{T(t)(x) - x}{t}\right] = 0,$$

for all $x \in X$.

Remark: In a similar manner we can define a C_0-group on X by replacing $\mathbb{R}_+$ with $\mathbb{R}$.

Definition 3.3 (see e.g. [20, p. 99, Definition 7.1.1]): Let $(X, (p_j)_{j\in J})$ be a complete, Hausdorff locally convex space. A family $F = (A_i)_{i\in\Gamma}$, $A_i \in B(X), \forall i$, is called equicontinuous, if for any $j_1 \in J$ there exists $j_2 \in J$ such that

$$p_{j_1}[A_i(x)] \leq p_{j_2}(x), \forall x \in X, i \in \Gamma.$$

According to e.g. [20, p. 100–103, Theorems 7.1.2, 7.1.3, 7.1.5, 7.1.6], we can state the following.

Theorem 3.4: Let $(X, (p_j)_{j\in J})$ be a complete, Hausdorff locally convex space and $A \in B(X)$ such that the countable family $\{A^k; k = 1, 2, \ldots, \}$ is equicontinuous. For $x \in X$ and $t \geq 0$, let us define $S_m(t, x) = \sum_{k=0}^{m} \frac{t^k}{k!} A^k(x)$. It follows:

(i) For each $x \in X$ and $t \geq 0$, the sequence $S_m(t, x), m = 1, 2, \ldots,$ is convergent in X, that is there exists an element in X dented by $e^{tA}(x)$, such that

$$\lim_{m\to+\infty} p_j(e^{tA}(x) - S_m(t, x)) = 0, \forall j \in J$$

and we write $e^{tA}(x) = \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k(x)$;

(ii) For any fixed $t \geq 0$, we have $e^{tA} \in B(X)$;

(iii) $e^{(t+s)A} = e^{tA}e^{sA}, \forall t, s \geq 0$;

(iv) For every $j \in J$ we have

$$\lim_{t\to 0^+} p_j\left[A(x) - \frac{e^{tA}(x) - x}{t}\right] = 0,$$

for all $x \in X$;

(iv) $\frac{d}{dt}[e^{tA}(x)] = A[e^{tA}(x)]$ and the function $e^{tA}(x_0) : \mathbb{R} \to X$ is the unique solution of the Cauchy problem $x'(t) = A[x(t)], t \in \mathbb{R}, x(0) = x_0$.

Remark: Theorem 3.4, (i)–(iii), show that $T(t) = e^{tA}, t \geq 0$ is a C_0-semigroup of operators as in Definition 3.2.

4. Almost Automorphic Groups and Semigroups on Fréchet Spaces

Everywhere in this section, $(X, (p_i)_{i\in\mathbb{N}}, d)$ will be a Fréchet space. First we recall the following.

Definition 4.1 (see [20], p. 51, Definition 3.1.1 or [21]): A function $f \in C(\mathbb{R}, X)$ is almost periodic if for each neighborhood of the origin U, there exists a real number l such that any interval of length l contains at least a point s such that

$$f(t+s) - f(t) \in U \ \textit{for every}\ t \in \mathbb{R}.$$

The number s depends on U and is called a U-translation or U-almost period of f. It is noted that the set of all U-almost periods of f is relatively dense. Recall that a set $P \subset \mathbb{R}$ is said to be relatively dense (r.d. shortly) in $\mathbb{R}$, if there exists $l > 0$ such that $[a, a+l] \bigcap P \neq \phi$ for every $a \in \mathbb{R}$.

It is well-known (see for instance [20] , p. 53, Remark 3.1.6) that the range of an almost periodic function with values in a Fréchet space X is relatively compact (r.c. shortly) in the complete metric space X.

Remark 4.2: In terms of the family of seminorms $(p_i)_{i\in\mathbb{N}}$, Definition 4.1 can easily rewritten as follows: for any $\varepsilon > 0$, there exists an r.d. set in $\mathbb{R}$, $(\tau)_\varepsilon$, such that

$$\sup_{t\in\mathbb{R}} p_i(f(t+\tau) - f(t)) \leq \varepsilon, \forall \tau \in (\tau)_\varepsilon, \forall i \in \mathbb{N}.$$

Here $\tau \in (\tau)_\varepsilon$ is called ε-almost period of f.

Similar to the case of Banach spaces (see e.g. [1, pp. 7–11]), we can develop a theory of Bochner's transform for Fréchet spaces, as follows.

Denote by $C_b(\mathbb{R}, X)$ the Fréchet space of all continuous bounded functions from $\mathbb{R}$ to X, endowed with the countably family of increasing seminorms $q_i(f) = sup_{t\in\mathbb{R}}\{p_i(t)\}$, for all $i \in \mathbb{N}$. Note that $f \in C_b(\mathbb{R}, X)$ is bounded if $f(\mathbb{R})$ is a bounded set in the metric space (X, d), which is equivalent to $sup\{p_i(f(t)); t \in \mathbb{R}\} < +\infty$, for all $i \in \mathbb{N}$.

The Bochner transform on $C_b(\mathbb{R}, X)$ is defined as in the case of Banach spaces, by $\tilde{f} : \mathbb{R} \to C_b(\mathbb{R}, X)$, $\tilde{f}(s)(t) = f(t+s)$, for all $t \in \mathbb{R}$ and we write $\tilde{f} = B(f)$. The properties of Bochner's transform on Fréchet spaces, can be summarized by the following.

Theorem 4.3:

(i) $q_i(\tilde{f}(s)) = q_i(f) = q_i(\tilde{f}(0))$, for all $s \in \mathbb{R}$ and $i \in \mathbb{N}$;

(ii)
$$q_i(\tilde{f}(s+\tau) - \tilde{f}(s)) = sup\{p_i(f(t+\tau) - f(t)); t \in \mathbb{R}\} = q_i(\tilde{f}(\tau) - \tilde{f}(0)),$$
for all $i \in \mathbb{N}$, $s, \tau \in \mathbb{R}$;

(iii) f is almost periodic if and only if, for any $s \in \mathbb{R}$, $\tilde{f}(s)$ is almost periodic, with the same set of ε-almost periods $(\tau)_\varepsilon$;

(iv) $\tilde{f}(s)$ is almost periodic if and only if there exists an r.d. sequence in $\mathbb{R}$, denoted by $\{s_n; n \in \mathbb{N}\}$, such that the set $\{\tilde{f}(s_n); n \in \mathbb{N}\}$ is r.c. in the complete metric space (X, d);

(v) $\tilde{f}(s)$ is almost periodic if and only if its range $R_{\tilde{f}(s)}$ is r.c. in the complete metric space (X, d);

(vi) (Bochner's criterion) f is almost periodic, if and only if, $R_{\tilde{f}(s)}$ is r.c. in the complete metric space (X, d), for any $s \in \mathbb{R}$.

Note that all the above remain valid if we consider functions $f \in C(\mathbb{R}_+, X)$.

Proof:

(i), (ii) are immediate.

(iii) Immediate from (ii).

(iv) If $\tilde{f}(s)$ is almost periodic, it follows (see e.g. Remark 3.1.6, p. 53 in [20]) that for every sequence $(s_n)_n$, the set $\{\tilde{f}(s_n); n \in \mathbb{N}\}$ is r.c. in the complete metric space (X, d).

Conversely, let us suppose that there exists an r.d. sequence $(s_n)_n$ such that the set $\{\tilde{f}(s_n); n \in \mathbb{N}\}$ is r.c. in the complete metric space (X, d). This is equivalent with the fact that $\{\tilde{f}(s_n); n \in \mathbb{N}\}$ is totally bounded in (X, d). In what follows, the proof is identical to that in [1], pp. 8–9, by replacing there the norm $||\tilde{f}(s)||$, by $q_i(\tilde{f}(s))$, $i \in \mathbb{N}$. Also, for relatively compactness in (X, d), one replace the spheres with respect to the norm $||\cdot||$ with the spheres with respect to the metric d (invariant to translations) and we take into account that the convergence of a sequence in (X, d) is equivalent with the convergence with respect to all p_i, $i \in \mathbb{N}$. Similarly, for the proof of continuity of $\tilde{f}(s)$, we take into account the above relationships (i) and (ii).

(v) The necessity follows from Theorem 3.1.5, p. 52 in [20]. The sufficiency is a direct consequence of (iv).

(vi) It is immediate from (iii) and (v). ■

The above (vi) Bochner's criterion in fact can be restated as follows.

Theorem 4.4 (see [20], p. 55, Theorem 3.1.8): A function $f \in C(\mathbb{R}, X)$ is almost periodic if and only if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(t + s_n))$ is uniformly convergent in $t \in \mathbb{R}$.

Now, we are in position to prove the following sufficient conditions for almost periodicity in Fréchet spaces.

Theorem 4.5: Let $f \in C_b(\mathbb{R}, X)$. If there exists a r.d. set of real numbers (s_n) such that:

(i) $\{f(s_n); n \in \mathbb{N}\}$ is r.c. in the complete metric space (X, d) and

(ii) for every $i \in \mathbb{N}$, there exist $j \in \mathbb{N}$ and a constant $C_{i,j} > 0$ such that $C_{i,j} \sup\{p_i[f(t+s_n)-f(t+s_m)]; t \in \mathbb{R}\} \leq p_j[f(s_n)-f(s_m)]$, for every $n, m \in \mathbb{N}$, then f is almost periodic.

Proof: The inequality in statement together with Theorem 4.3, (ii), obviously implies $p_j(f(s_n) - f(s_m)) \geq C_{i,j} q_i(\tilde{f}(s_n) - \tilde{f}(s_m))$.

Since by hypothesis $\{f(s_n); n \in \mathbb{N}\}$ is r.c., it has a convergent subsequence $(f(s'_n))_n$, which is Cauchy sequence in the complete metric space (X, d), i.e. it is convergent. Combined with Theorem 4.3, (iv), it follows that $\tilde{f}(s)$ is almost periodic, which by Theorem 4.3, (ii), implies that f is almost periodic. ■

Remark 4.6: It is shown in [7, Theorem 3.1], that the space $AP(X)$ of all almost periodic functions $\mathbb{R} \to X$ is a Fréchet space. It is clear that $AP(X) \subset AA(X)$.

Although in general, the concepts of almost periodicity and almost automorphy are not equivalent, Theorem 4.5 us allows to prove the equivalence between the almost periodicity and almost automorphy of the "orbits" of a group/semigroup. Thus we present.

Theorem 4.7: Let $(T(t))_{t\in\mathbb{R}}$ be a family of uniformly bounded group of bounded linear operators on a Fréchet space $(X, (p_i)_{i\in\mathbb{N}}, d)$ and $x_0 \in X$ be given.

Then the following are equivalent:

(i) $t \to T(t)x_0$ is almost automorphic;

(ii) $t \to T(t)x_0$ is almost periodic.

Proof: It suffices to prove that (i) implies (ii). Since $T(t)_{t\in\mathbb{R}}$ is uniformly bounded, for every i, there exists j and $M_i > 0$ such that $p_i(T(t)x_0) \leq M_{i,j} p_j(x_0)$, for all $t \in \mathbb{R}$. Also, $R_{T(t)x_0}$ is relatively compact since $T(t)x_0$ is almost automorphic as function of t (see Theorem 2.4, (v)). Thus given an r.d. sequence of real numbers (s'_n), we can find a subsequence (s_n) such that $(T(s_n)x_0)_{n\in\mathbb{N}}$ is Cauchy. Now in view of the following inequality

$$C_{i,j} p_i[T(t+s_n)x_0 - T(t+s_m)x_0] \leq p_j[T(s_n)x_0 - T(s_m)x_0],$$

for all $t \in \mathbb{R}$, (where $C_{i,j} = \frac{1}{M_{i,j}}$) we conclude that $T(t)x_0$ is almost periodic by Theorem 4.5. ■

Definition 4.8: A motion $x \in C(\mathbb{R}, X)$ is said to be strongly stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $p_i(x(t_1) - x(t_2)) < \delta$ for every i implies $p_i(x(t+t_1) - x(t+t_2)) < \epsilon$ for every i and $t \in \mathbb{R}$.

Example 4.9: If $(T(t))_{t\in\mathbb{R}}$ be a family of uniformly bounded group of continuous linear operators, then the function $x(t) := T(t)e$ for some $e \in X$ is a strongly stable motion in X.

Theorem 4.10: If $x \in C(\mathbb{R}, X)$ is a strongly stable motion with a relatively compact range in X, then $x \in AP(X)$.

Proof: It is a direct consequence of Theorem 4.4. ■

Definition 4.11 (see [20]): A function $f \in C([0, \infty), X)$ is said to be asymptotically almost automorphic if it admits the (unique) decomposition $f = g + h$ where $g \in AA(X)$ and $h \in C([0, \infty), X)$ with $\lim_{t\to\infty} h(t) = 0$. If $g \in AP(X)$, then f is called asymptotically almost periodic g and h are called principal term and corrective term respectively of f.

It is clear that if f is asymptotically almost periodic, then it is asymptotically almost automorphic. Although the converse is not true in general, we will prove that in case of uniformly bounded semigroups, the response is affirmative.

Theorem 4.12: Let $(T(t))_{t\in\mathbb{R}^+}$ be a family of uniformly bounded semigroup of continuous linear operators. If $t \to T(t)x_0$ is asymptotically almost automorphic then it is asymptotically almost periodic.

Proof: Let (s''_n) be a given sequence in $\mathbb{R}^+$. Then we can extract a subsequence (s'_n) such that $(g(s'_n))$ is convergent, where g is the principal term of $T(t)x_0$. Since $h \in C([0, \infty), X)$, we can extract a subsequence (s_n) such that $h(s_n)$ is convergent (the situation when $s_n \to +\infty$ is covered by the property of h as the corrective term of $T(t)x_0$, i.e $h(s_n) \to 0$). This implies that $(T(s_n)x_0)$ is convergent, too. We now end the proof by using Theorem 4.5, exactly as in the proof of Theorem 4.7. ■

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