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Abelian - Tauberian Theorem for Laplace Transform of Hyperfunctions

Deepthi A.N.*[1], Mangalambal.N.R**

**Centre for Research in Mathematical Science,*
St.Joseph's College,Irinjalakuda,Thrissur ,Kerala

***Department of Mathematics, St.Joseph's College,*
Irinjalakuda,Thrissur,Kerala, India

Abstract

Urs Graf applied Laplace transform to Sato's hyperfunctions. In this paper we have proved an Abelian- Tauberian type theorem for Laplace transform of Hyperfunctions.

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INTRODUCTION

Mikio Sato[6] has introduced the idea of hyperfunctions to mention his generalization of the concept of ordinary and generalized functions.Urs Graf use Sato's idea, which uses the classical complex function theory to generalize the notion of function of a real variable and has applied various transforms like Laplace transform, Fourier transform, Hilbert transform, Mellin transforms, Hankel transform to a class of hyperfunctions in his book 'Introduction to Hyperfunctions and their Integral transforms'. [1]
Tauberian theory was first developed by Norbert Wiener[7] in 1932. Various types

[1]*Corresponding author*
Department of Mathematics, Sree Narayana College, Nattika, Kerala-680 566, India

of abelian tauberian theorems are proved by many authors for integral transforms. Using Wieners's Tauberian theorem, Shikao Ikehara proved a Tauberian theorem for Dirichlet series, which is known as Wiener Ikehara Theorem. In 1980, using contour integration, Newmann invented new method to prove Tauberian theorems. Korevaar further developed Newmann's method.
In this paper we have proved Abelian Tauberian theorem for the integral of Laplace transform for hyperfunction of bounded exponential growth using the Abelian Tauberian theorem for Laplace transform of meaure functions[5].

1. PRELIMINARIES

We denote the upper and lower half-plane of the complex plane $\mathbb{C}$ by $\mathbb{C}_+ = \{z \in \mathbb{C} : Iz>0\}, \mathbb{C}_- = \{z : Iz<0\}$ respectively.

Definition 1.1[1]: For an open interval I of the real line, the open subset $N(I) \subset \mathbb{C}$ is called a *complex neighborhood* of I, if I is a closed subset of $N(I)$.
We let $N_+(I) = N(I) \cap \mathbb{C}_+$ and $N_-(I) = N(I) \cap \mathbb{C}_-$. $\mathfrak{O}(N(I) \setminus I)$ denotes the ring of holomophic functions in $N(I) \setminus I$. For a given interval I a function $F(z) \in \mathfrak{O}(N(I) \setminus I)$ can be written as

$$F(z) = \begin{cases} F_+(z) & \text{for } z \in N_+(I), \\ F_-(z) & \text{for } z \in N_-(I) \end{cases}$$

where $F_+(z) \in \mathfrak{O}(N_+(I))$ and $F_-(z) \in \mathfrak{O}(N_-(I))$ are called upper and lower componet of $F(z)$ respectively. In general the upper and lower component of $F(z)$ need not be related to each other. If they are analytic continuations from each other we call $F(z)$ a global analytic function on $N(I)$ and we can write $F_+(z) = F_-(z) = F(z)$.

Defintion 1.2[1]: Two functions $F(z)$ and $G(z)$ in $\mathfrak{O}(N(I) \setminus I)$ are equivalent if for $z \in N_1(I) \cap N_2(I)$, $G(z) = F(z) + \phi(z)$, with $\phi(z) \in \mathfrak{O}(N(I))$ where $N_1(I)$ and $N_2(I)$ are complex neighborhoods of I of $F(z)$ and $G(z)$ respectively.

Definition 1.3[1]: An equivalence class of functions $F(z) \in \mathfrak{O}(N(I) \setminus I)$ defines a hyperfunction $f(x)$ on I. Which is denoted by $f(x) = [F(z)] = [F_+(z), F_-(z)]$. $F(z)$ is called defining or generating function of the hyperfunction. The set of all hyperfunctions defined on the interval I is denoted by $\mathfrak{B}(I)$.

$$\mathfrak{B}(I) = \mathfrak{O}(N(I) \setminus I) \setminus \mathfrak{O}(N(I))$$

A real analytic function $\phi(x)$ on I is defined by the fact that $\phi(x)$ can analytically be continued to a full neighborhood U containing I i.e. we then have $\phi(z) \in \mathfrak{O}(U)$. For any complex neighborhood $N(I)$ containing U we may then write $\mathfrak{B}(I) = \mathfrak{O}(N(I) \setminus I) \setminus \mathcal{A}(I)$, where $\mathcal{A}(I)$ is the ring of all real analytic functions on I. Thus a hyperfunction $f(x) \in \mathfrak{B}(I)$ is determined by a defining function $F(z)$ which is holomorphic in an adjacent neighborhood above and below the interval I, but is only determined upto a real analytic function on I.

The value of a hyperfunction at a regular point x is

$$f(x) = F(x + i0) - F(x - i0) = \lim_{\epsilon \to 0^+} \{F_+(x + i\epsilon) - F_-(x - i\epsilon)\}$$

provided the limit exists.

Example[1]: Dirac delta function at $x = 0$ is represented in terms of hyperfunction as $\delta(x) = [\frac{-1}{2\pi iz}]$. Here the defining function is $F(z) = \frac{-1}{2\pi iz}.F(z)$ is defined except at $z = 0$. At $z = 0$, $F(z)$ has an isolated singularity, which is a pole of order 1. For every real number $x \neq 0$ the limit $\lim_{\epsilon \to 0^+} \{F_+(x + i\epsilon) - F_-(x - i\epsilon)\}$ exists and equal to 0.

Definition 1.4[1]: A hyperfunction $f(x)$ is called holomorphic at $x = a$, if the lower and upper component of the defining function can analytically be continued to a full(two- dimensional) neighborhood of the real point a i.e. the upper/ lower component can analytically be continued across a into the lower/upper half-plane.

Definition 1.5[1]: Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction, holomorphic at both end points of the finite interval $[a, b]$, then the (definite)integral of $f(x)$ over $[a, b]$ is defined and denoted by

$$\int_a^b f(x)dx = \int_{\gamma_{a,b}^+} F_+(z)dz - \int_{\gamma_{a,b}^-} F_-(z)dz = -\oint_{(a,b)} F(z)dz$$

where the contour $\gamma_{a,b}^+$ runs in N_+ from a to b above the real axis, and the contour $\gamma_{a,b}^-$ is in N_- from a to b below the real axis.

Example[1]: $\int_{-\infty}^{\infty} \delta(x - a)dx = -\oint \frac{-1}{2\pi i(z-a)}dz = 1$

Definition 1.6[1]: Let Σ_0 be the largest open subset of the real line where the hyperfunction $f(x) = [F(z)]$ is vanishing. Its complement $K_0 = \mathbb{R} \setminus \Sigma_0$ is said to be the support of the hyperfunction $f(x)$ denoted by $suppf(x)$.

Let Σ_1 be the largest open subset of the real line where the hyperfunction $f(x) = [F(z)]$ is holomorphic. Its complement $K_1 = \mathbb{R} \setminus \Sigma_1$ is said to be the singular support of the

hyperfunction $f(x)$ denoted by $singsuppf(x)$.
Consider open sets $J = (a, 0) \cup (0, b)$ with some $a<0$ and some $b>0$ and compact subsets $K = \left[a', a''\right] \cup \left[b', b''\right]$ with $a<a' \leq a''<0$ and $0<b' \leq b''<b$. Also consider the following open neighborhoods $[-\delta, \infty) + iJ$ and $(-\infty, \delta] + iJ$ of $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively for some $\delta>0$
Introduce the subclass $\mathfrak{O}\,(\mathbb{R}_+)$ of hyperfunctions $f(x) = [F(z)]$ on $\mathbb{R}$ satisfying
(i) The support $suppf(x)$ is contained in $[0, \infty)$
(ii) Either the support $suppf(x)$ is bounded on the right by a finite number $\beta>0$ or we demand that among all equivalent defining functions, there is one, $F(z)$ defined in $[-\delta, \infty) + iJ$ such that for any compact $K \subset J$ there exist some real constant $M'>0$ and σ' such that $|F(z)| \leq M' e^{\sigma' \Re z}$ holds uniformly for all $z \in [0, \infty) + iK$
Because $suppf(x) \subset \mathbb{R}_+$ and since the singular support $singsuppf$ is a subset of the support, we have $singsuppf \subset \mathbb{R}_+$. Therefore $f(x)$ is a holomorphic hyperfunction for all $x<0$. Moreover, the fact that $F_+(x + i0) - F_-(x - i0) = 0$ for all $x<0$ shows that $F(z)$ is real analytic on the negative part of the real axis. Hence $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ implies that $\chi_{(-\epsilon,\infty)} f(x) = f(x)$ for any $\epsilon>0$.

Definition 1.7[1]: We call the subclass of hyperfunctions $\mathfrak{O}(\mathbb{R}_+)$ the class of *rightsided originals.*
In the case of an unbounded support $suppf(x)$, let $\sigma = inf\sigma'$ be the greatest lower bound of all σ' where the infimum is taken over all σ' and all equivalent defining functions satisfying (ii). This number $\sigma_- = \sigma_-(f)$ is called the growth index of $f(x) \in \mathbb{R}_+$. It has the properties
(i)$\sigma_- \leq \sigma'$
(ii) For every $\epsilon>0$ there is a σ' with $\sigma_- \leq \sigma' \leq \sigma_- + \epsilon$ and an equivalent defining function $F(z)$ such that $|F(z)| \leq M' e^{\sigma' \Re z}$ uniformly for all $z \in [0, \infty) + iK$.
In the case of a bounded support supp $f(x)$,we set $\sigma_-(f) = -\infty$

Definition 1.8[1]: The Laplace transform of a right-sided original $f(x) = [F(z)] \in \mathfrak{O}(\mathbb{R}_+)$ is now defined by

$$\widehat{f}(s) = \mathcal{L}[f(x)](s) = -\int_{\infty}^{(0+)} e^{-sz} F(z) dz.$$

The image function $\widehat{f}(s)$ of $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ is holomorphic in the right half-plane $\Re s>\sigma_-(f)$
Similarly, we introduce the class $\mathfrak{O}(\mathbb{R}_-)$ of hyperfunctions specified by
(i) The support $suppf(x)$ is contained in $\mathbb{R}_- = (-\infty, 0]$
(ii) Either the support $suppf(x)$ is bounded on the left by a finite number $\alpha<0$, or we

demand that among all equivalent defining functions there is one, denoted by $F(z)$ and defined in $(-\infty, \delta] + iJ$ such that for any compact subset $K \subset J$ there are some real constants $M''>0$ and σ''such that $|F(z)| \leq M'' e^{\sigma'' \Re z}$ holds uniformly for $z \in (-\infty, 0] + iK$.

Definition 1.9[1]: The set $\mathfrak{O}(\mathbb{R}_-)$ is said to be the class of left-sided originals.
In the case of an unbounded support let $\sigma_+ = sup\sigma''$ be the least upper bound of all σ'', where the supremum is taken over all σ'' and all equivalent defining functions satisfying (ii). The number $\sigma_+ = \sigma_+(f)$ is called the growth index of $f(x) \in \mathfrak{O}(\mathbb{R}_-)$. It has the properties
(i) $\sigma'' \leq \sigma_+$.
(ii) For every $\epsilon>0$ there is a σ'' such that $\sigma_+ - \epsilon \leq \sigma'' \leq \sigma_+$ and a definig function $F(z)$ such that $|F(z)| \leq M'' e^{\sigma'' \Re z}$ uniformly for $z \in (-\infty, 0] + iK$.
If the support $suppf(x)$ is bounded, we set $\sigma_+(f) = +\infty$

Definition 1.10[1]: The Laplace transform of a left-sided original $f(x) = [F(z)] \in \mathfrak{O}(\mathbb{R}_-)$ is defined by

$$\widehat{f}(s) = \mathcal{L}[f(x)](s) = -\int_{-\infty}^{(0+)} e^{-sz} F(z)dz.$$

The image function $\widehat{f}(s)$ of $f(x) \in \mathfrak{O}(\mathbb{R}_-)$ is holomorphic in the left half-plane $\Re s<\sigma_+(f)$

Definition 1.11[1]: With $g(x) \in \mathfrak{O}(\mathbb{R}_-)$,$f(x) \in \mathfrak{O}(\mathbb{R}_+)$,$h(x) = g(x) + f(x)$,

$$\mathcal{L}[h(x)](s) = \widehat{g(x)}(s) + \widehat{f(x)}(s), \sigma_-(f)<\Re s<\sigma_+(g), \text{ provided } \sigma_-(f)<\sigma_+(g).$$

Definition 1.12[1]: Hyperfunctions of the subclass $\mathfrak{O}(\mathbb{R}_+)$ are said to be of bounded exponential growth as $x \to \infty$ and hyperfunctions of the subclass $\mathfrak{O}(\mathbb{R}_-)$ are said to be of bounded exponential growth as $x \to -\infty$.
An ordinary function $f(x)$ is called of bounded exponential growth as $x \to \infty$, if there are some real constants $M'>0$ and σ' such that $|f(x)| \leq M' e^{\sigma' x}$ for sufficently large x. It is called of bounded exponential growth as $x \to -\infty$, if there are some real constants $M''>0$ and σ'' such that $|f(x)| \leq M'' e^{\sigma'' x}$, for sufficently negative large x
A function or a hyperfunction is of *bounded exponential growth*, if it is of bounded exponential growth for $x \to -\infty$ as well as for $x \to \infty$ .Thus a hyperfunction or ordinary function $f(x)$ has a Laplace transform, if it is of bonded exponential growth, and if $\sigma_-(f)<\sigma_+(f)$

Proposition 1.13[1]: If $f(x) = [F(z)]$ is a hyperfunction of bounded exponential growth which is holomorphic at $x = c$, then

$$-\int_{-\infty}^{(c+)} e^{-sz} F(z)dz = \int_{-\infty}^{c} e^{-sx} f(x)dx,$$

$$-\int_{\infty}^{(c+)} e^{-sz} F(z)dz = \int_{c}^{\infty} e^{-sx} f(x)dx, \text{ thus}$$

$$-\{\int_{-\infty}^{(c+)} e^{-sz} F(z)dz + \int_{\infty}^{(c+)} e^{-sz} F(z)dz\} = \int_{-\infty}^{\infty} e^{-sx} f(x)dx$$

Proposition 1.14[1]: Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth with an arbitrary support and holomorphic at some point $x = c$. If in addition $\sigma_- = \sigma_-(\chi_{(0,\infty)} f(x)) < \sigma_+ = \sigma_+(\chi_{(-\infty,0)} f(x))$, then its Laplace transform is given by

$$\mathcal{L}[f(x)](s) = \mathcal{L}[\chi_{(-\infty,c)} f(x)](s) + \mathcal{L}[\chi_{(c,\infty)} f(x)](s) = \int_{-\infty}^{c} e^{-sx} f(x)dx + \int_{c}^{\infty} e^{-sx} f(x)dx = \int_{-\infty}^{\infty} e^{-sx} f(x)dx$$

Definition 1.15[1]: Consider hyperfunctions depending on a continuous parameter α or an integral parameter k. The continuous parameter α varies in some open region ω of the complex plane and α_0 is a limit point of ω. Integral parameter k may vary in $\mathbb{N}$ or $\mathbb{Z}$.Then $f(x, \alpha) = [F(z, \alpha)], \alpha \in \omega; f_k(x) = [F_k(z)], k \in \mathbb{N}$ or $\mathbb{Z}$. We say that a family of holomorphic functions $F(z, \alpha)$,or a sequence of holomorphic functions $F_k(z)$ defined on a common domain $N \subset \mathcal{C}$ *converges uniformly in the interior of* N to $F(z)$ as $\alpha \to \alpha_0$, or $k \to \infty$, respectively if $F(z, \alpha)$ or $F_k(z)$ converges uniformly to $F(z)$ in every compact sub domain of N. This uniform convergence in the interior of N is also called *compact convergence in* N.

Definition 1.16[1]: Let $f(x) = [F_+(z), F_-(z)]$ be defined on I such that, for every k,equivalent defining functions $G_k(z)$ of $F_k(z)$ exist, such that $G_{+k}(z)$ and $G_{-k}(z)$ are uniformly convergent in the interior of $N_+(I)$ and $N_-(I)$ to $F_+(z)$ and $F_-(z)$ respectively. Then we write $f(x) = \lim_{k\to\infty} f_k(x)$, and say that the sequence of hyperfunctions $f_k(x)$ converges in the sense of hyperfunctions to $f(x)$. If a limit in the sense of hyperfunctions exists, it is unique.

Definition 1.17: A function $h : (0, \infty) \to (0, \infty)$ is said to be slowly varying at infinity if $lim_{p\to\infty} \frac{h(px)}{h(p)} = 1$ for all $x>0$

2. ABELIAN -TAUBERIAN THEOREM FOR LAPLACE TRANSFORM OF HYPERFUNCTIONS

We first develop the background for deriving the Continuity theorem of Hyperfunctions which then leads to the Abelian-Tauberian theorem for Hyperfunctions.

2.1 Measurable Hyperfunctions

Definition 2.1.1: A hyperfunction $f(x) = [F(Z)] = [F_+(z), F_-(z)]$ is said to be a measurable hyperfunction if the defining function $F(z) \in [F(z)]$ are all complex Lebesgue measurable functions.

Note: Here onwards we are considering sequence of hyperfunctions $(f_n(x)) = ([F_n(z)])$, where the sequence of defining functions $(F_n(z))$ are defined on a common domain $N \subset \mathbb{C}$

Lemma 2.1.2: (Fatou's lemma for Hyperfunctions)
Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth. Then $liminf \int f_n(x)dx \geq \int liminf f_n(x)dx$
Proof: Applying Fatou's lemma for measurable functions to the sequence of defining functions of $(f_n(x))$ we get
$liminf \int F_n(z)dz \geq \int liminf F_n(z)dz$
Also it holds for every $G_n(z) \in [F_n(z)]$.
Hence the result follows. □

Theorem 2.1.3:(Monotone Convergence Theorem for Hyperfunctions)
Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth.If $(f_n(x))$ is monotonic increasing and $(f_n(x)) \to f(x)$,where $f(x) = [F(x)]$ then $\int f(x)dx = lim \int f_n(x)dx$
Proof: Let $f(x) = lim_{n\to\infty} f_n(x)$.Then by lemma 2.1.2 we have
$\int f(x)dx = \int lim f_n(x)dx$
$= \int liminf f_n(x)dx$
$= \int liminf F_n(z)dz$
$\leq liminf \int F_n(z)dz$
$= liminf \int f_n(x)dx$
Since $(f_n(x))$ is monotonic increasing and $(f_n(x)) \to f(x)$ in the sense of

hyperfunction we have $f_n(x) \leq f(x)$.
Hence $\int f_n(x)dx \leq \int f(x)dx$
Then $limsup \int f_n(x)dx \leq \int f(x)dx$
So $\int f(x)dx \leq liminf \int f_n(x)dx \leq limsup \int f_n(x)dx \leq \int f(x)dx$
Thus $\int f(x)dx = lim \int f_n(x)dx$ □

Theorem 2.1.4:(Dominated Convergence Theorem for Hyperfunctions)
Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth.If $|f_n(x)| \leq g(x)$, where $g(x) = [G(z)]$ is a real valued hyperfunction and $lim_{n\to\infty} f_n(x) = f(x), f(x) = [F(z)]$ then $f(x)$ is integrable and $lim \int f_n(x)dx = \int f(x)dx$
Proof:Applying Dominated convergence theorem for measurable functions to the sequence $(F_n(z))$ of defining functions of $(f_n(x))$ we have $F(z)$ is integrable and
$lim \int F_n(z)dz = \int F(z)dz$
Then using the convergence in the sense of hyperfunctions we get $f(x)$ is integrable and $lim \int f_n(x)dx = \int f(x)dx$. □

Theorem 2.1.5:(Bounded convergence theorem for Hyperfunctions)
Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth,defined on $(0, \infty)$. If $|f_n(x)| \leq P$ and $lim_{n\to\infty} f_n(x) = f(x), f(x) = [F(z)]$ then
$lim \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$
Proof: Follows from Bounded convergence theorem for real valued measurable functions □

2.2 Continuity Theorem for Hyperfunction

Lemma 2.2.1:
Let $f(x) = [F(z)]$ and $g(x) = [G(z)]$ are two holomorphic hyperfunctions of bounded exponential growth with Laplace transforms $\hat{f}(s) = \mathcal{L}[f(x)](s)$ and $\hat{g}(s) = \mathcal{L}[g(x)](s)$. If they have a common vertical strip of convergence then $\hat{f}(s) = \hat{g}(s)$ implies $f(x) = g(x)$
Proof:Suppose $\hat{f}(s) = \hat{g}(s)$
$\Rightarrow \mathcal{L}[f(x)](s) = \mathcal{L}[g(x)](s)$
$\Rightarrow \int_0^\infty e^{-sz}F(z)dz = \int_0^\infty e^{-sz}F(z)dz$

$\Rightarrow \int_0^\infty e^{-sz}(F(z) - G(Z))dz = 0$
$\Rightarrow F(z) - G(z) = 0$
$\Rightarrow [F(z)] = [G(z)]$
$\Rightarrow f(x) = g(x)$ □

Theorem 2.2.2:(Continuity theorem for Hyperfunctions)
Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth,defined on $(0, \infty)$.
(a) Let $f(x) = [F(z)]$ be a measurable hyperfunction with support contained in $(0, \infty)$ such that $f_n(x) \to f(x)$ for all ponts x at which f_n's and f are holomorphic. If there exists $t \geq 0$ such that $sup_{n\geq 1}\mathcal{L}[f_n(x)](t) < \infty$ then $\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s)$ as $n \to \infty$ for all $s > t$
(b) Suppose there exists $t \geq 0$ such that $\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s)$ as $n \to \infty$ for all $s > t$ then $f_n(x) \to f(x)$ for all ponts x at which f_n's and f are holomorphic if the Laplace transforms of f_n's and f have a common vertical strip of convergence.
Proof:
(a) Let $M = sup_{n\geq 1}\mathcal{L}[f_n(x)](t) < \infty$.Then for any $s > t$ and $x \in (0, \infty)$
$\int_0^\infty e^{-sx} f_n(x)dx \to \int_0^\infty e^{-sx} f(x)dx$ by dominated convergence theorem for hyperfunctions.
Let $s > t$ and $\epsilon > 0$ such that f is holomorphic at $y \in (0, \infty)$ with $Me^{-(s-t)y} \leq \epsilon$.

$$\begin{aligned}\int_0^y e^{-sx} f_n(x)dx &\leq \mathcal{L}[f_n(x)](s)\\ &\leq \int_0^y e^{-sx} f_n(x)dx + e^{-(s-t)y}\int_y^\infty e^{-tx} f_n(x)dx\\ &\leq \int_0^y e^{-sx} f_n(x)dx + \epsilon\end{aligned}$$

Then
$$\begin{aligned}\int_0^y e^{-sx} f(x)dx &\leq liminf_{n\to\infty}\mathcal{L}[f_n(x)](s)\\ &\leq limsup_{n\to\infty}\mathcal{L}[f_n(x)](s)\\ &\leq \int_0^y e^{-sx} f(x)dx + \epsilon\end{aligned}$$

Letting $y \to \infty$ along holomorphic points of $f(x) = [F(z)]$

$$\begin{aligned}\int_0^\infty e^{-sx} f(x)dx &\leq liminf_{n\to\infty}\mathcal{L}[f_n(x)](s)\\ &\leq limsup_{n\to\infty}\mathcal{L}[f_n(x)](s)\\ &\leq \int_0^\infty e^{-sx} f(x)dx + \epsilon\end{aligned}$$

$i.e.\mathcal{L}[f(x)](s) \leq liminf_{n\to\infty}\mathcal{L}[f_n(x)](s) \leq limsup_{n\to\infty}\mathcal{L}[f_n(x)](s) \leq \mathcal{L}[f(x)](s) + \epsilon$
Since $\epsilon > 0$ is arbitrary,
$\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s)$ as $n \to \infty$ for all $s > t$
(b) Suppose that $\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s)$ as $n \to \infty$ for all $s > t$ and the Laplace transforms of f_n's and f have a common vertical strip of convergence.By 2.6 Lemma and dominated convergence theorem for hyperfunctions

$$\begin{aligned} f_n(x) &= \int_0^\infty e^{sx}\mathcal{L}[f_n(x)](s)ds \\ &\to \int_0^\infty e^{sx}\mathcal{L}[f(x)](s)ds \\ &= f(x) \end{aligned}$$

□

2.3 Abelian-Tauberian Theorem for Laplace Transform of Hyperfunctions

Now we are going to prove abelian tauberian theorem for the Laplace transform of hyperfunctions.To avoid formulas consisting of reciprocals we are introducing two positive variables p and q such that $pq = 1$. Then $q \to 0$ when $p \to \infty$

Theorem 2.3.1:
Let $f(x) = [F(z)]$ be a measurable, holomorphic hyperfunction on $(0, \infty)$ having compact support and bounded exponential growth.If the Laplace transform $\hat{f}(s) = [f(x)](s)$ is bounded for $s>0$ then the following conditions are equivalent.

(a) $\frac{\mathcal{L}[f(x)](qs)}{\mathcal{L}[f(x)](q)} \to \frac{1}{s^{\alpha+1}}$ as $q \to 0$

(b) $\frac{f(px)}{f(p)} \to x^\alpha$ as $p \to \infty$

Also $\mathcal{L}[f(x)](q) \sim f(p)\alpha!$, $\alpha \geq 0$ is an integer

Proof: **(a)⇒ (b)**
Suppose $\frac{\mathcal{L}[f(x)](qs)}{\mathcal{L}[f(x)](q)} \to \frac{1}{s^{\alpha+1}}$ as $q \to 0$. Then by theorem 2.2.2 $\frac{f(px)}{\mathcal{L}[f(x)](q)} \to \frac{x^\alpha}{\alpha!}$

Letting $x = 1$ we have $\frac{f(p)}{\mathcal{L}[f(x)](q)} \to \frac{1}{\alpha!}$ ....(1)

So $\frac{\alpha! f(p)}{\mathcal{L}[f(x)](q)} \to 1$ as $p \to \infty$

$\mathcal{L}[f(x)](q) \sim f(p)\alpha!$.......(2)

Substituting (2) in (1)

$\frac{f(px)}{f(p)\alpha!} \to \frac{x^\alpha}{\alpha!}$ as $p \to \infty$, $i.e. \frac{f(px)}{f(p)} \to x^\alpha$ as $p \to \infty$
(b)⇒ (a)
Suppose $\frac{f(px)}{f(p)} \to x^\alpha$ as $p \to \infty$

Then by theorem 2.2.2, $\frac{\mathcal{L}[f(x)](qs)}{f(p)} \to \frac{\alpha!}{s^{\alpha+1}}$........(3)

But $f(p) \sim \frac{\mathcal{L}[f(x)](q)}{\alpha!}$........(4)

Substituting (4) in (3)

$\frac{\mathcal{L}[f(x)](qs)\alpha!}{\mathcal{L}[f(x)](q)} \to \frac{\alpha!}{s^{\alpha+1}}, i.e. \frac{\mathcal{L}[f(x)](qs)}{\mathcal{L}[f(x)](q)} \to \frac{1}{s^{\alpha+1}}$ as $q \to 0$ □

We can express the above theorem in terms of slowly varying function also.

Theorem 2.3.2:
Let $f(x) = [F(z)]$ be a measurable, holomorphic hyperfunction on $(0, \infty)$ having compact support and bounded exponential growth. If the Laplace transform $\hat{f}(s) = [f(x)](s)$ is bounded for $s>0$ then the following conditions are equivalent.
(a) $\mathcal{L}[f(x)](s) \sim \frac{1}{s^{\alpha+1}} h(\frac{1}{s})$ as $s \to 0+$
(b)$f(x) \sim \frac{x^{\alpha+1}}{\alpha!} h(x)$ as $x \to \infty$
where $h : (0, \infty) \to (0, \infty)$ is a slowly varying function at infinity and $\alpha \geq 0$ is an integer
Proof:**(a)⇒ (b)**
Suppose $\mathcal{L}[f(x)](s) \sim \frac{1}{s^{\alpha+1}} h(\frac{1}{s})$ as $s \to 0+$

Then $\frac{\mathcal{L}[f(x)](\frac{s}{t})}{\mathcal{L}[f(x)](\frac{1}{t})} \sim \frac{1}{s^{\alpha+1}} \frac{h(\frac{t}{s})}{h(t)}$
$\sim \frac{1}{s^{\alpha+1}}$ as $t \to \infty$
Using theorem 2.3.1 and putting $q = \frac{1}{t}$ we have

$f(t) \sim \frac{\mathcal{L}[f(x)](\frac{1}{t})}{\alpha!}$
$\sim \frac{t^{\alpha+1}}{\alpha!} h(t)$ as $t \to \infty$
Similarly we can prove **(b)⇒ (a)** □

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