Divergence measure and its relation to nonspecificity of fuzzy sets

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Abstract

Divergence measure is a tool used to quantify the discrimination of two subsets of universal set. In this paper, we proposed some relations between the divergence measure based on cardinalities of a fuzzy set and uncertainty due to nonspecificity of fuzzy set. Due to that relation, a new class of divergence measures are also obtained. The application of above measure in the field of image segmentation is also mentioned.

AMS subject classification:
Keywords: Divergence measure, Uncertainty, Nonspecificity, Image Segmentation.

1. Introduction

Information retrieval from an uncertain scenario is a main research area during the last century. Uncertainty measures are used widely in the process of information acquisition. Shannon's entropy [2] based on probability theory is one of the important uncertainty measures handled the information gaining process. The assumption that probability theory is capable of capturing any kind of uncertainty was challenged only in the second half of the 20th century by the introduction of Fuzzy set theory by Lotfi Zadeh [19]. But later different fuzzy measures such as Possibility theory [10, 19], Dempster-Shafer

theory [12, 22], Plausibility theory [10, 12], Measures of fuzziness [13, 14, 16], Entropy [4, 5, 6, 9, 15, 17, 18] etc. were developed to quantify different kind of uncertainties involved the information system. A good survey of different kind of uncertainties and uncertainty measures is given in [12].

But there are another type of measures which is used in literature [1–6] to differentiate between any two subsets of a universal set X with finite or infinite cardinality, known as Divergence Measures. In mathematical words it can be expressed as a function $D : X \times X \to R^+$. This type of measures are used in information theory and Probability theory from the beginning of last century. Out of these Kullback-Leibler divergence is the most prominent one and it is a non-symmetric measure of the difference between two probability distributions P and Q. In [3] Kullback-Leibler divergence is defined as follows:

$$D_{KL}(P, Q) = \sum_{x \in X} P(x) \log \frac{P(x)}{Q(x)} \tag{1}$$

The symmetrised form of above measure in [2] as follows:

$$D_J(P, Q) = D_{KL}(P, Q) + D_{KL}(Q, P) \tag{2}$$

Later Vajda [21], Toussaint [22] and Jianhua Lin [2] came up with other types of divergence for two probability distributions on X based on Variational distances and they establishes relation between their measures and Kullback-Leibler divergence. Divergence measures are also developed to differentiate two fuzzy subsets of a Fuzzy set 'F(X)' and is known as 'Fuzzy Divergence'. Even though different types of divergence measures for fuzzy sets are available in literature [1–6], those measures which are relevant to the article are reviewing below.

In 1993, Bhandari and Pal [5] introduced a divergence measure based on the subsethood measure is:

$$D(A, B) = \sum_{i=1}^{n} \left[(\mu_A(x_i) - \mu_B(x_i)) \ln \frac{\mu_A(x_i)}{\mu_B(x_i)} + (\mu_B(x_i) - \mu_A(x_i)) \ln \frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)} \right] \tag{3}$$

In 1999, Fan and Xie [4] came up with another type of divergence measure based on exponential function is:

$$I(A, B) = \sum_i \left(1 - (1 - \mu_A(x_i)) e^{\mu_A(x_i) - \mu_B(x_i)} - \mu_A(x_i) e^{\mu_B(x_i) - \mu_A(x_i)} \right) \tag{4}$$

In 2002, S.Montes etc. all [1] introduced an axiomatic definition for the divergence of fuzzy measures based on some natural properties are given below:

$Axiom\,1 : D(A, B) = D(B, A)$
$Axiom\,2 : D(A, A) = 0$
$Axiom\,3 : \max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)\,\forall A, B, C \in \Im(X)$

In this paper, we studied the important properties of the divergence measure introduced in [6] and establish its relationship between Kosko's subsethood measure and divergence measure given by (3). Also we obtained a relation between the divergence measure and uncertainty. As the consequences of above relation, a new class of divergence measures are also formed. The rest of this paper is organized as follows: Section 2 provides definitions and important properties of the divergence measure. Nonspecificity of fuzzy sets and its relation to the new class of divergence is discussed in Section 3. The application of divergence measure in the field of image segmentation is explained in Section 4 and Concluding remarks are in Section 5.

2. Divergence measure and uncertainty

Throughout this paper we use the following conventions. Let X be the universal set with finite number of elements say 'n'.Let $\Im(X)$ denote the set of all fuzzy subsets of X; $\mu_A(x_i)$ is the membership function of $A = \Im(X)$;P(X) denote the power set (crisp) of X which contain 2^n elements; $\left|\tilde{A}\right|_\alpha$ denote the fuzzy cardinality of a fuzzy set $A = \Im(X)$ for some particular $\alpha = \Lambda(A)$; $|A|$ denote the scalar cardinality of fuzzy set $A = \Im(X)$; Assume that for a crisp set $\left|\tilde{A}\right|_\alpha = 0$ and $|A|$ is the usual cardinality of that set itself.

Definition 2.1. Given a finite set X with A as its fuzzy set, then the relative cardinality of A is defined as

$$Card(A) = \frac{|A|}{|X|} = \frac{\sum \mu_A(x)}{|X|}$$

Proposition 2.2. The relative cardinality must be lie between 0 and 1. ($0 \leq card(A) \leq 1$)

Proof. Because $|A| \leq n$, Card(A) $\leq$ 1. The lower part holds since A is a fuzzy set. ∎

Definition 2.3. Uncertainty quantity of fuzzy set A is defined as

$$U(A) = -\log_2 Card(A)$$

By the definition itself we can say $U(A) \geq 0$ because card $(A) \leq 1$.

Definition 2.4. Given a finite set X and a fuzzy set A on X we calculate the average uncertainty quantity of the fuzzy set with

$$H(A) = -\sum_{i=1}^{n} \frac{1}{n} \log_2 Card(A)$$

Definition 2.5. [6] For two fuzzy sets A, B $\in \Im(X)$, Fuzzy divergence between A and B is defined as

$$D_R(A, B) = \frac{2}{|A| + |B|} \sum_i \left[\alpha_i \max\left\{ 0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right| \right\} \right]$$

The following theorems and propositions are immediate consequences of **Definition 2.5 [6]**.

Theorem 2.6. $D_R(A, B)$ is a distance measure on $\Im(X)$ (see [4] for definition of distance measure).

Proof. Given

$$D_R(A, B) = \frac{2}{|A| + |B|} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right]$$

$For\,DP4:$

$\forall A, B, C \in \Im(X)$ and If $A \subset B \subset C$ then $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$

ie for particular α_i, obviously $\left| \tilde{A} \right|_{\alpha_i} \leq \left| \tilde{B} \right|_{\alpha_i} \leq \left| \tilde{C} \right|_{\alpha_i}$

Hence $\left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \leq \left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{C} \right|_{\alpha_i} \right|$

$D_R(A, B) \leq D_R(A, C)$

$|||^{ly} D_R(B, C) \leq D_R(A, C)$

For DP3 :

$\left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \geq 0$ and $\alpha_i \in [0, 1]$ $we've$

$\sum_i \left[\alpha_i \max \left\{ 0, \left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right] \leq 1$

Hence $\max \left\{ \frac{2}{|A| + |B|} \sum_i \left[\alpha_i m 0, \left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right] = \frac{2}{|A| + |B|} \right.$

$\forall D, D' \in P(X), D_R(D, D') = \frac{2}{|D| + |D'|}$

Since the summation is immaterial for crisp sets D and D'

Hence DP3

DP1 and DP2 are very clear from the definition itself.

∎

Theorem 2.7. $D_R(A, B)$ is a metric on $\Im(X)$.

Proof. To prove that $D_R(A, B)$ is a metric, it is enough to prove the triangle inequality only by theorem 2.1 we've $D_R(A, B) \leq D_R(A, C) \rightarrow (i)$ $D1_R(B, C) \leq D_R(A, C) \rightarrow (ii)$ $(i) + (ii) \Rightarrow D_R(A, B) + D_R(B, C) \leq 2D_R(A, C)$ Again from (i) & (ii)

$$\frac{1}{D_R(A, B)} \geq \frac{1}{D_R(A, C)} \rightarrow (iii)$$

$$\frac{1}{D_R(B,C)} \geq \frac{1}{D_R(A,C)} \rightarrow (iv)$$

$$(iii) + (iv) \Rightarrow \frac{1}{D_R(A,B)} + \frac{1}{D_R(B,C)} \geq \frac{2}{D_R(A,C)}$$

$$\frac{D_R(A,B) + D_R(B,C)}{D_R(A,B)D_R(B,C)} \geq \frac{2}{D_R(A,C)}$$

$$\frac{D_R(A,B) + D_R(B,C)}{D_R^2(A,C)} \geq \frac{2}{D_R(A,C)}$$

$$D_R(A,B) + D_R(B,C) \geq \frac{2D_R^2(A,C)}{D_R(A,C)}$$

$$D_R(A,B) + D_R(B,C) \geq D_R(A,C)$$

Hence proof. ∎

Proposition 2.8. Relation between Kosko's Subsethood measure [9] and proposed divergence measure.

Theorem 2.9. $D_R(A,B)$ reduces to $D(A,B)$ proposed divergence measure reduces to bhandari and pal [3] divergence measure.

Proof. By proposition 2.1 we've

$$S(A,B) = \frac{\sum count\ (A \cap B)}{K \sum count\, A} D_R(A,B),$$

$$K = \sum_i \alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \rightarrow (L)$$

$$S(B,A) = \frac{\sum count\ (A \cap B)}{K \sum count\, A} D_R(B,A),$$

$$K = \sum_i \alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \rightarrow (M)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{S(B,A)}{S(A,B)} = \frac{\sum count\, A}{\sum count\, B} \times \frac{D_R(B,A)}{D_R(B,A)}$$

$$i.e \Rightarrow \frac{S(B,A)}{S(A,B)} = \frac{\sum count\, A}{\sum count\, A} \rightarrow (N)$$

From (N) we can easily arrive at the defnition of $D(A,B)$ in [3]. ∎

2.1. Divergence Measure and its relation to uncertainty

Divergence measures are used to discriminate in between any two sets. Information about the sets is very important in the discrimination procedure and in most cases they are incomplete, imprecise, vague or deficient in some other way. All these information deficiency may result in different types of uncertainty. Here we are discussing about some relations between the proposed divergence measure and uncertainty in fuzzy sets.

From definition 4 we've

$$D_R(A, B) = \frac{2}{|A| + |B|} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right]$$

$$= \frac{2}{n \left[card(A) + card(B) \right]} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right]$$

by definition 1

$$card(A) + card(B) = \frac{2}{n D_R(A, B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right] \rightarrow (5)$$

Case 1: Let

$$card(A) = card(B)$$

then

$$2\,card(A) = \frac{2}{n D_R(A, B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right]$$

$$-\log_2 card(A) = -\log_2 \left(\frac{1}{n D_R(A, B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right] \right)$$

$$U(A) = -\log_2 \left(\frac{1}{n D_R(A, B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right] \right) \rightarrow (6)$$

Case 2: Let

$$card(B) = 0$$

$$-\log_2 card(A) = -\log_2 \left(\frac{2}{n D_R(A, B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} \right| \right\} \right] \right)$$

$$U(A) = -\log_2 \left(\frac{2}{n D_R(A, B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left\| \tilde{A} \right\|_{\alpha_i} \right| \right\} \right] \right) \rightarrow (7)$$

Case 3: Let

$$card(A) = 0$$

$$U(B) = -\log_2 \left(\frac{2}{nD_R(A,B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right] \right) \rightarrow (8)$$

Case 4:

$$card(A) + card(B) = card(A+B)$$

$$card(A+B) = \frac{2}{nD_R(A,B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right]$$

$$U(A+B) = -\log_2 \left(\frac{2}{nD_R(A,B)} \sum_i \left[\alpha_i \max \left\{ 0, \left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{B} \right|_{\alpha_i} \right| \right\} \right] \right) \rightarrow (9)$$

Hence uncertainty involved in the fuzzy sets A + B can be expressed as logarithmic function of divergence measure.

3. Nonspecificity

Nonspecificity is a kind of Uncertainty involved in a set of possible alternatives. It is connected with cardinalities of relevant sets of alternatives. If A and B are two sets of alternatives such that $A \subset B$, then the nonspecificity of A is smaller than nonspecificity of B. Nonspecificity of set containing single element is zero. Measures of nonspecificity and their usefulness were discussed in detail by Dubois and Prade in [9]. Higashi and Klir [10] introduced a measure U of nonspecificity of fuzzy set characterised by axioms given below:

1. for all $A \in \Im(X), U(A) \in [0, \infty)$

2. $U(A) = 0$ iff there exists $x \in X$ such that $A = \{x\}$,

3. if $A \subset B$, then $U(A) \leq U(B)$

Uncertainty measure in definition 2 is obtained from the relative cardinality of fuzzy sets and the following theorem establishes that it is also a nonspecificity of fuzzy sets characterised by above axioms.

Theorem 3.1. $U(A) = -\log_2 Card(A)$ satisfies axioms in the definition.

Proof. Axiom 1: If $Card(A) = 1 \Rightarrow U(A) = 0$ $If Card(A) \neq 1$ then $U(A) > 0$ Hence $U(A) \in [0, \infty)$
Axiom 2:

$$U(A) = 0 \Leftrightarrow -\log_2 Card(A) = 0 \Leftrightarrow Card(A) = 1 \Leftrightarrow A = \{x\}$$

Axiom 3:

$$A \subseteq B \Rightarrow Card(A) \leq Card(B)$$
$$\Rightarrow -\log_2 Card(A) \leq -\log_2 Card(B)$$

$$\Rightarrow U\left(A\right) \le U\left(B\right)$$

Hence theorem.   ∎

Proposition 3.2.

$$D_R\left(A, B\right) = \frac{1}{n} \sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]$$

Proof. By axiom 2

$$U\left(A\right) = 0 \Rightarrow \frac{1}{n D_R\left(A, B\right)} \sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right] = 1$$

$$\Rightarrow D_R\left(A, B\right) = \frac{1}{n} \sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]$$

∎

Proposition 3.3.

$$D_R\left(A, B\right) = \frac{2}{n} \sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i}\right|\right\}\right]$$

Proof. Similar to proposition 1.   ∎

Proposition 3.4.

$$D_R\left(A, B\right) = \frac{2}{n} \sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]$$

Proof. Similar to proposition 1.   ∎

Proposition 3.5.

$$U\left(A + B\right) < U\left(A\right) + U\left(B\right) \, if \, U\left(A\right) = U\left(B\right)$$

Proof.

$$U\left(A + B\right) = -\log_2\left(\frac{2}{n D_R\left(A, B\right)} \sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]\right)$$

$$< -2\log_2\left(\frac{1}{n D_R\left(A, B\right)} \sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]\right)$$

$$U\left(A + B\right) < U\left(A\right) + U\left(B\right)$$

∎

Proposition 3.6.

$$D_R\left(A' \cup C, B' \cup C\right) = D_R\left(A \cap C', B \cap C'\right) \forall A, B, C \in \Im(X)$$

Proof.
$$A \cap C' = \min\{a, c'\} = \min\{(a')', 1 - c\} = \max\{a', c\} = A' \cup C$$

Hence result. ∎

Proposition 3.7. For any two fuzzy subsets of X, $D_R(A \cup B, A \cap B) = D_R(A, B)$.

Proof. By defnition

$$D_R(A \cup B, A \cap B) = \frac{1}{|A \cup B| + |A \cap B|} \sum_i \alpha_i \max\{0, \left||(A \cup B)|_{\alpha_i} - |(A \cap B)|_{\alpha_i}\right|\}$$

$$A \cup B = \max\{A(x), B(x)\} \text{ and } A \cap B = \min\{A(x), B(x)\}$$
$$If \ A \cup B = A(x) \text{ and } A \cap B = A(x)$$

then we've
$$\left|(A\tilde{\cup}B)\right|_{\alpha_i} = \left|(A\tilde{\cap}B)\right|_{\alpha_i} = \left|\tilde{A}\right|_{\alpha_i}$$

$$D_R(A \cup B, A \cap B) = 0 = D_R(A, B) \text{ since } A = B$$

Similarily we can prove the result when

$$A \cup B = B(x) \text{ and } A \cap B = B(x)$$

If $A \cup B = A(x)$ and $A \cap B = B(x)$

$$\left|(A\tilde{\cup}B)\right|_{\alpha_i} = \left|\tilde{A}\right|_{\alpha_i}$$

and
$$\left|(A\tilde{\cap}B)\right|_{\alpha_i} = \left|\tilde{B}\right|_{\alpha_i}$$
$$|A \cup B|_{\alpha_i} = |A|_{\alpha_i}$$

and
$$|A \cap B|_{\alpha_i} = |B|_{\alpha_i}$$

Hence we've

$$D_R(A \cup B, A \cap B) = \frac{1}{|A|_{\alpha_i} + |B|_{\alpha_i}} \sum_i \alpha_i \max\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\}$$

$$= D_R(A, B)$$

∎

Proposition 3.8. Let A, B, C be three fuzzy subsets of X then

$$D_R (A \cup B, C) \leq D_R (A, C) + D_R (B, C)$$

Proof.

$$D_R (A, C) + D_R (B, C) - D_R (A \cup B, C)$$

$$= \frac{1}{|A|_{\alpha_i} + |C|_{\alpha_i}} \sum_i \alpha_i \max\{0, \left| \left| \tilde{A} \right|_{\alpha_i} - \left| \tilde{C} \right|_{\alpha_i} \right| \} + \frac{1}{|B|_{\alpha_i} + |C|_{\alpha_i}} \sum_i \alpha_i \max\{0, \left| \left| \tilde{B} \right|_{\alpha_i} - \left| \tilde{C} \right|_{\alpha_i} \right| \}$$

$$- \frac{1}{|A \cup B|_{\alpha_i} + |C|_{\alpha_i}} \sum_i \alpha_i \max\{0, \left| \left| A \tilde{\cup} B \right|_{\alpha_i} - \left| \tilde{C} \right|_{\alpha_i} \right| \}$$

$$\geq 0$$

Hence result.   ■

4. Application to the field of Image Segmentation

The important problem for the vision system is the identification of objects on an image. This operation is easy for the human observer, but very difficult for machines. The division of an image into meaningful regions is known as segmentation. Out of the different methods [23-27], thresholding becomes a simple but effective tool to separate objects from the background. The output of thresholding is a binary image where gray level of 0 (black) indicates the background and a gray level of 1 (white) indicates the foreground.

4.1. The Proposed Method

Let $X = \{(i, j) : i = 0, 1, \ldots, M - 1; j = 0, 1, \ldots, N - 1\}$, $K = \{0, 1, 2, \ldots, l - 1\}$, where M, N and l are three positive integers. Thus a digitized image defines a mapping $I : X \to K$. Let I(x, y) be the gray level value of the image at the pixel (x, y). Let $A : X \to [0, 1]$ be a fuzzy set for the given image X. The pixel value of an image is replaced by the fuzzy membership of Z- function. Hence image X with size M×N having l gray levels can be defined as an array of memberships.
The object and background of given image can also be considered as fuzzy subsets of A(x). Let $\alpha \in \Lambda(A)$, level set of A such that it divides the given fuzzy set into two fuzzy subsets say K and L. Compute the divergence measure of K with respect to L ($ie D_R (K, L)$) and L with respect to K ($ie D_R (L, K)$). The value of $\alpha \in \Lambda(A)$, which makes the measure into an equilibrium can be selected as an optimum value and the corresponding pixel value can be choosen as an optimal threshold.

Algorithm
Input: X
Output: Segmented image, S
STEP 1: Let $B = \{0, 1, 2, \ldots, l - 1\}$ be the distinct pixel values of X

STEP 2: Convert B into a fuzzy set
STEP 3: Assume a α_s and divide fuzzy set into two fuzzy subsets A and B
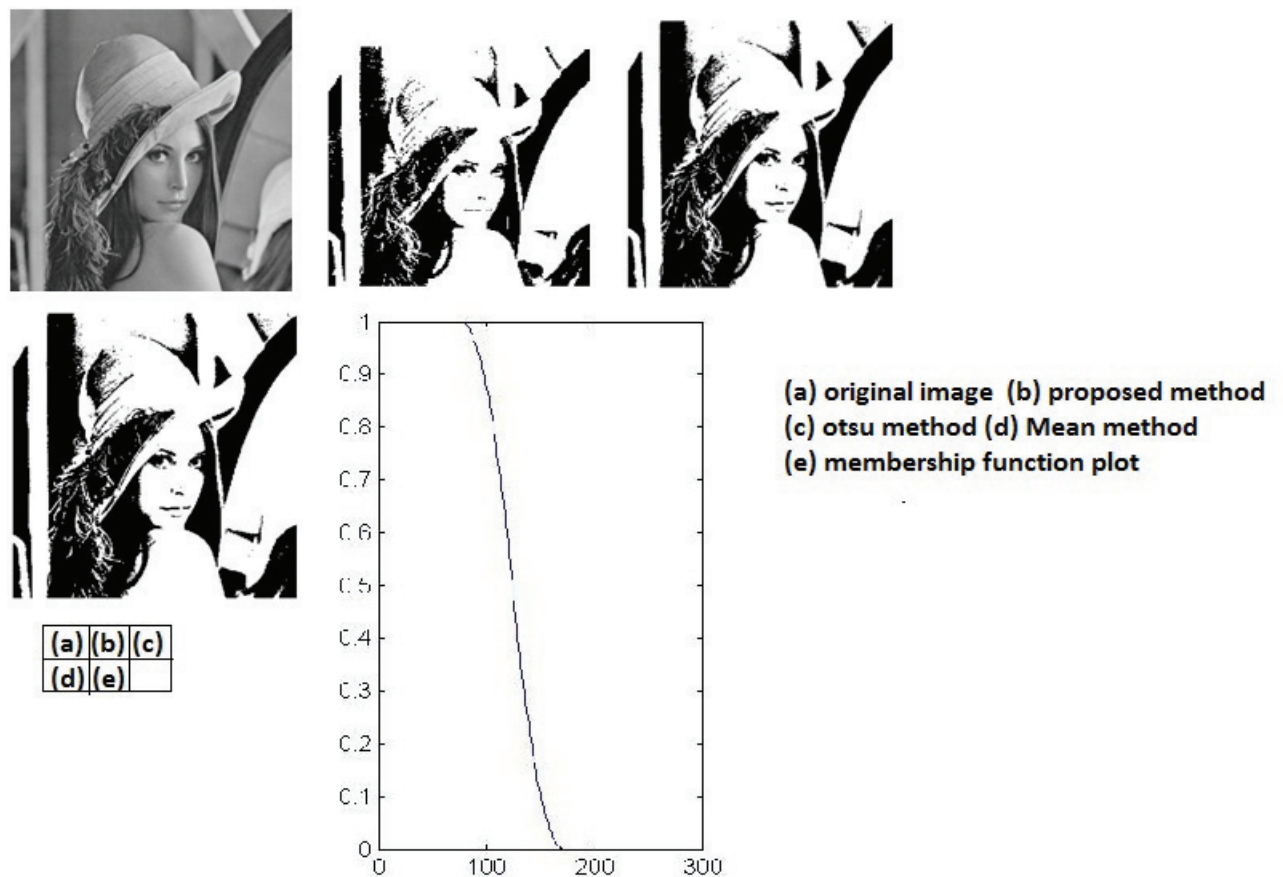STEP 4: Compute the divergence measure between A and B
STEP 5: Change the value of α_s and GOTO STEP 3
STEP 6: Pixel value corresponding to α_s, which make divergence measure into an equilibrium is selected as the optimal threshold
STEP 7: Segment image using value got in STEP 6.

4.2. Experimental Results

(a) original image (b) proposed method
(c) otsu method (d) Mean method
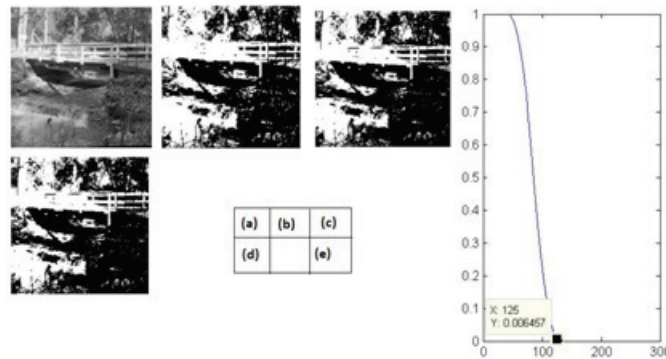(e) membership function plot

Fig 1. Lena image

To verify the efficiency of the method, experiments have been carried on many gray scale images and promising results are obtained.For the illustration of the proposed method, we use Lena and Bridge images. **Fig 1** and **Fig 2** show the results, with (a) showing the original image (b) showing the result using proposed method (c) showing the result using Otsu method (d) showing the result using mean method and (e) showing the membership function plot.

The original image is converted into a binary image, where gray level of 0 (black) indicates background and 1(white) will indicate the foreground. The threshold obtained during the experiments of proposed method together with other methods for five images are summarized in **Table 1**.The computation time is fairly small in comparison with otsu method[28]. All the segmentation results are obtained with the help of **MATLAB**.

Fig. 1 (a) is the gray scale Lena image and (b) is the result using the proposed method. We can see that the image is clearly converted into a binary image. **Fig. 1**(c) and (d) is the result using Otsu method and mean method. The visual comparison of proposed method with other methods doesn't show any differences.

Table 1

Image/Method	Otsu	Mean	Proposed
LENA	117	117	108
BARBA	125	120	98
BRIDGE	125	125	116
RICE	125	125	105
PEPPERS	102	102	93

a) original image (b) proposed method (c) otsu method
(d) Mean method (e) membership function plot

Fig 2. Bridge image

Fig. 2(a) is the gray scale bridge image and (b) is the binary image obtained using the proposed method. We can easily distinguishes the foreground (bridge) and the background from (b). Fig. 2(c) and (d) is the result using Otsu and mean method. From above tables , we can also conclude that the thresholds obtained in the proposed method are also reliable with other methods. The divergence measure and corresponding α value computed for different images are presented in **Table 2**.

Table 2

image	α	$D_R(K, L)$
LENA	0.779	9.26
BARBA	0.7163	0.84
BRIDGE	0.8106	12.78
RICE	0.5624	5.93
PEPPERS	0.6762	10.23

From the above discussions, it appears that the overall performance of the proposed method is consistently satisfactory compared to the other thresholding techniques [28].

5.   Conclusion

A new definition of divergence measure between two fuzzy sets is proposed along with its justification. Its properties are found to include Kosko's and Pal's divergence measure between fuzzy sets. Also the uncertainty due to nonspecificity and divergence measure is connected, which gives another class of divergence. Some desirable properties of new class of divergence measures are also derived. An algorithm for converting a gray scale image into binary image (as an example of application of the new divergence measure) is proposed. The results are compared with those of existing thresholding methods and found to be superior for a wide class of images.

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