Chapter 10

Hyperbolic trajectories

(e > 1)

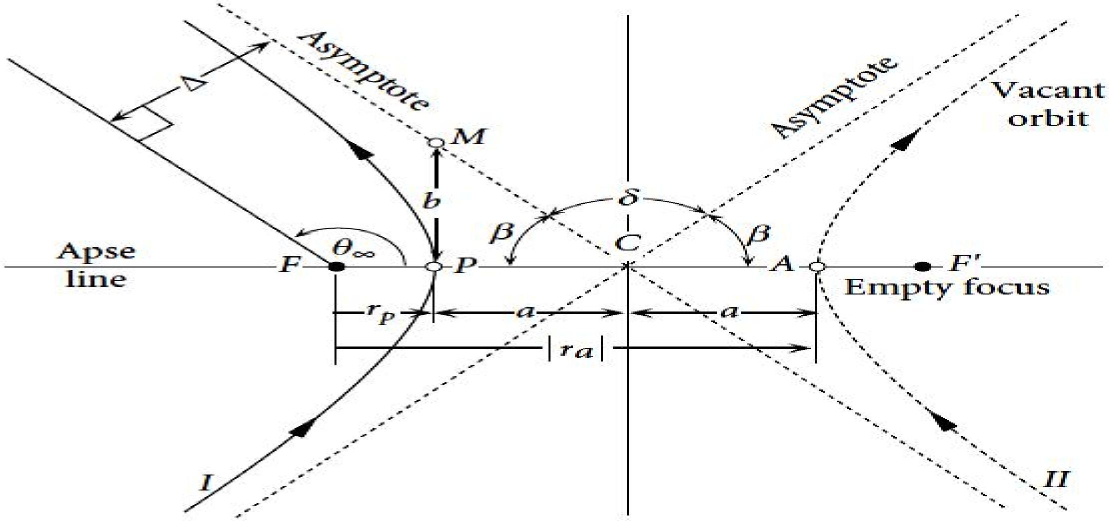
CHAPTER CONTENT

10- HYPERBOLIC TRAJECTORIES (e >1)

10- HYPERBOLICTRAJECTORIES (e >1)

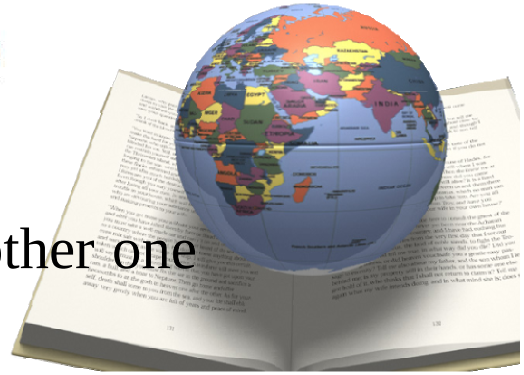
★If $e > 1$, the orbit formula describes the geometry of the hyperbola

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \qquad (1)$$

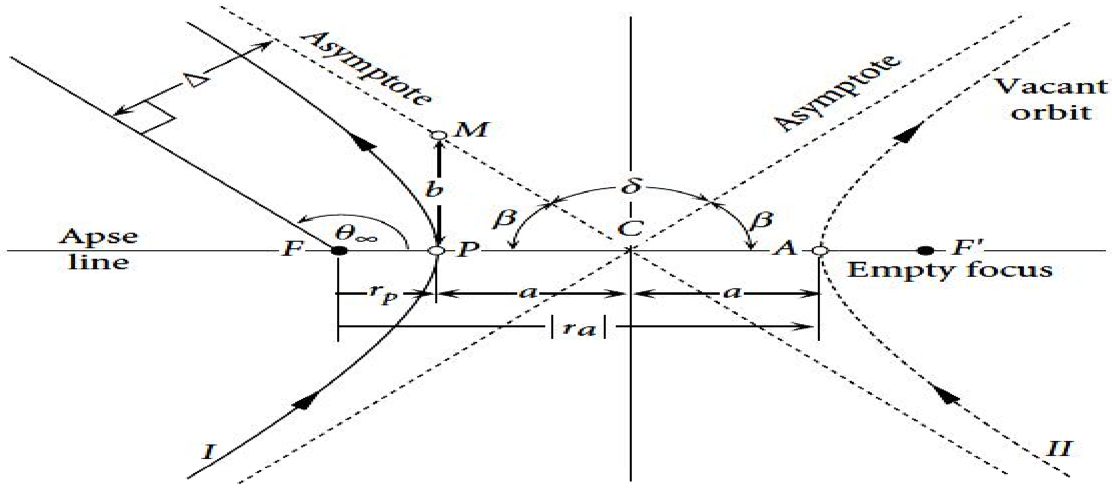


✶ The system consist of two symmetric curves

✶ One of the occupied by the orbiting body, the other one is its empty, mathematical image



10- HYPERBOLICTRAJECTORIES (e >1)



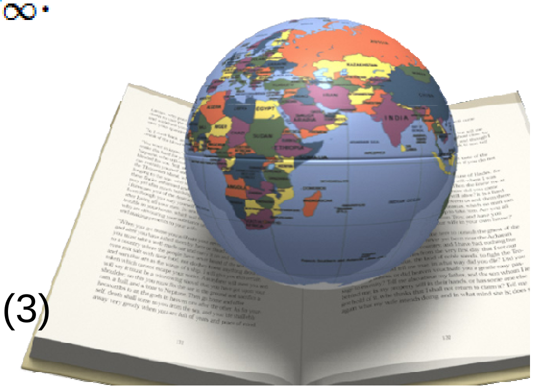
- Clearly: $\lim r \to \&$

  $\cos\theta \to -1/e$

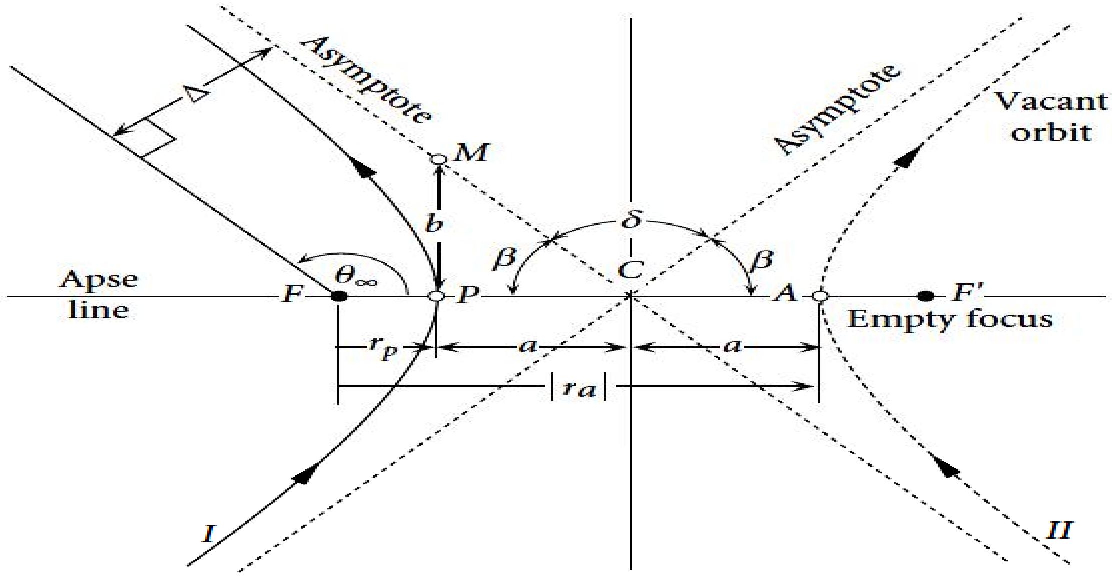
- We denote this value of true anomaly since the radial distance approaches infinity as the true anomaly approaches θ_∞.

$$\theta_\infty = \cos^{-1}(-1/e) \qquad (2)$$

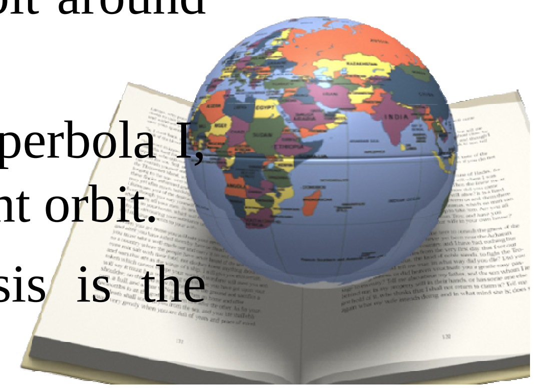
- θ_∞ is known as the true of the asymptote.
- Observe that θ_∞ lies between $90°$ and $180°$
- From trigonometry it follow that $\sin\theta_\infty = \dfrac{\sqrt{e^2-1}}{e}$ (3)



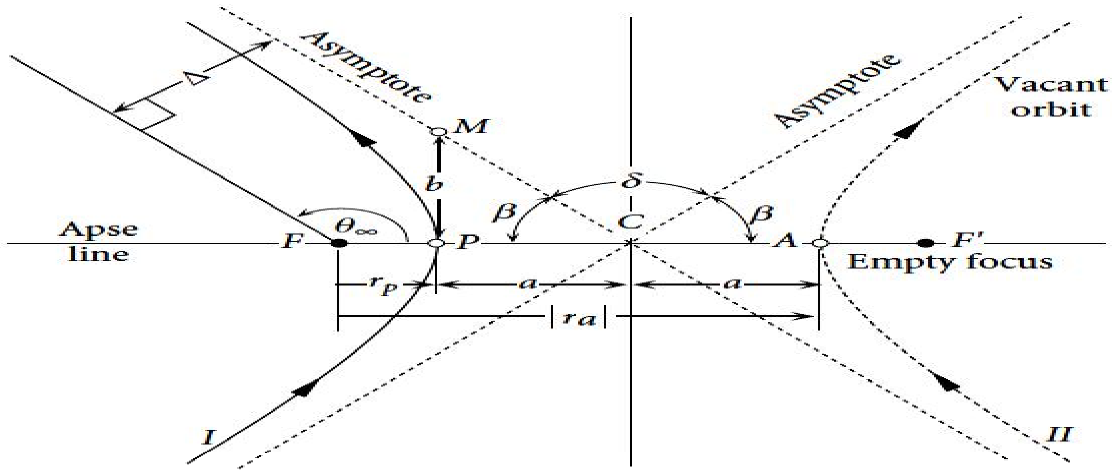
10- HYPERBOLICTRAJECTORIES (e >1)



- For $-\theta_\infty < \theta < \theta_\infty$, the physical trajectory is the occupied hyperbola I (on the left)
- For $\theta_\infty < \theta < (360° - \theta_\infty)$, hyperbola II- the vacant orbit around the empty focus F' - is traced out. (NOTE17,P69,{1})
- Periapsis P lies on the apse line on the physical hyperbola I, whereas apoapsis A lies on the apse line on the vacant orbit.
- The point halfway between periapsis and apoapsis is the center C of the hyperbola.



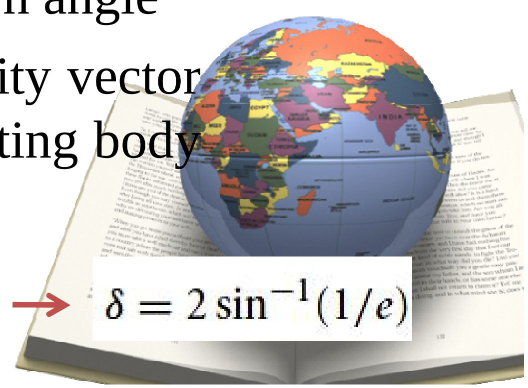
10- HYPERBOLICTRAJECTORIES (e >1)



- The asymptotes intersect at C, making angle β with the apse line.

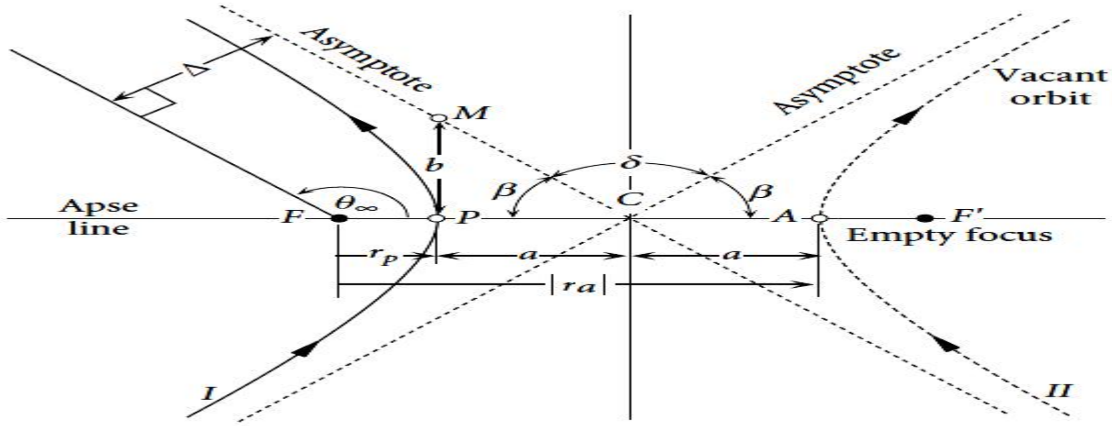
$$\beta = 180° - \theta_\infty \longrightarrow \cos\beta = -\cos\theta_\infty \longrightarrow \beta = \cos^{-1}(1/e) \quad (2)$$

- The angle δ between the asymptotes is called the turn angle
- The turn angle is the angle through which the velocity vector of the orbiting body is rotated as it rounds the attracting body at F and heads back towards infinity.



$$\delta = 180° - 2\beta, \rightarrow \sin\frac{\delta}{2} = \sin\left(\frac{180° - 2\beta}{2}\right) = \sin(90° - \beta) = \cos\beta \overset{\text{Eq. 2.89}}{=} \frac{1}{e} \rightarrow \delta = 2\sin^{-1}(1/e)$$

10- HYPERBOLICTRAJECTORIES (e >1)



- The distance r_p from the focus F to the periapsis is given by equation:

$$r_a = \frac{h^2}{\mu}\frac{1}{1+e} \qquad (6)$$

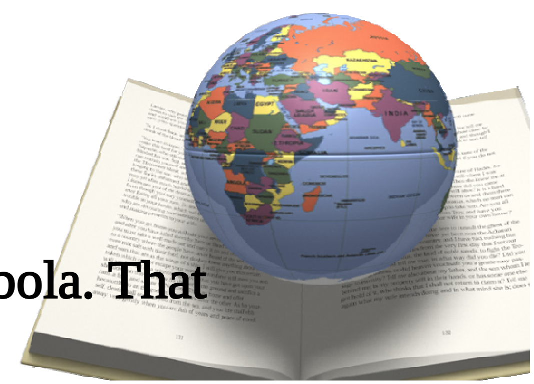
- The radial coordinate r_a of apoapsis is found by setting $\theta = 180°$ in equation:

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \qquad (7)$$

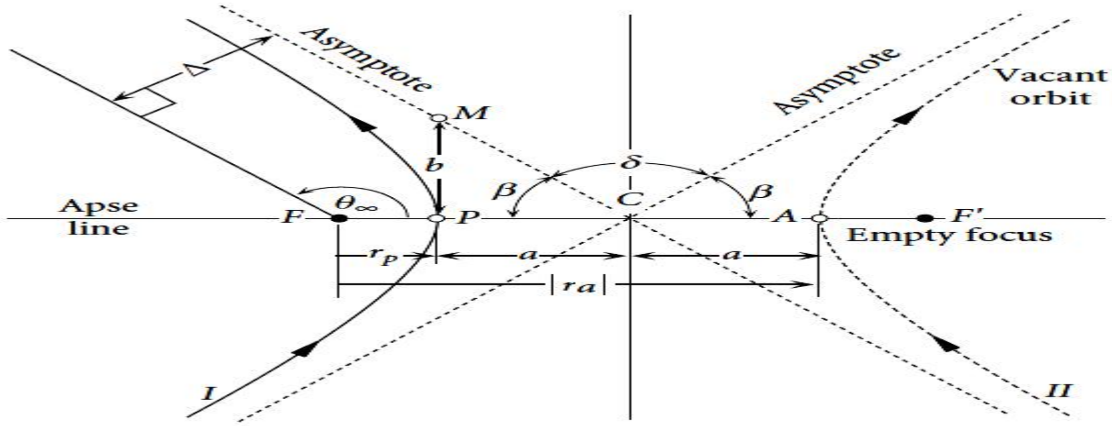
- so

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e}$$

- Observe that r_a is negative, since $e > 1$ for the hyperbola. That means the apoapse lies to the right of the focus F



10- HYPERBOLICTRAJECTORIES (e >1)



✶ We see that the distance 2a from periapse P to apoapse A is:

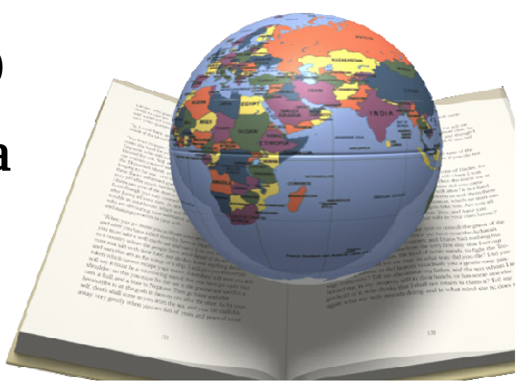
$$2a = |r_a| - r_p = -r_a - r_p$$

✶ Substituting equation (6) , (7) yields

$$2a = -\frac{h^2}{\mu}\left(\frac{1}{1-e} + \frac{1}{1+e}\right) \longrightarrow a = \frac{h^2}{\mu}\frac{1}{e^2-1} \qquad (8)$$

✶ So the orbit formula may be written for the hyperbola

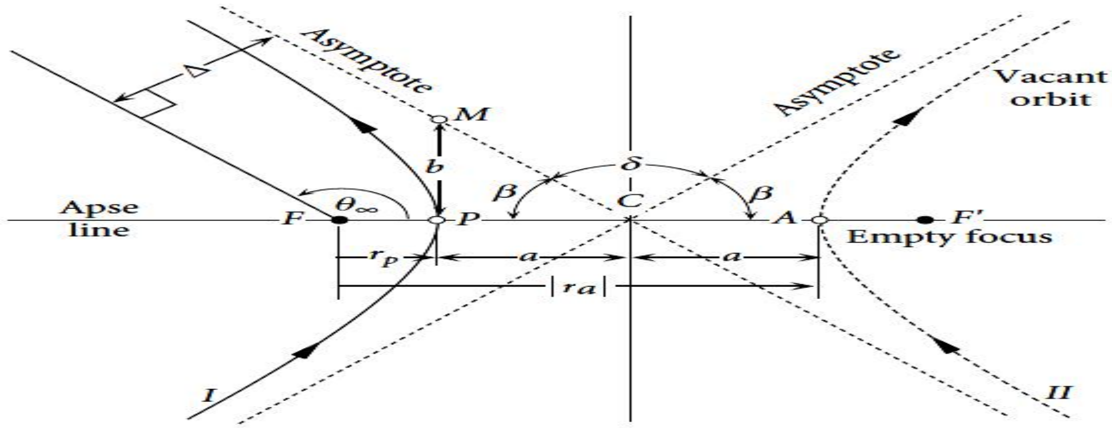
$$r = a\frac{e^2-1}{1+e\cos\theta} \qquad (9)$$



10- HYPERBOLICTRAJECTORIES (e >1)

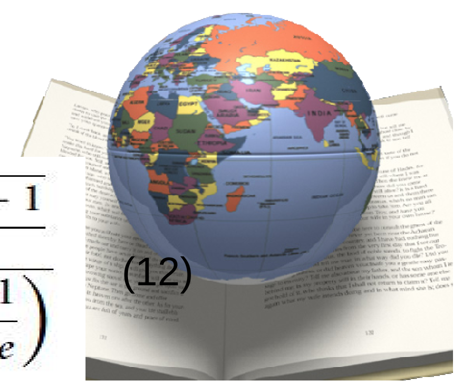
- From equation (g) it follows that: $r_p = a(e-1)$ (10)

$$r_a = -a(e+1) \quad (11)$$

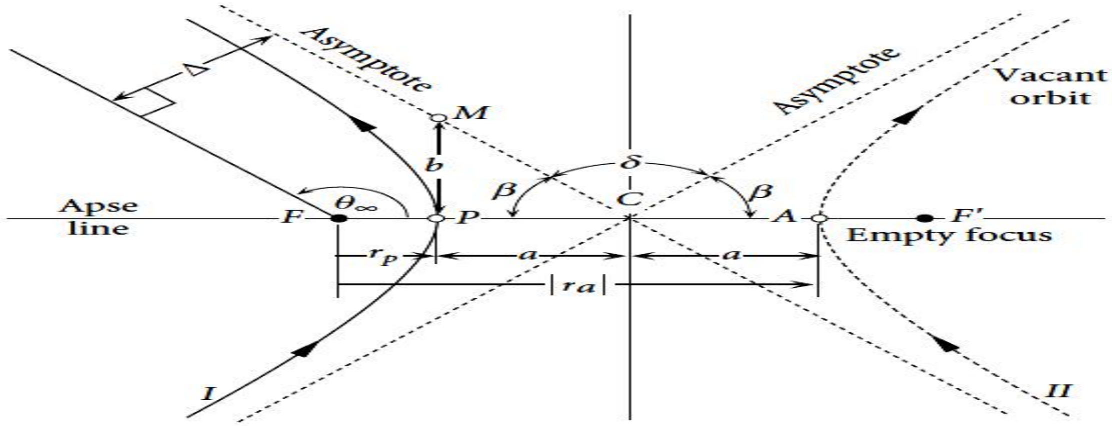


- The distance b, from periapsis to an asymptote measured perpendicular to the apse line; is the semiminor axis of the hyperbola
- The length b is

$$b = a\tan\beta = a\frac{\sin\beta}{\cos\beta} = a\frac{\sin(180-\theta_\infty)}{\cos(180-\theta_\infty)} = a\frac{\sin\theta_\infty}{-\cos\theta_\infty} = a\frac{\frac{\sqrt{e^2-1}}{e}}{-\left(-\frac{1}{e}\right)} \quad (12)$$



10- HYPERBOLICTRAJECTORIES (e >1)



- ✶ The distance Δ between the asymptote and a parallel line through the focus is called the aiming radius
- ✶ We see that

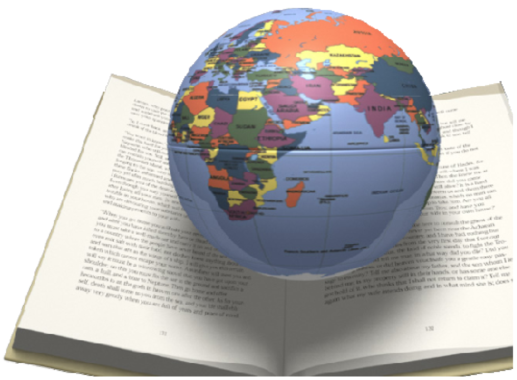
$$\Delta = (r_p + a)\sin\beta$$

(10) ⟶ $\Delta = ae\sin\beta$

(4) ⟶ $\Delta = ae\dfrac{\sqrt{e^2-1}}{e}$

(3) ⟶ $\Delta = ae\sin\theta_\infty = ae\sqrt{1-\cos^2\theta_\infty}$

(2) ⟶ $\Delta = ae\sqrt{1-\dfrac{1}{e^2}}$

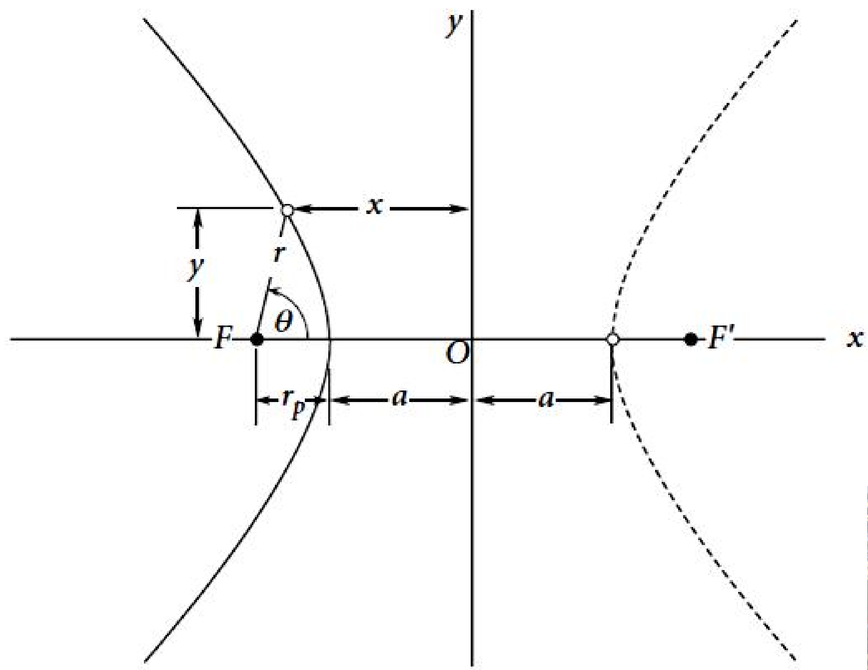


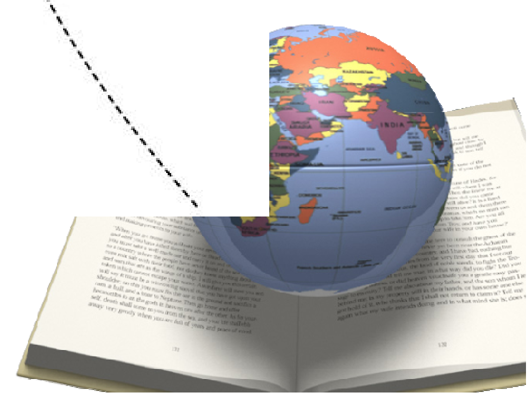
10- HYPERBOLICTRAJECTORIES (e >1)

✶ Finally: $\Delta = a\sqrt{e^2 - 1}$ (13)

✶ Comparing this result with equation 12, it is clear that the aiming radius equals the length of the semiminor axis of the hyperbola.

✶As with the ellipse and the parabola, we can express the polar form of the equation of the hyperbola in a cartesian coordinate system whose origin is in this case midway between the two foci.



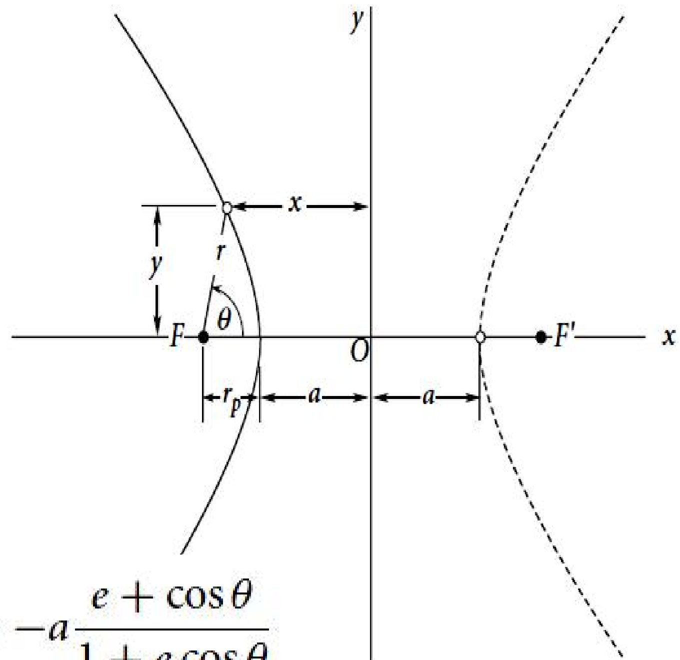


10- HYPERBOLICTRAJECTORIES (e >1)

✶ From the figure it is apparent that:

$$x = -a - r_p + r\cos\theta \qquad (14)$$

$$y = r\sin\theta \qquad (15)$$

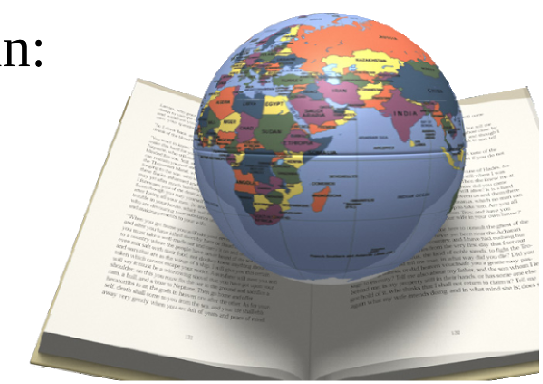


✶ Using equation (9),(10), (14) we obtain:

$$x = -a - a(e-1) + a\frac{e^2-1}{1+e\cos\theta}\cos\theta = -a\frac{e+\cos\theta}{1+e\cos\theta}$$

✶ substituting equation (9) and (12) in (15) we obtain:

$$y = \frac{b}{\sqrt{e^2-1}}\frac{e^2-1}{1+e\cos\theta}\sin\theta = b\frac{\sqrt{e^2-1}\,\sin\theta}{1+e\cos\theta}$$



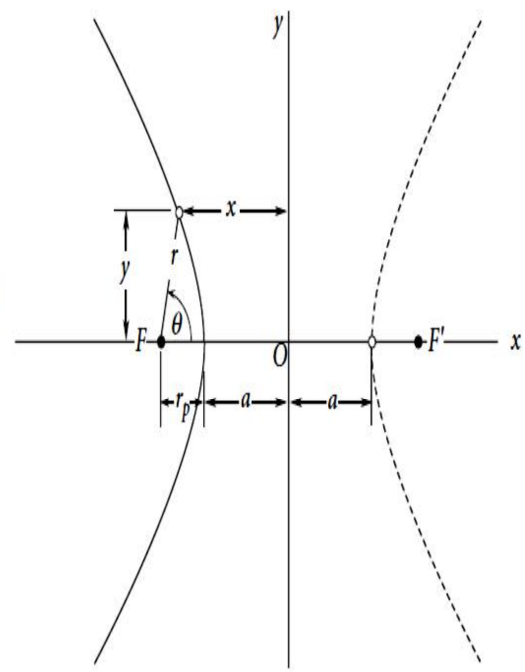
10- HYPERBOLICTRAJECTORIES (e >1)

✶ It follows that:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{e+\cos\theta}{1+e\cos\theta}\right)^2-\left(\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta}\right)^2$$

$$=\frac{e^2+2e\cos\theta+\cos^2\theta-(e^2-1)(1-\cos^2\theta)}{(1+e\cos\theta)^2}$$

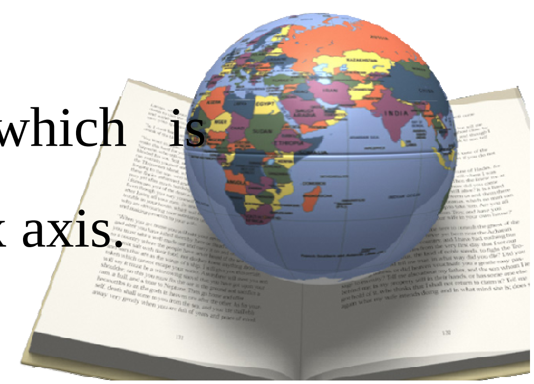
$$=\frac{1+2e\cos\theta+e^2\cos^2\theta}{(1+e\cos\theta)^2}=\frac{(1+e\cos\theta)^2}{(1+e\cos\theta)^2}$$



✶ That is,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (16)$$

✶ this is the familiar equation of hyperbola which is symmetric about x and y exes, with intercept on the x axis.



10- HYPERBOLICTRAJECTORIES (e >1)

- The specific energy of the hyperbolic trajectory is:

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}(1-e^2)$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2-1}$$

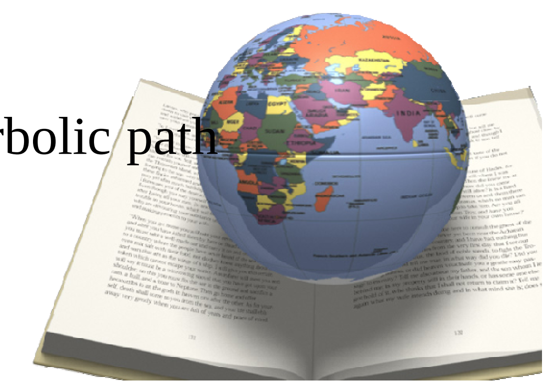
$$\Rightarrow \quad \varepsilon = \frac{\mu}{2a} \qquad (17)$$

- The specific energy of a hyperbolic orbit is clearly positive and independent of the eccentricity.
- The conservation of energy for a hyperbolic trajectory is:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \qquad (18)$$

- Let v_∞ denote the speed at which a body on a hyperbolic path arrives at infinity so:

$$(18) \rightarrow \quad v_\infty = \sqrt{\frac{\mu}{a}} \qquad (19)$$



10- HYPERBOLICTRAJECTORIES (e >1)

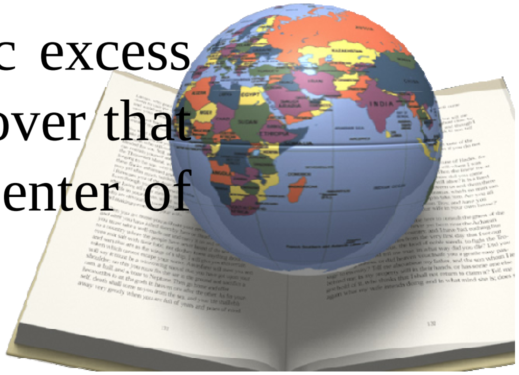
- In terms of v_∞ we may write equation (18) as:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^2}{2}$$

- v_∞ is called the hyperbolic excess speed.
- Substituting the expression for escape speed, we obtain for a hyperbolic trajectory

$$v^2 = v_{esc}^2 + v_\infty^2 \qquad (19)$$

- This equation clearly shows that the hyperbolic excess speed v_∞ represent the excess kinetic energy over that which is required to simply escape from the center of attraction.



10- HYPERBOLICTRAJECTORIES ($e > 1$)

- The square of v_∞ is denoted C_3 , and is known as the characteristic energy

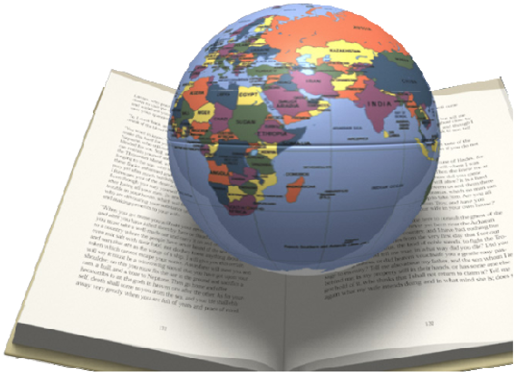
$$C_3 = v_\infty^2 \qquad (20)$$

- C_3 is a measure of the energy required for an interplanetary mission and C_3 is also a measure of **maximum energy a launch vehicle can import to a spacecraft of a given mass**

$$C_3)_{\text{launchvehicle}} > C_3)_{\text{mission}}$$

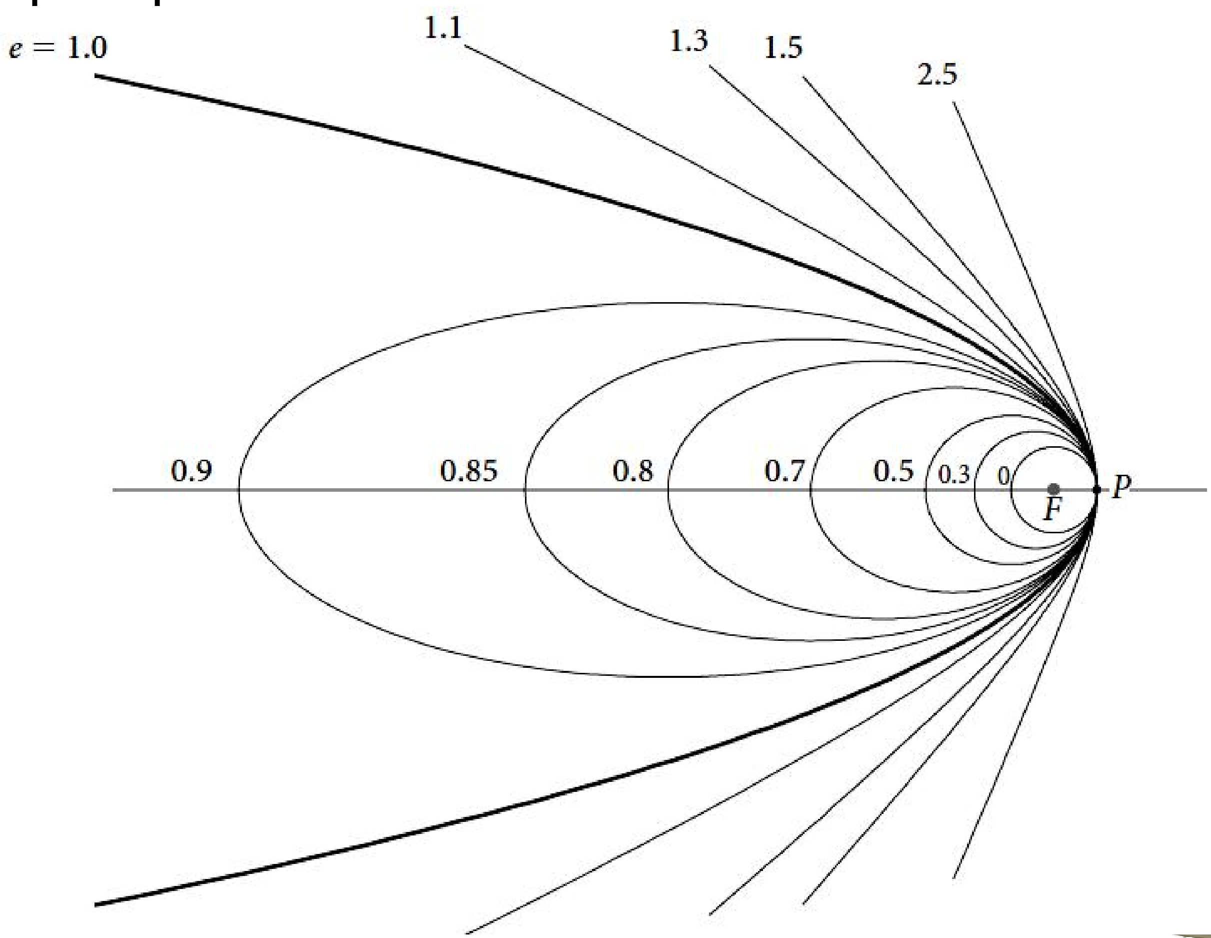
- v_∞ **can be find also:**

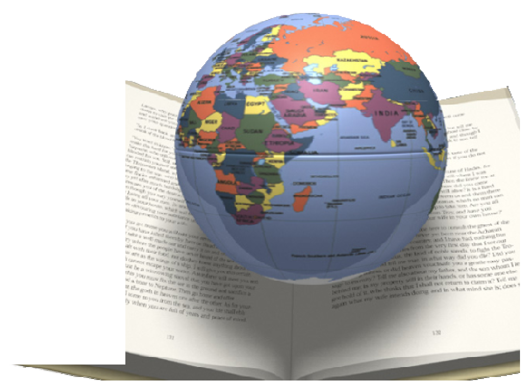
$$v_\infty = \frac{\mu}{h} e \sin\theta_\infty = \frac{\mu}{h}\sqrt{e^2 - 1} \qquad \textbf{(21)}$$



10- HYPERBOLICTRAJECTORIES (e >1)

- The figure shows a range of trajectories, from a circle through hyperbolas, all having common focus and periapsis





10- HYPERBOLICTRAJECTORIES ($e > 1$)

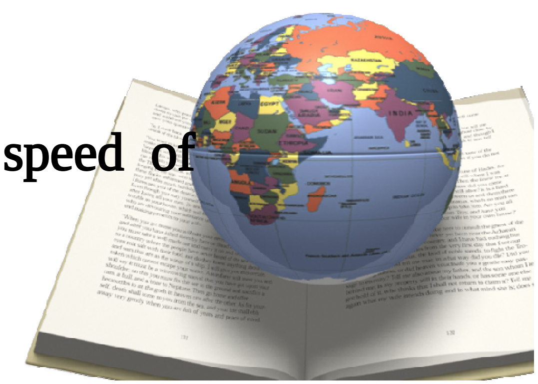
EXAMPLE ?.1

- At given point of a spacecraft's geocentric trajectory, the radius is 14600km, the speed is 8.6km/s, and the flight path angle is $50°$. Show that the path is a hyperbola and calculate the following: (a) C_3, (b) angular momentum, © true anomaly, (d) eccentricity, (e) radius of perigee, (f) turn angle, (g) semimajor axis, and (h) aiming radius.

  to determine the type of the trajectory, calculate the escape speed at the given radius.

$$v_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2 \cdot 398\,600}{14\,600}} = 7.389\,\text{km/s}$$

Since the escape speed is less than the spacecraft's speed of 8.6km/s, the path is a hyperbola.



10- HYPERBOLICTRAJECTORIES (e >1)

EXAMPLE ?.1

(a) the hyperbolic excess velocity v_∞ is found from equation (19),

$$v_\infty^2 = v^2 - v_{esc}^2 = 8.6^2 - 7.389^2 = 19.36\,\text{km}^2/\text{s}^2$$

From **equation (20) it follows that**

$$\underline{C_3 = 19.36\,\text{km}^2/\text{s}^2}$$

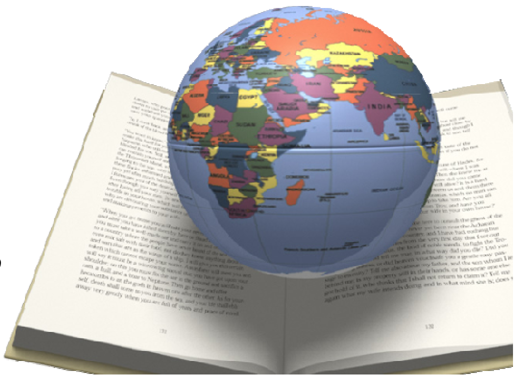
(b) Knowing the speed and the flight path angle, we can obtain both v_r **and** $v_\perp$:

$$v_r = v \sin\gamma = 8.6 \sin 50° = 6.588\,\text{km/s} \qquad \text{(a)}$$

$$v_\perp = v \cos\gamma = 8.6 \cdot \cos 50° = 5.528\,\text{km/s} \qquad \text{(b)}$$

Then equation * provides us with the angular momentum,

$$h = r v_\perp = 14\,600 \cdot 5.528 = \underline{80\,710\,\text{km}^2/\text{s}} \qquad \text{(c)}$$



10- HYPERBOLICTRAJECTORIES ($e > 1$)

EXAMPLE ?.1

(c) Evaluating the orbit equation at the given location on the trajectory, we get

$$14\,600 = \frac{80\,710^2}{398\,600}\frac{1}{1 + e\cos\theta}$$

From **which**

$$e\cos\theta = 0.1193 \qquad \text{(d)}$$

The radial component of velocity is given by equation $v_r = \dfrac{\mu}{h}e\sin\theta$, $v_r = \mu e\sin\theta/h$, so that with (a) and (c), we obtain

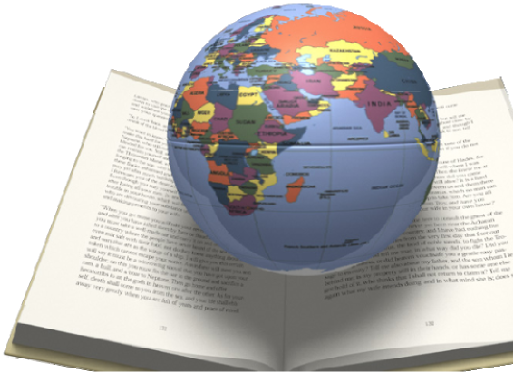
$$6.588 = \frac{398\,600}{80\,170}e\sin\theta$$

or

$$e\sin\theta = 1.334 \qquad \text{(e)}$$

Computing the ratio of (e) to (d) yields

$$\tan\theta = \frac{1.334}{0.1193} = 11.18 \quad \Rightarrow \quad \underline{\theta = 84.89°}$$



10- HYPERBOLICTRAJECTORIES (e >1)

EXAMPLE ?.1

(d) We substitute the true anomaly back into either (d) or (e) to find the eccentricity,

$$\underline{e = 1.339}$$

(e) The radius of perigee can now be found from the orbit equation,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e\cos(0)} = \frac{80\,710^2}{398\,600}\frac{1}{1+1.339} = \underline{6986\ \text{km}}$$

(f) The formula for turn angle is equation $\delta = 2\sin^{-1}(1/e)$, from which

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.339}\right) = \underline{96.60°}$$

(g) The semimajor axis of the hyperbola is found in equation

$$a = \frac{h^2}{\mu}\frac{1}{e^2-1}$$

(h) According to equation $b = a\sqrt{e^2-1}$, the aiming radius is

$$\Delta = a\sqrt{e^2-1} = 20\,590\sqrt{1.339^2-1} = \underline{18\,340\ \text{km}}$$