16 Feb '24

Descriptive Complexity

We have seen the following about FO so far:

1. Syntax & semantics of FO
2. The notions of satisfiability, unsatisfiability & validity for FO sentences & FO theories.
3. Three well-known results of classical model theory:
   a) Compactness theorem
   b) Löwenheim–Skolem theorem
   c) Lindström's theorem.
4. Relations between structures:
   a) Substructure
   b) Embeddability
   c) Isomorphism
5. FO definability of relations (b) & (c)

(6) Using classical results to show inexpressibility of properties in FO.

- Connectedness of arbitrary graphs

- Evenness of sizes of finite sets

(7) Failure of the compactness theorem in the finite.

Revisiting the notions / results 'over' a given class

The notions and results seen earlier can more generally be seen over a given class of structures. In doing so, some of them stay unaffected while some change (possibly significantly).

Fix a class $\mathcal{C}$ of σ-structures.

① Satisfiability over $\mathcal{C}$

We say an FO sentence φ over a vocabulary σ is satisfiable over $\mathcal{C}$ if there exists a σ-structure $M \in \mathcal{C}$ such that

$$M \models \varphi$$

② Unsatisfiability over $\mathcal{C}$

We say an FO sentence φ over a vocabulary σ is unsatisfiable over $\mathcal{C}$ if it is not satisfiable over $\mathcal{C}$

③ **Validity over C**

We say an FO sentence φ over a vocabulary σ is valid over C if for all σ-structures $M \in C$

$$M \models \varphi$$

Notes:

ⓐ If φ is unsatisfiable over C, it could still be satisfiable (so satisfiable over $\overline{C}$ - the complement of C). However if φ is unsatisfiable, then it certainly is unsatisfiable over C.

ⓑ Similarly:

- validity over C $\underset{\checkmark}{\overset{\times}{\rightleftarrows}}$ validity

- Satisfiability over C $\underset{\times}{\overset{\checkmark}{\rightleftarrows}}$ Satisfiability

(c) Satisfiability over C can be alternatively also referred to as any of the following:

- Satisfiability restricted to C
- — " — relativized to C
- — " — modulo C
- — " — under a promise of C

Similarly for the other notions.

(d) If C is definable by an FO theory T, then 'C' in the usages above can be replaced with T.

E.g. Satisfiability modulo T (SMT)

(e) If C = the class of all finite σ-structures, then

→ Satisfiability / unsatisfiability / validity over C is also referred to as finite satisfiability / unsatisfiability / validity.

→ A sentence / theory that is satisfiable / unsatisfiable / valid over C is said to be finitely satisfiable / unsatisfiable / valid

Important algorithmic results:

Theorem. Let σ be a finite vocabulary containing at least one binary relation symbol.

① (Gödel, 1930) The class of all valid FO sentences over σ is recursively enumerable (r.e.).

② (Turing, 1936) The class of all satisfiable FO sentences over σ is undecidable.

③ By ① & ②, over arbitrary structures,

→ FO validity is r.e. and not co-r.e

→ FO satisfiability is co-r.e and not r.e.

④ (Trakhtenbrot, 1960) The class of all finitely satisfiable FO sentences over σ is undecidable.

(P.T.O.)

(5) Since the class of finitely satisfiable FO sentences over σ is clearly r.e., it follows from (4) that over finite structures,

→ FO validity is co-r.e. and not r.e

→ FO satisfiability is r.e and not r.e.

Observe: The statuses of the validity & satisfiability problems for FO turned opposite in going from arbitrary to finite structures.

Lets examine what happens to the 3 classical model theoretic results under such a movement.

Classical model theoretic results as properties of classes of structures

Fix a class C of σ-structures. Define the following properties of C.

① Compact (C) : Let T be an FO theory over σ. Then TFAE :

ⓐ Every finite subset of T is satisfiable over C.

ⓑ T is satisfiable over C.

② Löwenheim - Skolem (C) : Let T be as above in ①. If T has an infinite model in C, then T has a countable model in C.

③ Lindström (C) : For the notion of abstract logics as in Lindström's theorem, FO is the only abstract logic that satisfies Compact (C) and Löwenheim - Skolem (C).

$\boxed{\text{①, ② \& ③ = Compactness, Löwenheim-Skolem \& Lindström's theorems hold over } C.}$

Results

1. If C = all σ-structures (arbitrary sizes) then all 3 properties defined above are true for C.

More generally, the foll. holds. (Exercise)

If C = a class of arbitrary structures definable by an FO theory T then again all the 3 properties defined above are true for C.

2. If C = all finite σ-structures, then
   - → Compact (C) is false
   - → Löwenheim - Skolem (C) is true trivially
   - → Lindström (C) is false.

More generally: (Exercise)

The same truth values of the properties hold when C is any infinite class (upto isomorphism) of finite σ-structures.

③ If C = a finite (upto isomorphism) class of finite σ-structures, then

→ Compact (C) is true

→ Löwenheim-Skolem (C) is true trivially

→ Lindström (C) is true.

(Exercise)

Notes:

Observe that as a consequence of ① & ② above, we get that

The class of all finite σ-structures is not definable by any FO theory (and hence also not definable by any FO sentence). The same also holds for any infinite class of finite σ-structures.

Definability over a class

Fix a class C of σ-structures.

(1) A subclass $\mathcal{D}$ of C is said to be FO **definable over C** if there exists an FO sentence φ such that if

$$\mathrm{Mod}_C(\varphi) = \text{the class of models of } \varphi \text{ that belong to } C$$

then $\mathrm{Mod}_C(\varphi) = \mathcal{D}$.

Equivalently,

$$\mathrm{Mod}_{All}(\varphi) \cap C = \mathcal{D}$$

$\boxed{All = \text{all } \sigma\text{-structures}}$

(2) A subclass $\mathcal{D}$ of C is said to be FO **axiomatizable over C** if there exists an FO theory T such that if

$$\mathrm{Mod}_C(T) = \text{the class of models of } T \text{ that belong to } C$$

then $\mathrm{Mod}_C(T) = \mathcal{D}$.

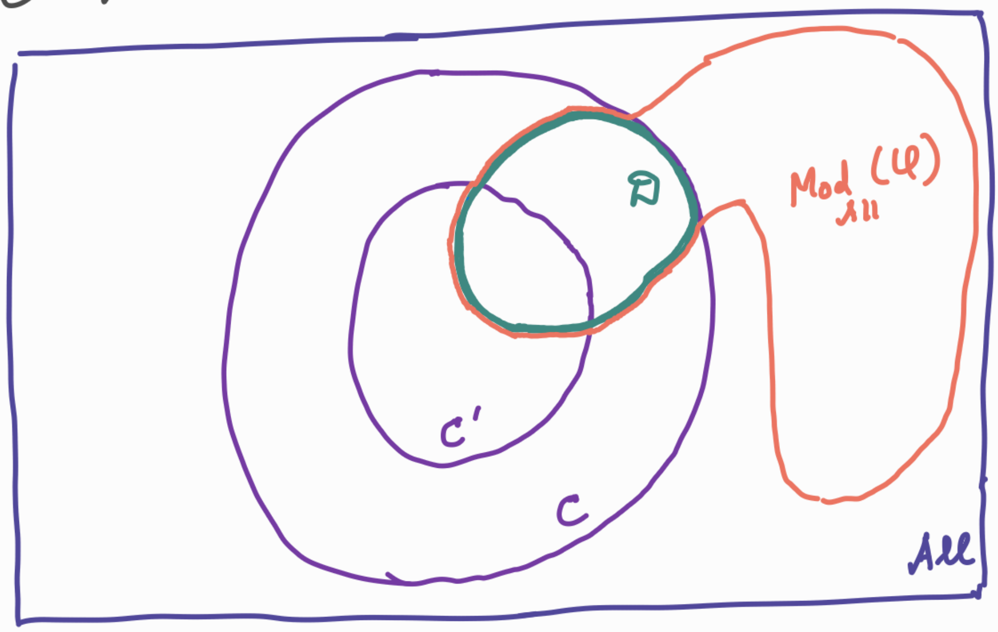
Equivalently: $\mathrm{Mod}_{All}(T) \cap C = \mathcal{D}$.

Notes:

(a) If $\mathcal{D}$ is definable / axiomatizable over $\mathcal{C}$, and $\mathcal{C}'$ is a subclass of $\mathcal{C}$, then $\mathcal{D} \cap \mathcal{C}'$ is definable/axiomatizable over $\mathcal{C}'$.

In particular, the same sentence / theory that defines / axiomatizes $\mathcal{D}$ over $\mathcal{C}$, also defines / axiomatizes $\mathcal{D} \cap \mathcal{C}'$ over $\mathcal{C}'$

(b) The converse however need not be true. That is, if $\mathcal{D} \cap \mathcal{C}'$ is definable over $\mathcal{C}'$ then $\mathcal{D}$ need not be definable over $\mathcal{C}$.



φ defines $\mathcal{D}$ over $\mathcal{C}$, and $\mathcal{D}' = \mathcal{D} \cap \mathcal{C}'$ over $\mathcal{C}'$.

Examples:

1) Let C = all arbitrary graphs &

C' = all finite graphs.

$\mathcal{D}$ = all graphs containing a triangle.

$\mathcal{D}$ is definable over C by the FO sentence

$$\varphi := \exists x \exists y \exists z \left(E(x,y) \wedge E(y,z) \wedge E(z,x) \right)$$

We see that φ also defines

$\mathcal{D} \cap C'$ = all finite graphs containing a triangle

over C'.

That is, $\mathrm{Mod}_{C'}(\varphi) = \mathcal{D} \cap C'$.

However,

$\mathcal{D} \cap C'$ is not even axiomatizable in FO (i.e. not defined by even an FO theory) over C.

2) Let C = all σ-structures where $\sigma = \{\leq\}$

C' = all finite σ-structures.

$\mathcal{D}$ = the class of all linear orders that are well-ordered.

(so do not contain infinite descending chains)

$\mathcal{D} \cap C'$ = all finite linear orders

$\mathcal{D} \cap C'$ is definable over C' by the FO sentence

φ = "$\leq$ is reflexive, symmetric, transitive & total"

However $\mathcal{D}$ is not axiomatizable in FO.

(We will see a proof this later.

Hint: $(\mathbb{N}, \leq)$ and $(\mathbb{N} + \mathbb{Z}, \leq)$ are FO-indistinguishable.)

Ehrenfeucht – Fraïssé games

Relational vocabulary:

We say σ is a relational vocabulary if σ does not contain any function symbols of positive arity. That is, σ contains only relation symbols and constant symbols (nullary function symbols), zero or more of each of these kinds of symbols.

If σ contains only relational symbols and no constant or function symbols, then σ is said to be a purely relational vocabulary.

Substructure induced by a set:

Let σ be a relational vocabulary.
Let A be a σ-structure (arbitrary), and let X be a subset of $\mathrm{Dom}(A)$ (= domain of A)

The substructure of A induced by X is the σ-structure B defined as follows:

(i) $\mathrm{Dom}(B) = X \cup \{c^A \mid c \text{ is a constant symbol of } \sigma\}$

Recall that c^A denotes the element of $\mathrm{Dom}(A)$ that interprets c.

(ii) For every relation symbol $R \in \sigma$ having arity $r \geq 0$:

$$R^B = R^A \cap (\mathrm{Dom}(B))^r$$

(iii) $c^B = c^A$ for all constant symbols $c \in \sigma$.

We denote the structure B as $A[X]$.
If X = set underlying a tuple $\bar{a}$ of A, we also denote B as $A[\bar{a}]$.

Recalling Isomorphism

Given σ-structures $\mathcal{A}$ & $\mathcal{B}$, we say $\mathcal{A}$ is isomorphic to $\mathcal{B}$, denoted $\mathcal{A} \cong \mathcal{B}$, if there is a bijective function $f : \mathrm{Dom}(\mathcal{A}) \longrightarrow \mathrm{Dom}(\mathcal{B})$, called an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, such that:

ⓐ For every relation symbol $R \in \sigma$ of arity $r \geq 0$, and elements $a_1, \ldots, a_r \in \mathrm{Dom}(\mathcal{A})$

$$(a_1, \ldots, a_r) \in R^{\mathcal{A}}$$

$$\text{iff}$$

$$(b_1, \ldots, b_r) \in R^{\mathcal{B}}$$

where $b_i = f(a_i)$ for $i \in [r]$

(b) For every function symbol $g \in \sigma$ of arity r, and elements $a_1 \ldots a_{r+1} \in Dom(A)$

$$g^A(a_1, \ldots, a_r) = a_{r+1}$$

iff

$$g^B(b_1, \ldots b_r) = b_{r+1}$$

where $b_i = f(a_i)$ for $i \in [r+1]$

Important fact: (Exercise)

If A and B are σ-structures, and φ is an FO sentence over σ, then:

$$A \cong B \longrightarrow (A \models \varphi \text{ iff } B \models \varphi)$$

That is, isomorphic structures are indistinguishable in FO.

More generally:

$$\left[(A \cong B) \wedge \left(\begin{array}{c} f \text{ is an isomorphism} \\ \text{from } A \text{ to } B \end{array} \right) \wedge \left(\begin{array}{c} f(\bar{a}) = \bar{b} \\ \text{for tuples } \bar{a}, \bar{b} \text{ from} \\ A, B \text{ resp. of length} \\ |\bar{x}| \end{array} \right) \right] \longrightarrow \begin{array}{c} A \models \varphi(\bar{a}) \\ \text{iff} \\ B \models \varphi(\bar{a}) \end{array}$$

for any FO formula $\varphi(\bar{x})$.

Partial Converse of Important Fact (Exercise):

Let $A \equiv B$ denote "$A \models \varphi$ iff $B \models \varphi$ for all FO sentences φ"

Show that:

$$\left[(A \equiv B) \wedge (\text{"Either } A \text{ or } B \text{ is finite"}) \right] \longrightarrow A \cong B$$

What happens if both A & B are infinite?

Partial Isomorphism:

Let σ be a relational vocabulary. (finite)
Let $c_1, \dots c_n \in \sigma$ be an enumeration of the constant symbols of σ for $n \geq 0$.
Let A and B be σ-structures, and let $\bar{a} = (a_1, \dots, a_m)$ and $\bar{b} = (b_1, \dots, b_m)$ be m-tuples of elements from A and B resp. for $m \geq 0$. We say the partial function f from A to B defined as

$$f(a_i) = b_i \qquad \forall i \in [m]$$

is a partial isomorphism from A to B

if the (partial) function g from A to B defined as

$$g = f \cup \{ (c_i^A, c_i^B) \mid i \in [n] \}$$

is an isomorphism from $A[\bar{a}]$ to $B[\bar{b}]$.

Notes :

1. The function g is a "full" isomorphism from $A[\bar{a}]$ to $B[\bar{b}]$.

2. Explicating the behaviour of g:
$g : \mathrm{Dom}(A') \longrightarrow \mathrm{Dom}(B')$ where $A' = A[\bar{a}]$ & $B' = B[\bar{b}]$ is such that :

(a) g is a bijection.

(b) For every relation symbol $R \in \sigma$ of arity $r \geq 0$, and elements $d_1, \ldots, d_r \in \mathrm{Dom}(A')$

$$(d_1, \ldots d_r) \in R^{A'}$$
$$\text{iff}$$
$$(e_1, \ldots e_r) \in R^{B'}$$

where $e_i = g(d_i)$ for $i \in [r]$.

(c) For $i, j \in [r]$ and d_i, d_j, e_i, e_j as above,

- $d_i = d_j$ iff $e_i = e_j$
- $c_\ell^{A'} = d_i$ iff $c_\ell^{B'} = e_i$ for $\ell \in [n]$

Observe that since A' is a substructure of A & B' a substructure of B, the conditions (a) & (b) above can equivalently be written as:

(a') For every relation symbol $R \in \sigma$ of arity $r \geq 0$, and elements $d_1, \ldots, d_r \in \mathrm{Dom}(A')$

$$(d_1, \ldots d_r) \in R^A \quad \text{iff} \quad (e_1, \ldots e_r) \in R^B$$

where $e_i = g(d_i)$ for $i \in [r]$

(b') For $i, j \in [r]$ and d_i, d_j, e_i, e_j as above,

- $d_i = d_j$ iff $e_i = e_j$
- $c_l^A = d_i$ iff $c_l^B = e_i$ for $l \in [n]$

Note: There can very well be a partial isomorphism between non-isomorphic structures.

Ehrenfeucht - Fraïssé (EF) games

Game arena : Two σ- structures A & B (σ is relational)

Players : Spoiler and Duplicator
Samson and Delilah
Abelard and Eloise/Heloise
Player 1 and Player 2
Shakuni and Draupadi
Durjan Singh and Sujan Singh
Seer and Deceiver
S and D

Rounds : m

Name (denotation) of the game : $\mathcal{G}(A, B, m)$

The game proceeds as follows.

In any round $l \in [m] = \{1, \dots m\}$, the following happen:

- Player S goes first and chooses one of the structures A or B. He then chooses an element of that structure. The element chosen is called the "move" of S in the round.

- In response, player D chooses the other structure (the one not chosen by S), and picks up an element of that structure. The element picked is called the move of D in the round.

So $\mathcal{G}(A, B, m)$ begins with the move of S in round 1, and ends with the move of D in round m.

At the end of $\mathcal{G}(A, B, m)$, let

$$a_1, \ldots, a_m \in Dom(A)$$

be the elements picked / chosen by any of the players in structure A.

Likewise let,

$$b_1, \ldots, b_m \in Dom(B)$$

be the elements picked / chosen by any of the players in structure B.

Let :

$$\bar{a} = (a_1, \dots, a_m)$$
$$\bar{b} = (b_1, \dots, b_m)$$

The pair $(\bar{a}, \bar{b})$ is called the position of $\mathcal{G}(A, B, m)$ after m rounds.

Winning condition :

We say $(\bar{a}, \bar{b})$ is a winning position for player D if the function

$$f : \bar{a} \longrightarrow \bar{b}$$

defined as

$$f(a_i) = b_i \qquad \forall i \in [m]$$

is a partial isomorphism from A to B.

If f as defined above is not a partial isomorphism from A to B, then the position is said to be winning for player S.

We say player D has a **winning strategy** in $\mathcal{G}(A, B, m)$ if regardless of the moves of player S, player D can make moves s.t. the resultant position of the game at the end of m rounds is winning for player D.

The notion of winning strategy for player S is defined similarly.

Let's play!

① $\sigma = \{\}$;

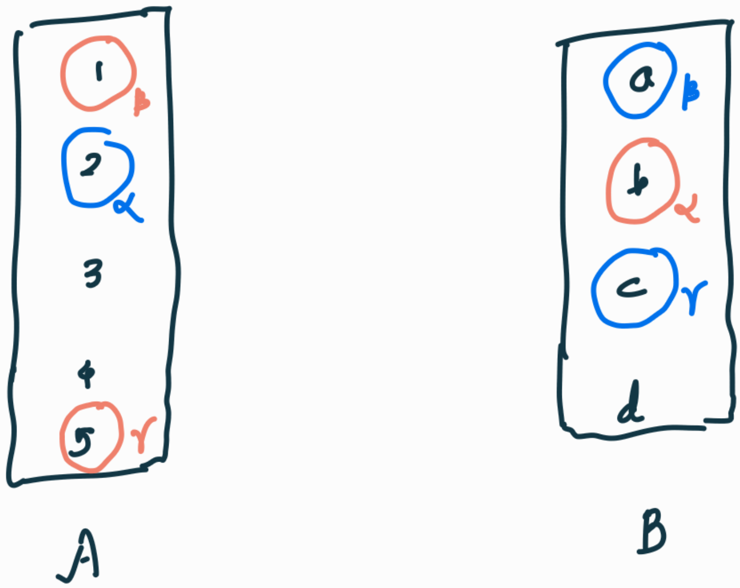
$A = \{1, 2, 3, 4, 5\}$; $B = \{a, b, c, d\}$

$m = 3$.

Play of $\mathcal{G}(A, B, 3)$

• – S (Abhisekh)

• – D (Pranshu)



$(\underset{\alpha}{\underline{2}}, \underset{\beta}{\underline{1}}, \underset{\gamma}{\underline{5}}) \in A^3$

$(\underset{\alpha}{\underline{b}}, \underset{\beta}{\underline{a}}, \underset{\gamma}{\underline{c}}) \in B^3$

$f : (2, 1, 5) \longrightarrow (b, a, c)$

is a partial iso between A & B.

So Pranshu (D) wins the game.

Indeed he has a winning strategy too.

Observe that if $m = 4$, D still always manages to win. That is D has a winning strategy.

But if $m = 5$, S has a winning strategy: in round i, S chooses the structure A & element i.

Verify that this strategy is indeed winning.

Appendix: Propositional logic as a fragment of FO

We describe below how propositional logic can be seen as a fragment of FO

Note that this is not immediate since variables in propositional logic take on Boolean values 'True' or 'False', while variables in FO take values from the domain of a given σ-structure. Since the domain can be any set, the values that FO variables can take on need not even be finite in number. Furthermore the domain might not contain the truth values 'True' & 'False' as elements.

Here is then how one can see a propositional formula as an FO formula, more specifically, an FO sentence.

$\boxed{\text{Each propositional variable is seen as a nullary predicate symbol.}}$

So for instance, the propositional formula

$$(p \wedge q) \vee \neg r$$

for prop variables p, q, r can be seen as the FO sentence

$$(P \wedge Q) \vee \neg R$$

over the vocabulary $\sigma = \{P, Q, R\}$ where P, Q, R are nullary predicate symbols.

Observe that the FO-sentence above is quantifier-free.

By the same token as above, Propositional logic formulae over a given set $\{p_1, p_2, \ldots p_n\}$ of propositional variables can be seen exactly as the set of quantifier-free FO sentences over the vocabulary

$$\sigma = \{P_1, P_2, \ldots P_n\}.$$

where the P_is are all nullary predicate symbols.

The correspondence illustrated in the example above is indeed a bijection between the mentioned sets of propositional and FO formulae/sentences.

How does the semantics of propositional logic correspond with the semantics of the above FO sentences?

Just as a k-ary predicate symbol is interpreted in any σ-structure as a set of k-tuples from the domain of the structure when $k > 0$, a nullary (0-ary) predicate symbol is interpreted in any σ-structure as a set of 0-tuples from the domain.

Since there is exactly one 0-tuple, namely the empty tuple '( )', from any domain, a nullary predicate symbol has only two possibilities for its interpretation in any σ-structure:

ⓐ the empty set { }

ⓑ the set containing the empty tuple, i.e. {( )}.

(a) above can be seen as the Boolean value 'False' while

(b) above can be seen as the Boolean value 'True'.

Observe that the domain does not play any role in the interpretation of a nullary predicate.

With the above then, for a given set of propositional variables $\{p_1, \ldots, p_n\}$, any assignment λ of Truth values to the p_i s corresponds to the σ-structure Λ defined as below for $\sigma = \{P_1 \ldots P_n\}$:

- The domain of Λ is empty.
- $P_i^{\Lambda} = \begin{cases} \{\} & \text{if } \lambda(p_i) = \text{False} \\ \{()\} & \text{otherwise.} \end{cases}$

Observe that for any propositional formula φ over $\{p_1, \ldots p_n\}$, if ψ is the FO sentence over $\sigma = \{P_1, \ldots P_n\}$ corresponding to φ as described earlier, then for any assignment λ of truth values to $\{p_1 \ldots p_n\}$,

$$\lambda \models \varphi \quad \text{iff} \quad \Lambda \models \psi$$

where

$$\Lambda \models P_i \quad \text{iff} \quad () \in P_i^{\Lambda}.$$