15 Mar '24

Descriptive Complexity

(Lecture 15)

In the last lecture, we saw:

① A simple inductive proof of the fact that the player D has a winning strategy in the EF game on m-rounds on two linear orders of lengths $\geq 2^m - 1$. Also a sketch of why the bound of $2^m - 1$ is optimal.

② Proof of the Ehrenfeucht–Fraïssé theorem: Player S's winning strategy $\longleftrightarrow$ Distinguishing sentence.

③ A characterization of the expressibility of properties of finite structures in FO. Extension to such expressibility over given classes of finite structures.

④ Examples of FO inexpressibility results in the finite.

Plan for today's lecture:

More inexpressibility results in the finite:

- Using EF games
- Using the method of interpretations

We first recall the following.

Characterization of FO expressibility

Thm: Let σ be a relational vocabulary. Let C be a class of (finite) σ-structures, and P a property of σ-structures of C. We assume C & P are isomorphism closed. Then TFAE:

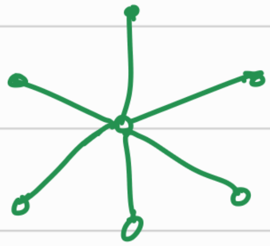
1. P is not expressible in FO over C.
2. For every $n \geq 1$, there exist σ-structures A_n & B_n s.t.
   - (i) A_n & B_n belong to C.
   - (ii) $A_n \in P$ but $B_n \notin P$
   - (iii) $A_n \simeq_n B_n$

     i.e. Player $\exists$ has a winning strategy in $\mathcal{G}(A_n, B_n, n)$.

More inexpressibility results

① **The class of even star graphs.**

Star graph $= K_{1,n}$.



$K_{1,6}$

(even star graph)

Even star graph $= K_{1,n}$ where n is even

Odd —"—— $=$ —"— n is odd.

Observe that star graphs are definable in the finite (i.e definable over finite structures)

$$\varphi_{star} := \exists x\, \forall y\, (x \neq y) \rightarrow \left(E(x,y) \wedge \forall z\, (z \neq x \rightarrow \neg E(y,z)) \right)$$

$$\wedge$$

$$\forall x\, \forall y \left[E(x,y) \rightarrow E(y,x) \wedge \neg E(x,x) \right]$$

By exactly the same reasoning as for sets we can show the following:

<u>Lemma:</u> Let $A_i = K_{1,n_i}$ for $i \in [2]$.
Let $m \geq 0$ be given. Then TFAE:

1. Player $\mathcal{D}$ has a winning strategy in $\mathcal{G}(A_1, A_2, m)$.
2. $m \leq \min\{n_1, n_2\} + 1$ or $n_1 = n_2$.

We can now show FO inexpressibility of even stars in the finite as below.

Let

$\sigma = \{E\}$

$\mathcal{C}$ = all finite σ-structures

$\mathcal{P}$ = all σ-structures in which E is the edge relation of an undirected graph (so E is irreflexive & symmetric)

and

the graph represented by E is an even star graph.

Given $n \geq 1$, let

A_n = the star graph $K_{1,2n}$

B_n = — " ——— $K_{1,2n+1}$

We verify that:

(i) $A_n, B_n \in \mathcal{C}$

(ii) $A_n \in \mathcal{P}$ but $B_n \notin \mathcal{P}$

(iii) $A_n \simeq_n B_n$

$\Rightarrow$ $\mathcal{P}$ is not FO expressible over finite σ-structures.

The same choices of A_n & B_n show also that:

(i) $\mathcal{P}$ is not FO expressible over finite (undirected) graphs

(ii) $\mathcal{P}$ is not FO expressible over star graphs

since both A_n & B_n are after all star graphs themselves.

Another way of seeing (i) & (ii) is as below:

(i) $\varphi_{\text{undirected}} := \forall x \, \neg E(x,x) \wedge$

$$\forall x \, \forall y \, \big(E(x,y) \rightarrow E(y,x)\big)$$

defines undirected finite graphs over finite σ-structures.

Then if P is definable over finite undirected graphs, by a sentence say ψ, then

$$\psi \wedge \varphi_{\text{undirected}}$$

would define P over all finite σ-structures which we saw earlier is impossible.

(ii) Likewise, if χ defined P over star graphs, then

$$\chi \wedge \varphi_{\text{star}}$$

would define P over all finite σ-structures, which again is a contradiction to our earlier result.

Exercise

Star graphs are indeed trees of height 1.

1. Show that trees of height 2 are definable in the finite in FO.
2. Show that even sized* trees of height 2 are not definable over finite σ-structures.
3. Conclude that these trees are not definable over all finite trees of height 2.

* An even sized tree is a tree whose number of nodes is even.

Transferring Inexpressibility results using definable transformations
a.k.a. interpretations

Let σ be a relational vocabulary.

Given a σ-structure A, an FO formula $\varphi(x)$ over σ with 1 free variable can be seen to define a subset S of the domain of A given by:

$$S = \varphi(A)$$
$$= \{ a \in Dom(A) \mid A \models \varphi[a] \}$$

Likewise an FO formula $\psi(x, y)$ over σ having 2 free variables can be seen to define a binary relation R on $Dom(A)$ given by

$$R = \psi(A, A)$$
$$= \{ (a_1, a_2) \in Dom(A) \times Dom(A) \mid A \models \psi[(a_1, a_2)] \}$$

If τ is a new vocabulary given by

$$\tau = \{E\}$$

for a binary predicate symbol E, then the pair $I := (\varphi(x), \psi(x,y))$ can be seen to define a τ-structure $B = I(A)$ as follows:

- $Dom(B) = S$
- $E^B = R \cap (S \times S)$

Where S & R are as defined earlier.
The pair $I := (\varphi(x), \psi(x,y))$ can then be seen to induce a definable transformation of a σ-structure into a τ-structure.
I is formally called an interpretation of τ-structures in σ-structures, or in short, a σ-τ interpretation.

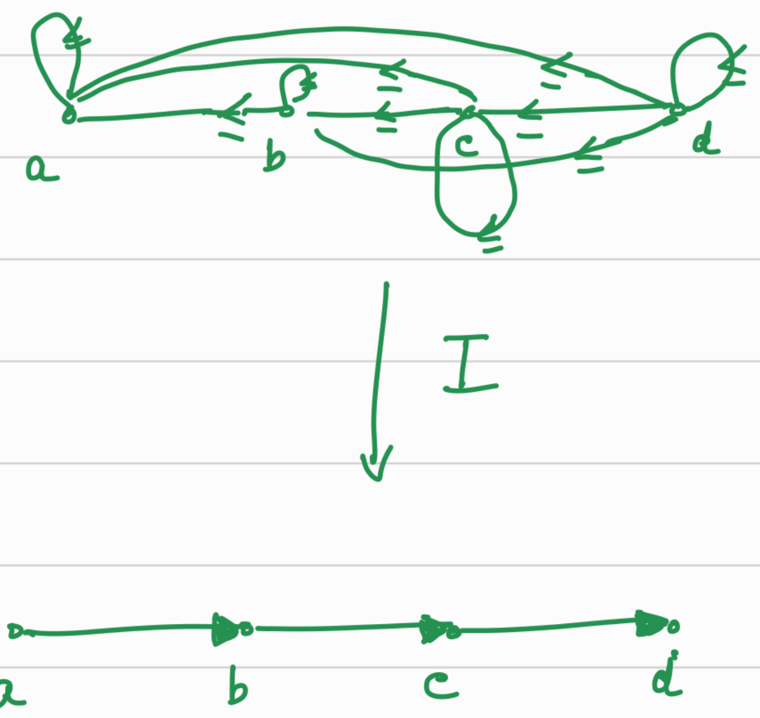
Examples:

① $\sigma = \{\leq\}$, $\tau = \{E\}$

$I = (\varphi(x), \psi(x,y))$ where

$$\varphi(x) := \text{True}$$
$$\psi(x,y) := \text{succ}(x,y)$$
$$\text{succ}(x,y) := \forall z\,(z \leq x \vee y \leq z)$$

Then if A is a linear order, I(A) is a directed path on the same vertices as A with edges as the successor relation of A



(2) $\sigma = \{\leq\}$, $\tau = \{E\}$

$I = (\varphi(x), \psi(x,y))$

$$\varphi(x) := \text{True}$$

$$\psi(x,y) := \text{2-succ}(x,y) \vee (\text{penult}(x) \wedge \text{first}(y)) \vee (\text{last}(x) \wedge \text{second}(y))$$

$$\text{2-succ}(x,y) := \exists z\,(\text{succ}(x,z) \wedge \text{succ}(z,y))$$

$$\text{penult}(x) := \exists z\,(\text{succ}(x,z) \wedge \text{last}(z))$$

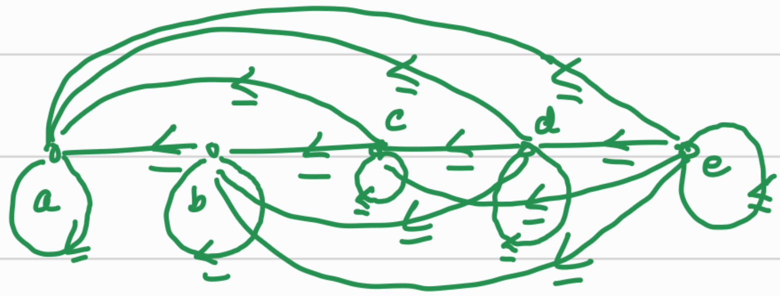
$$\text{last}(x) := \forall z\,(z \leq x)$$

$$\text{first}(x) := \forall z\,(x \leq z)$$

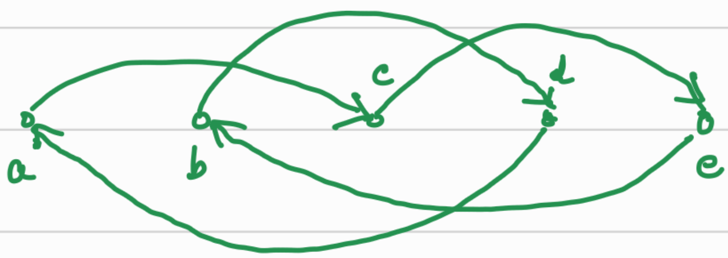
$$\text{second}(x) := \exists z\,(\text{first}(z) \wedge \text{succ}(z,x))$$

- I transforms a linear order of odd length[*] into a single directed cycle on the elements of the linear order.
- I transforms a linear order of even length into a disjoint union of two directed cycles each of length half the length of the linear order and the union of whose elements is the set of elements of the linear order.

([*] length of linear order = no. of points in the linear order)



$$\Big\downarrow I$$



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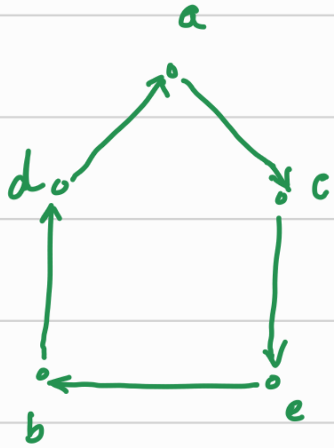
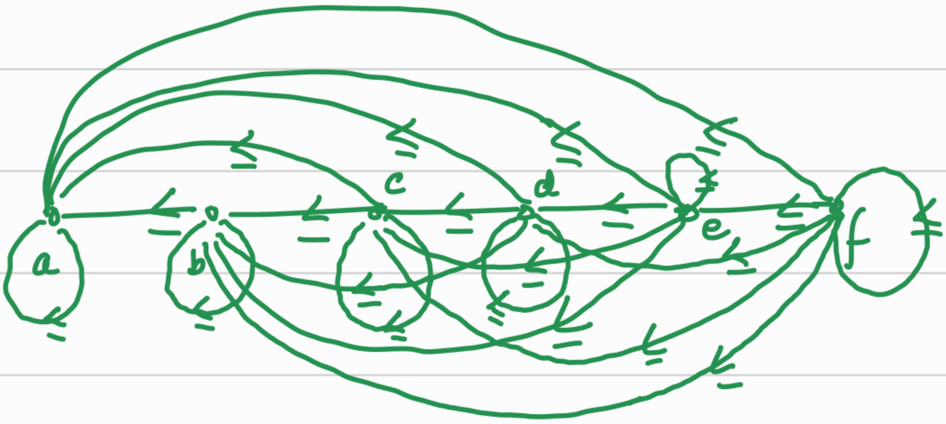
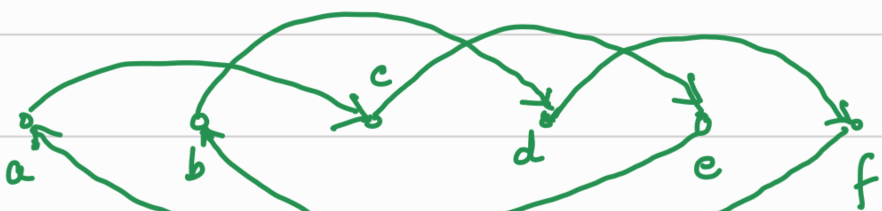




$$\Big\downarrow I$$



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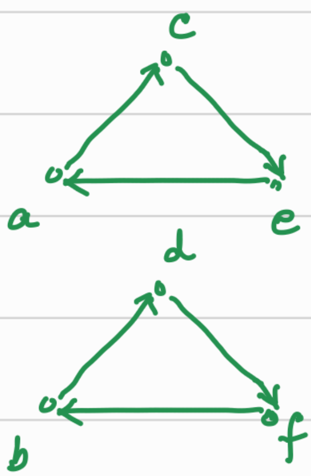

In the same way as the interpretation I transforms a σ-structure into a τ-structure, it also transforms, in the other direction, an FO sentence θ over τ into an FO sentence $I(\theta)$ over σ. (We are admittedly overloading the notation $I(\cdot)$ for the ease of exposition)

$I(\theta)$ is obtained from θ by doing the following replacements of subformulae of θ inductively:

Base Case: $E(x, y) \longmapsto \varphi(x, y)$

$$x = y \longmapsto x = y$$

Induction: $\alpha_1 \oplus \alpha_2 \longmapsto I(\alpha_1) \oplus I(\alpha_2) \quad \oplus \in \{\wedge, \vee\}$

$$\neg\alpha \longmapsto \neg I(\alpha)$$

$$(\alpha) \longmapsto (I(\alpha))$$

$$\exists x\, \alpha \longmapsto \exists x\, (\varphi(x) \wedge I(\alpha))$$

$$\forall x\, \alpha \longmapsto \forall x\, (\varphi(x) \longrightarrow I(\alpha))$$

A beautiful result about interpretations, one that is also called the fundamental theorem of interpretations, or alternatively the backwards translation theorem, is the following. We state it for the special case of $\tau = \{E\}$ for ease of exposition, though the theorem is true more generally for any relational τ.

Theorem:

Let σ be a relational vocabulary and $\tau = \{E\}$. Let $I = (\varphi(x), \psi(x,y))$ be a σ-τ interpretation. Then for any σ-structure A, and any FO sentence θ over τ:

$$A \models I(\theta) \text{ iff } I(A) \models \theta$$

The proof is not difficult and is left as an exercise. The idea is to use induction on the structure of θ.

We can now use the backwards translation theorem to transfer inexpressibility results across properties of structures. This is very much analogous to the idea of reductions from complexity theory. (We will see more of this later in the course.)

(i) Even length directed paths.

$\sigma = \{E\}$

$\mathcal{P} =$ class of σ- structures in which E is interpreted as a directed path of even length.

Suppose $\mathcal{P}$ is definable in the finite via an FO sentence β.

Let I be the interpretation in example 1 above.

Claim: $\gamma := I(\beta) \wedge$ "Axioms for linear orders" defines the class Q of even length linear orders.

Proof:

(a) Let $L \in \mathcal{Q}$.

Then $L \models$ "Axioms for linear order"

& $I(L) \in \mathcal{P}$ given the domain-preserving transformation that I induces.

By Backwards Translation theorem,

$$L \models I(\beta) \text{ iff } I(L) \models \beta$$

Since $I(L) \models \beta$ as β defines $\mathcal{P}$, we have $L \models I(\beta)$ & hence $L \models \gamma$.

(b) Let $L \models \gamma$

Then L is a linear order. Also

$L \models I(\beta)$ whereby

$I(L) \models \beta$ (backwards translation)

$\Rightarrow I(L)$ has even length.

Since $I(L)$ & L have the same domain, L has even length. Then $L \in \mathcal{Q}$. $\square$ (claim)

Since we know $\mathcal{Q}$ is inexpressible in FO, β cannot exist. Then $\mathcal{P}$ is also inexpressible in FO.

(ii) <u>Connectivity in the finite</u>

$\sigma = \{E\}$

P = The class of all (finite) σ-structures in which E is interpreted as the edge relation of an undirected finite graph that is connected.

Suppose P is definable in the finite by an FO sentence ξ. Consider the interpretation I as defined in example 2 above.

Let J be the following interpretation from τ-structures to τ-structures.

$$J = (\varphi'(x), \psi'(x,y))$$

$$\varphi'(x) := \text{True}$$
$$\psi'(x,y) := E(x,y) \vee E(y,x)$$

<u>Claim</u>: $\gamma := I(J(\xi)) \wedge$ "Axioms for linear orders" defines the class Z of odd length linear orders.

<u>Proof</u>:

ⓐ Let $L \in \mathcal{Z}$. Then L satisfies the axioms of linear orders.

Since L has odd length, $I(L)$ is a single connected directed cycle.

Then $J(I(L))$ is a single undirected cycle and hence connected.

Since ξ defines P in the finite,

$$J(I(L)) \models \xi$$

$$\Rightarrow I(L) \models J(\xi) \qquad \text{(backwards translation)}$$

$$\Rightarrow L \models I(J(\xi)) \qquad (\text{—''} \text{ ——})$$

$$\Rightarrow L \models \gamma$$

ⓑ Suppose $L \models \gamma$

Then L is a lin. order.

Also $L \models I(J(\xi))$

$$\Rightarrow I(L) \models J(\xi) \qquad \text{(backwards translation)}$$

$\Rightarrow J(I(L)) \models \xi$ (backwards translation)

$\Rightarrow J(I(L))$ is a connected undirected graph since β defines P.

$\Rightarrow I(L)$ is a directed graph whose underlying undirected graph is connected.

Recall that L is a linear order.

If L has even length, then as seen earlier, $I(L)$ is the disjoint union of two directed cycles.

Then $J(I(L))$ is the disjoint union of two undirected cycles.

In particular, $J(I(L))$ is disconnected.

But this contradicts our inference earlier that $J(I(L))$ is connected.

$\Rightarrow L$ cannot be of even length.

$\Rightarrow L$ is an odd length linear order.

$\Rightarrow L \in Z$

In summary,

$$L \in Z \quad \text{iff} \quad L \models \gamma \qquad \square_{\text{(claim)}}$$

$\Rightarrow$ γ defines Z - a contradiction to the FO inexpressibility of Z in the finite.

$\Rightarrow$ ξ does not exist.

$\Rightarrow$ ρ is inexpressible in FO in the finite.

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