Strongly continuous semigroups on some Fréchet spaces

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Joint work with L. Frerick (Universität Trier), E. Jordá (Universidad Politécnica de Valencia), and J. Wengenroth (Universität Trier)

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$$A : D(A) \to X, x \mapsto \lim_{h \downarrow 0} \frac{1}{h}(T(h)x - x),$$

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X sequentially complete $\Rightarrow$ $D(A)$ dense in X and $(A, D(A))$ has a closed graph

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$$\Rightarrow \Big(T(s)f\Big)(0) = f(s) \neq 0 = \sum_{j=0}^{\infty} \frac{s^j}{j!}(A^j f)(0)\text{''} = \Big(\exp(sA)f\Big)(0)\text{''}$$

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E. Borel: $C^\infty(\mathbb{R}) \to \mathbb{K}^{\mathbb{N}_0}, f \mapsto (f^{(j)}(0))_{j\in\mathbb{N}_0}$ surjective

$$\Rightarrow \exists\, f \in C^\infty(\mathbb{R}) : \sum_{j=0}^{\infty} \frac{s^j}{j!}(A^j f)(0) \text{ diverges for every } s > 0$$

Conejero (2007): Is every C_0-semigroup T on the Fréchet space $\omega = \mathbb{K}^{\mathbb{N}}$ of the form

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Not satisfied, e.g. by $T(s)x = (e^{ns}x_n)_{n\in\mathbb{N}}$

Let $(X_n, \pi_m^n)_{n \leq m \in \mathbb{N}}$ be a (countable) projective spectrum of Banach spaces, i.e. $(X_n, \|\cdot\|_n)$ Banach spaces with unit balls B_n,

$$\pi_m^n : X_m \to X_n,\, n \leq m \in \mathbb{N}$$

norm decreasing operators with

$$\forall\, n \leq m \leq k : \pi_m^n \circ \pi_k^m = \pi_k^n \text{ and } \pi_n^n = id_{X_n}$$

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$$X = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n;\, \pi_m^n(x_m) = x_n \text{ for all } n \leq m\}$$

its projective limit,

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$(X_n, \pi_m^n)_{n \leq m \in \mathbb{N}}$ is called strict iff π_m^n is surjective - hence open - for all $n \leq m$

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Examples:

- countable products $X = \Pi_{n=1}^{\infty} Y_n$ of Banach spaces Y_n, with $X_m := \Pi_{n=1}^{m} Y_n$ $\pi_m^n : X_m \to X_n, (x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$
- $L_{loc}^p(\Omega) \cong \Pi_{n=1}^{\infty} L_{loc}^p(K_n)$, $(K_n)_n$ compact exhaustions of $\Omega \subseteq \mathbb{R}^d$ open
- $C^n(\Omega)$ for $n \in \mathbb{N}_0$

Theorem

Let T be a semigroup on the quojection X.

i) If T is uniformly continuous then its generator is continuous and everywhere defined, and for all $x \in X$ and $s \geq 0$

$$T(s)x = \exp(sA)x = \sum_{j=0}^{\infty} \frac{s^j}{j!} A^j x.$$

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Consequence 1: $\omega = \mathbb{K}^{\mathbb{N}}$ is a quojection which is Montel

$\Rightarrow$ C_0-semigroups on ω are uniformly continuous

$\Rightarrow$ Conejero's question is answered in the affirmative

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Not true in general: $Af = f'$ generates the shift semigroup on $C^\infty(\mathbb{R})$ but $A^2 f = f''$ does not generate a C_0-semigroup on $C^\infty(\mathbb{R})$

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Remark: Countability of the strict projective spectrum is only needed to ensure that X is barrelled and that each π_m lifts bounded sets. The theorem is thus true for arbitrary strict projective limits of Banach spaces which are barrelled and satisfy this lifting property, e.g. $\mathbb{K}^I$.

For a quojection X we have

i) $\forall\, m \in \mathbb{N} : \pi_m : X \to X_m$ is surjective, hence open,

ii) (Dierolf, Zarnadze, 1984)
$\forall\, m \in \mathbb{N}\, \exists\, D_m \subseteq X$ bounded : $B_m \subseteq \pi_m(D_m)$, i.e. π_m lifts bounded sets

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i) T u.c. $\Rightarrow D(A) = X$, $A \in L(X)$, and $T(s)x = \exp(sA)x$.

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Steps of the proof:

Claim 1: T C_0 with generator $(A, D(A))$ then

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Claim 3: If $A \in L(X)$ satisfies condition in ii) then

$$\forall\, x \in X, s \geq 0 : \exp(sA)x = \sum_{j=0}^{\infty} \frac{s^j}{j!} A^j x \text{ conv. abs.+unif. on bdd sets}$$

and $(\exp(sA))_{s\geq 0}$ is u.c. semigroup with generator A

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T C_0, X barrelled $\Rightarrow$ T locally equicont., thus

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Since for every $k \in \mathbb{N}_0$

$$\forall\, x \in D(A^k) : A^k x = \frac{d^k}{ds^k} T(s) x_{|s=0}$$

Claim 1 follows.

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For $t_0 = 1$ and $n = n_0$ choose m as above. With a little trick - where it is needed that π_m lifts bounded sets: $\exists y \in X, t > 0 :$
$\pi_m(x_0) = \pi_m(C_t y)$

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$C_t \in L(X)$, $C_t(X) \subseteq D(A)$, $A(C_t x) = \frac{1}{t}(T(t)x - x)$, and
$C_t \rightarrow_{t\rightarrow 0} id_X$ unif. on bounded sets
Fix $x_0 \in X$ and $n_0 \in \mathbb{N}$. In the proof of Claim 1 we have shown

$$\forall t_0 > 0, n \,\exists\, m \,\forall\, x : (\pi_m(x) = 0 \Rightarrow \forall\, s \in [0, t_0] : \pi_n(T(s)x) = 0)$$

For $t_0 = 1$ and $n = n_0$ choose m as above. With a little trick - where it is needed that π_m lifts bounded sets: $\exists y \in X, t > 0 :$
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n_0 arbitrary $\Rightarrow x_0 \in D(A)$, x_0 arbitrary $\Rightarrow D(A) = X$, Closed Graph Theorem $\Rightarrow A \in L(X)$

Claim 3: If $A \in L(X)$ satisfies

$$(*)\ \ \forall\, n\, \exists\, m\, \forall\, k, x \in X : \pi_m(x) = 0 \Rightarrow \pi_n(A^k x) = 0.$$

then

$$\forall\, x \in X, s \geq 0 :\ \exp(sA)x = \sum_{j=0}^{\infty} \frac{s^j}{j!} A^j x \text{ conv. abs.+unif. on bdd sets}$$

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$\Rightarrow \ \|\pi_n(A^j x)\|_n = \|\tilde{A}_j(\pi_m(x))\|_n \leq \lambda^j \|\pi_m(x)\|_m$, which implies Claim 3.

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