Lecture 7a
Polytropes

Prialnik Chapter 5
Glatzmaier and Krumholz Chapter 9
Pols 4

We have reached a point where we have most of the information required to solve realistic stellar models. A critical piece still mixing is the nuclear physics, but it turns out that many of the observed properties of stars, except their lifetimes and radii, reflect chiefly the need to be in hydrostatic and thermal equilibrium, and not the energy source.

Historically, prior to computers, stellar structure was often calculated using polytropes. Even today. they provide a valuable tool for understanding many stellar properties.

The Stellar Structure Equations

$$\frac{dm}{dr} = 4\pi r^2 \rho \qquad \frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho \qquad \frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L(r)}{4\pi r^2} \qquad \frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{L(r)}{(4\pi r^2)^2}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho\varepsilon \qquad \frac{dL(m)}{dm} = \varepsilon$$

Still to be defined - ε

In addition we have or will have the physics equations

$$P = P(T, \rho, \{Y_i\}) = \left.\frac{\rho}{\mu} N_A k T\right)_{ions} + P_e + \frac{1}{3} a T^4$$

$$\kappa = \kappa(T, \rho, \{Y_i\}) \approx \kappa_0 \, \rho^a T^b$$

$$\varepsilon = \varepsilon(T, \rho, \{Y_i\}) \approx \varepsilon_0 \, \rho^m T^n \qquad (TBD)$$

and the boundary conditions

at r = 0 L(r)=0, m = 0

at r = R $P(R) \approx 0$, m= M, $T = \left(\frac{L(R)}{4\pi R^2 \sigma} \right)^{1/4} \approx 0$

These are 7 equations in 7 unknowns: ρ, T, P, L, ε, κ, and r which, in principle, given M and initial composition, can be solved as a function of time for the boundary conditions. Most of our theoretical knowledge of stellar evolution comes from doing just that.

Aside: Boundary Conditions

- Radiative Zero BC:

  Ideally we would use some atmospheric model to tell us what the temperature BC is at the surface

  The T change over the whole star is so large that the difference between 0 and the real T_{eff} at surface is small – except when computing a luminosity.

  To get effective T at the end, use:

$$T_{\rm eff} = \left(\frac{L}{4\pi R^2 \sigma} \right)^{1/4}$$

Polytropes

We assume a global relation between pressure and density:

$$P \propto \rho^{\gamma} \quad \gamma = (n+1)/n$$

n is called the polytropic index. Don't confuse this with the adiabatic index which is local.

Some examples: $P \propto \rho^{5/3}$ or $\rho^{4/3}$ $\quad n = \text{constant}$

- Fully convective (adiabatic):
- White dwarfs (completely degenerate):
- Pressure is a mix of gas + radiation, but the ratio is constant throughout

Polytropes

- Consider HSE

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho \qquad \ddot{r} = 0$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm(r)$$

- Differentiating again and dividing by r^2:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm(r)}{dr} = -4\pi G r^2 \rho(r)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G \rho(r)$$

Polytropes

- Now make this dimensionless

Central density: ρ_c

Define: θ such that $\rho(r) = \rho_c \theta^n(r)$

Then:

$$P(r) = K\rho^{\gamma}(r)$$
$$= K\ \rho_c^{\gamma}\,\theta^{n\gamma}(r)$$
$$= K\,\rho_c^{1+1/n}\,\theta^{n+1}(r)$$

$$\gamma = \frac{n+1}{n} = 1 + \frac{1}{n}$$

Given n, K, and ρ_c these two equations define the distribution of pressure and density in the star

Use these assumptions in the equation for hydrostatic equilibrium

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \qquad P = K\rho^\gamma = K\rho_c^\gamma\theta^{\gamma n} = K\rho_c^\gamma\theta^{n+1}$$

$$\frac{K\rho_c^\gamma}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n}\frac{d\theta^{n+1}}{dr}\right) = -4\pi G\rho_c\theta^n \qquad \rho = \rho_c\theta^n$$

$$P_c\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\theta^n}\frac{d\theta^{n+1}}{dr}\right) = -4\pi G\rho_c\theta^n$$

$$\frac{P_c}{\rho_c^2}\frac{(n+1)}{r^2}\frac{d}{dr}\left(\frac{r^2\theta^n}{\theta^n}\frac{d\theta}{dr}\right) = -4\pi G\theta^n$$

$$\left[\frac{P_c(n+1)}{4\pi G\rho_c^2}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

$$\left[\frac{P_c(n+1)}{4\pi G\rho_c^2}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

Note that the quantity in brackets has units length^2

Define it to be α^2 then

$$\frac{\alpha^2}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$ with $$\alpha = \left[\frac{P_c(n+1)}{4\pi G\rho_c^2}\right]^{1/2}$$

$$\frac{dyne\, gm^2\, cm^6}{cm^2\, dyne\, cm^2 gm^2} = cm^2$$ for the dimension of α^2

Now define $r = \alpha\xi$ with ξ a dimensionless radius-like variable. Then

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

This is the Lane Emden equation. It involves only dimensionless quantities and can be solved for a given n to give $\theta(\xi)$. n does not have to be an integer.

Basically all that went into it was hydrostatic equilibrium, mass conservation, and a power law equation of state

Polytrope boundary conditions

1) $\theta^n = \dfrac{\rho}{\rho_c}$ So at $\dfrac{r}{\alpha} = \xi = 0 \quad \theta = 1$

2) $\xi \equiv \dfrac{r}{\alpha}$ so $d\xi = \dfrac{1}{\alpha} dr \Rightarrow \dfrac{d\theta}{d\xi} = \alpha \dfrac{d\theta}{dr}$

but $\rho \equiv \rho_c \theta^n$ so $\dfrac{d\rho}{dr} = n\rho_c\, \theta^{n-1} \dfrac{d\theta}{dr}$

$\Rightarrow \quad \dfrac{d\theta}{dr} = \dfrac{1}{n\rho_c\, \theta^{n-1}} \dfrac{d\rho}{dr}$ and $\dfrac{d\theta}{d\xi} = \dfrac{\alpha}{n\rho_c\, \theta^{n-1}} \dfrac{d\rho}{dr}$

but in spherical symmetry for hydrostatic equilibrium

ρ is a local maximum at r = 0, so $\dfrac{d\rho}{dr} = 0$

So at at $\xi = 0 \quad \dfrac{d\theta}{d\xi} = 0$

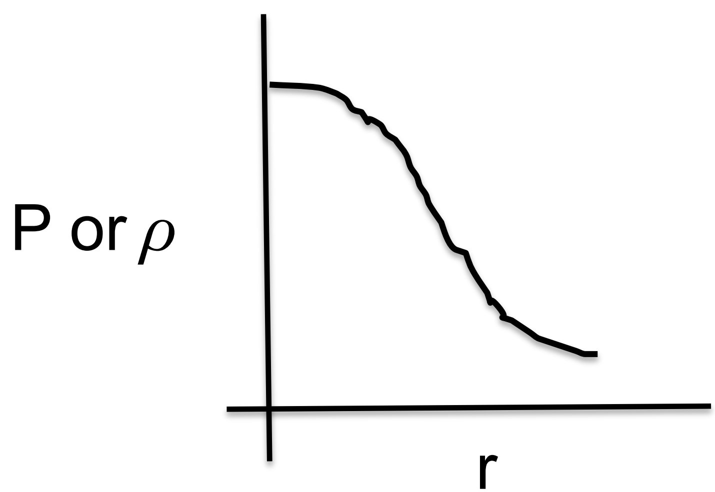
That is

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2} \rightarrow 0 \text{ as } r \rightarrow 0$$

because m(r) $\propto r^3$ approaches 0 faster than r^2

And since $P = K\rho^\gamma$ $\quad \frac{dP}{dr} = \gamma K \rho^{\gamma-1} \frac{d\rho}{dr}$

so $\frac{d\rho}{dr}$ also $\rightarrow 0$. The density and pressure have local maxima at the center of the star - no "cusps"



Polytrope boundary conditions

3) At the surface r = R the density (first) goes to zero so

$R_* = \alpha\xi_1$ where ξ_1 is where $\theta(\xi)$ first reaches 0

For any value of n, the mass

$$M = \int_0^R 4\pi r^2 \rho\, dr = 4\pi\alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n\, d\xi$$

$r = \alpha\xi \qquad R = \alpha\xi_1$

$\rho = \rho_c \theta^n$

But the Lane Emden equation says

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n \quad \text{or} \quad \xi^2\theta^n = -\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right)$$

So $\quad \mathrm{M} = -4\pi\alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi$

$$= -4\pi\alpha^3 \rho_c \int_0^{\xi_1} d\left(\xi^2 \frac{d\theta}{d\xi}\right) = -4\pi\alpha^3 \rho_c\, \xi_1^2 \left.\frac{d\theta}{d\xi}\right)_{\xi_1} = M$$

$\frac{d\theta}{d\xi} = 0$ at $\xi = 0$

The quantity $\xi_1^2 \left.\frac{d\theta}{d\xi}\right)_{\xi_1}$ is uniquely determined by n and

is given in tables. So we have an equation connecting

M, α. and ρ_c for a given polytropic index, n.

Cox and Guilli (1965)

	ξ_1	$\xi_1^2 \frac{d\theta}{d\xi}$	$D_n = \frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
1	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2	4.35287	2.41105	11.40254
3	6.89685	2.01824	54.1825
4	14.97155	1.79723	622.408
4.5	31.8365	1.73780	6189.47
5	∞	1.73205	∞

Further α can be expressed in terms of ρ_c and K

$$\alpha = \left[\frac{P_c(n+1)}{4\pi G\rho_c^2}\right]^{1/2} = \left[\frac{K\rho_c^{\frac{n+1}{n}}(n+1)}{4\pi G\rho_c^2}\right]^{1/2} = \left[\frac{K\rho_c^{\frac{1-n}{n}}(n+1)}{4\pi G}\right]^{1/2}$$

$$M = -4\pi\alpha^3\rho_c\,\xi_1^2\left.\frac{d\theta}{d\xi}\right)_{\xi_1} = -4\pi\left[\frac{K\rho_c^{\frac{1-n}{n}}(n+1)}{4\pi G}\right]^{3/2}\rho_c\,\xi_1^2\left.\frac{d\theta}{d\xi}\right)_{\xi_1}$$

$$= -\frac{(n+1)^{3/2}}{\sqrt{4\pi}}\xi_1^2\left.\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}\rho_c^{\frac{3-3n}{2n}+1}$$

$$M = -\frac{(n+1)^{3/2}}{\sqrt{4\pi}}\xi_1^2\left.\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}\rho_c^{\frac{3-n}{2n}}$$

which is another useful form which gives M in terms of ρ_c (or vice versa) for a given K and n. Note that M is independent of ρ_c if n = 3.

Another interesting quantity that can be obtained from the tables is the ratio of central density to average density - how compact the core of the star is.

$$
\begin{aligned}
D_n &\equiv \frac{\rho_c}{\overline{\rho}} \\
&= \rho_c \frac{4\pi R^3}{3M} \\
&= \frac{4\pi}{3}\rho_c \left(\alpha\xi_1\right)^3 \left[-4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\Theta}{d\xi}\right)_{\xi_1}\right]^{-1} \\
&= -\left[\frac{3}{\xi_1}\left(\frac{d\Theta}{d\xi}\right)_{\xi_1}\right]^{-1}
\end{aligned}
$$

See the previous table. The case n = 0, $D_n = 1$ is the sphere of constant density case

Analytic solutions

For $n = 0, 1, 5$, and only for these values, there exist analytic solutions to the Lane Emden equation.

Unfortunately none of them correspond to common stars, but the solutions help to demonstrate how polytropes can be solved and they can used for interesting approximate cases.

n = 0

Technically the case n = 0 is a singularity

$$P = K\rho^{1+1/n} \quad \text{diverges as n} \to 0$$

This reflects the fact that constant density can only be maintailed in the face of gravity if the fluid is incompressible (like the ocean; P can vary but not ρ).

Nevertheless, some interesting properties of polytropes can be illustarted with n = 0 so long as we don't use the pressure-density relation explicitly

n = 0 - the constant density case

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n = -1$$

$$\xi^2\frac{d\theta}{d\xi} = -\int\xi^2 = -\frac{\xi^3}{3} + C$$

$$\frac{d\theta}{d\xi} = -\frac{\xi}{3} + \frac{C}{\xi^2} \Rightarrow \int d\theta = -\frac{1}{3}\int\xi\, d\xi + \int\frac{C d\xi}{\xi^2}$$

$$\theta = -\frac{\xi^2}{6} - \frac{C}{\xi} + D$$ but $\theta=1$ at $\xi=0$

So C = 0 and D =1 and

$$\theta = 1 - \frac{\xi^2}{6}$$

http://www.vistrails.org/index.php/User:Tohline/SSC/Structure/Polytropes

The first zero of $\theta = 1 - \dfrac{6}{\xi^2}$ is at $\xi_1 = \sqrt{6}$

$$\xi = \frac{r}{\alpha}$$

so $\quad R = \sqrt{6}\,\alpha$

$$P(r) = P_c \theta^{n+1}(r) = P_c \theta(r) \quad \text{for n = 0}$$

$$P(r) = P_c\left(1 - \frac{\xi^2}{6}\right) = P_c\left(1 - \frac{r^2}{\alpha^2 6}\right)$$

but $\alpha = \dfrac{R}{\xi_1} = \dfrac{R}{\sqrt{6}}$ so

$$P(r) = P_c\left(1 - \frac{r^2}{R^2}\right)$$

For ideal gas pressure and constant composition

T would have the same dependence (ρ=constant)

$$P = \frac{\rho N_A k T}{\mu} \Rightarrow T(r) = T_c\left(1 - \frac{r^2}{R^2}\right)$$

but also $\alpha = \left[\frac{P_c(n+1)}{4\pi G\rho_c^2}\right]^{1/2} = \left[\frac{P_c}{4\pi G\rho_c^2}\right]^{1/2} = \frac{R}{\sqrt{6}}$ so

$$P_c = \frac{2\pi R^2 G\rho_c^2}{3} = \frac{G\left(\frac{4}{3}\pi R^3\rho_c\right)\rho_c}{2R} = \boxed{\frac{GM\rho}{2R}}$$

This agrees with what can be obtained by integration of the equation of hydrostatic equilibrium.

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

$$\int_c^{surf} dP = P_c = -GM\rho\int_0^R \frac{dr}{r^2} = \frac{GM\rho}{2R}$$

but from the polytropic equation we learn how P varies with r

Recall M= $-4\pi\alpha^3\rho_c\,\xi_1^2 \left.\frac{d\theta}{d\xi}\right)_{\xi_1}$

This last quantity, $\xi_1^2 \left.\frac{d\theta}{d\xi}\right)_{\xi_1}$, is a number that

depends on the poytropic index n. For n = 0 it is just -2$\sqrt{6}$ (see table) and so

$$\text{M} = 4\pi \frac{R^3}{6\sqrt{6}} 2\sqrt{6}\,\rho_c = \frac{4\pi}{3} R^3 \rho$$

as one would expect for constant density. So we understand sars of constant density quite well.

Other analytic solutions exist for n = 1 and 5

n = 1

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta$$

The solution is order zero Spherical Bessel function

$$\theta(\xi) = \frac{Sin\,\xi}{\xi} \Rightarrow \xi_1 = \pi \text{ at } \theta = 0 \qquad \alpha = \frac{R}{\pi}$$

$$nb.\ \lim_{\xi \to 0} \frac{Sin\,\xi}{\xi} = 1 \quad \text{(l'Hopital's rule)}$$

n=1 continued

$$\alpha = \left[\frac{P_c(n+1)}{4\pi G\rho_c^2}\right]^{1/2} = \left[\frac{P_c}{2\pi G\rho_c^2}\right]^{1/2} = \frac{R}{\pi}$$

So $$\mathrm{P}_c = \frac{2G}{\pi}\rho_c^2 R^2$$

Also $$\xi \equiv \alpha \mathrm{r} = \frac{\pi r}{R} \quad \Rightarrow \quad \theta = \frac{Sin(\xi)}{\xi} = \frac{R}{\pi r}\ Sin(\frac{\pi r}{R})$$ so

$$P = P_c\theta^2 == P_c\left(\frac{R}{\pi r}Sin\left(\frac{\pi r}{R}\right)\right)^2$$

$$\rho = \rho_c\theta = \rho_c\left(\frac{R}{\pi r}Sin\left(\frac{\pi r}{R}\right)\right)$$

n=1 continued

$$M = -4\pi\alpha^3\rho_c\,\xi_1^2\left.\frac{d\theta}{d\xi}\right)_{\xi_1}$$

$$\xi_1^2\left.\frac{d\theta}{d\xi}\right)_{\xi_1} = -\pi \quad \text{for n = 1}$$

$$\text{so M} = -4\pi\alpha^3\rho_c\,\xi_1^2\left.\frac{d\theta}{d\xi}\right)_{\xi_1} = 4\pi^2\rho_c\left(\frac{R}{\pi}\right)^3 = \frac{4}{\pi}\rho_c R^3$$

$$\text{but} \quad P_c = \frac{2G}{\pi}\rho_c^2 R^2 \qquad \text{so } R^3 = \left(\frac{\pi P_c}{2G\rho_c^2}\right)^{3/2}$$

$$\text{and } P_c = K\rho_c^{\frac{n+1}{n}} = K\rho_c^2 \qquad \text{so } R^3 = \left(\frac{\pi K}{2G}\right)^{3/2}$$

$$R = \left(\frac{\pi K}{2G}\right)^{1/2}$$

independent of M!

n=1 continued

The density adjusts to keep the same radius no matter what M is. R is independent of M.

This may seem a bit strange but actally neutron stars are approximately polytropes with index $0.5 < n < 1$. Central density rises with M but radius does not vary much.

n = 5

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^5$$

$$\theta = \left[1 + \frac{1}{3}\xi^2\right]^{-1/2} \quad \text{which} \rightarrow 0 \text{ only as } \xi \rightarrow \infty$$

$$\frac{\rho}{\rho_c} = \left[1 + \frac{1}{3}\xi^2\right]^{-5/2} \qquad M = \left[\frac{2 \bullet 3^4\ K^3}{\pi G^3}\right]\rho_c^{-1/5}$$

see 12 pages back

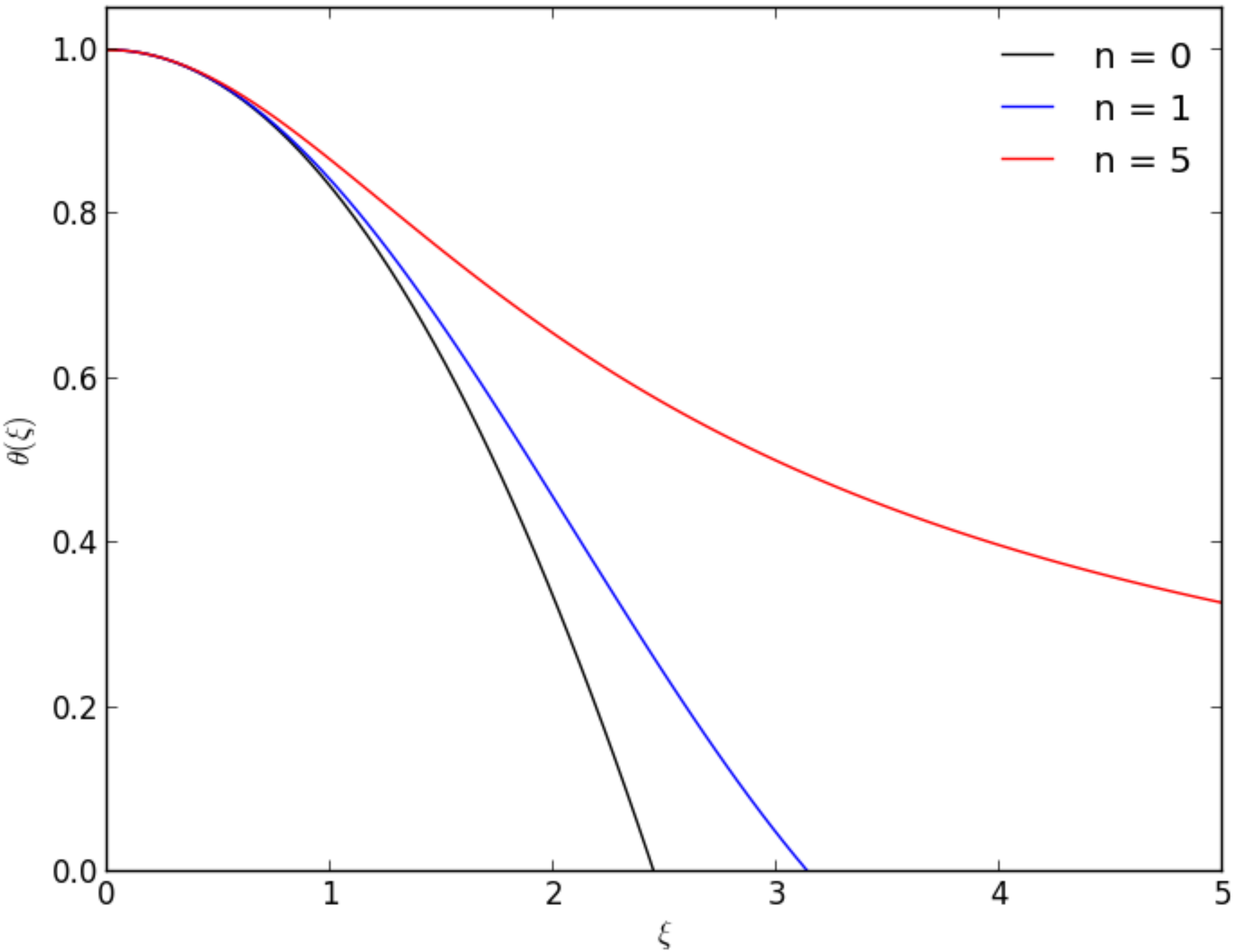
$$P_c = \left[\frac{\pi M^2 G^3}{2 \bullet 3^4}\right]^{1/3} \rho_c^{4/3}$$

This configuration, which is not very physical has an infinite radius and a finite mass, central density and central pressure. It is infinitely centrally concentrated and we will see later has infinite binding energy. There are no solutions, analytic or otherwise for n > 5. Most physically relevant polytropes have 1.5 < n < 3.

Analytic Solutions

Note tendency of all to agree with n = 0 at the origin

$\theta(\xi)$



The General Case
(0 < n < 5)

The general solution requires numerical integration of the Lane Emden equation. There are tools available for this purpose. A simple program is also given at the class website

http://nucleo.ces.clemson.edu/home/online_tools/polytrope/0.8/

From Pols (2011)

In his notation $z_n = \xi_1 \quad \theta_n = -\xi_1^2 \left(\frac{d\theta}{d\xi} \right)_{\xi_1}$

Table 4.1. Numerical values for polytropic

n	z_n	Θ_n	$\rho_c/\bar{\rho}$
0	2.44949	4.89898	1.00000
1	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2	4.35287	2.41105	11.40254
3	6.89685	2.01824	54.1825
4	14.97155	1.79723	622.408
4.5	31.8365	1.73780	6189.47
5	∞	1.73205	∞

In the general case, we know M, n, and K (e.g. from the EOS):

First get α from the given mass

$$M = -4\pi\alpha^3 \left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^{n/(n-1)} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1} \Rightarrow \alpha \text{ as function of M, K, and n}$$

Where we have replaced ρ_c using the definition of α

$$\alpha = \left[\frac{P_c(n+1)}{4\pi G\rho_c^2}\right]^{1/2} = \left[\frac{K\rho_c^{(n+1)/n}(n+1)}{4\pi G\rho_c^2}\right]^{1/2} = \left[\frac{K\rho_c^{(1-n)/n}(n+1)}{4\pi G\rho_c^2}\right]^{1/2}$$

$$\rho_c = \left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^{n/(n-1)}$$

Once one has n, α and K, $\rho_c, \bar{\rho},$ and R easily follow.

$$\rho_c = \left[\frac{(n+1)K}{4\pi G\alpha^2} \right]^{n/(n-1)}$$

$$\frac{\rho_c}{\bar{\rho}} = D_n$$

$$R = \left(\frac{3M}{4\pi\bar{\rho}} \right)^{1/3}$$

Note the existence of a mass-radius relation

$$M = -4\pi\alpha^3 \left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^{n/(n-1)} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

$$\left[\frac{M}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)} = \left(4\pi\alpha^3\right)^{(n-1)}\left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^n$$

$$\left[\frac{M}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)} = \left(4\pi\right)^{-1}\alpha^{3n-3-2n}\left[\frac{(n+1)K}{G}\right]^n$$

move α to other side

$$\left[\frac{M}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)} \alpha^{3-n} = \frac{1}{4\pi G}\left(\frac{1}{G}\right)^{n-1}\left((n+1)K\right)^n$$

$$\left[\frac{M}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)}\alpha^{3-n}=\frac{1}{4\pi G}\left(\frac{1}{G}\right)^{n-1}\left((n+1)K\right)^{n}$$

move G^{n-1} to other side

$$\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)}\alpha^{3-n}=\frac{\left[(n+1)K\right]^{n}}{4\pi G}$$

Substituting the value of R $\alpha = R/\xi_1$

$$\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n}=\frac{\left[(n+1)K\right]^{n}}{4\pi G}$$

For a given n and K, the right hand side is constant and

$M \sim R^{(n-3)/(n-1)}$ *or* $R \sim M^{(n-1)/(n-3)}$

$$\left[\frac{GM}{-\xi_1^2\left(d\theta / d\xi\right)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n} = \frac{\left[\left(n+1\right)K\right]^n}{4\pi G}$$

Note the existence of "singularities" at n = 1 and n =3

For n = 1 the mass dependence drops out and the radius is independent of the mass (the central density just adjusts).

Even more interesting for n = 3, the radiis drops out and one just a critical mass whose radius is undefined. This will have important implications for the maximum mass of white dwarfs and for massive stars.

Mass radius relation for white dwarfs:

For a non-relativistic white dwarf γ=5/3 which implies n = 3/2

$$\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{1/2}\left(\frac{R}{\xi_1}\right)^{3/2}=\frac{\left[\left(5/2\right)K\right]^{3/2}}{4\pi G}$$

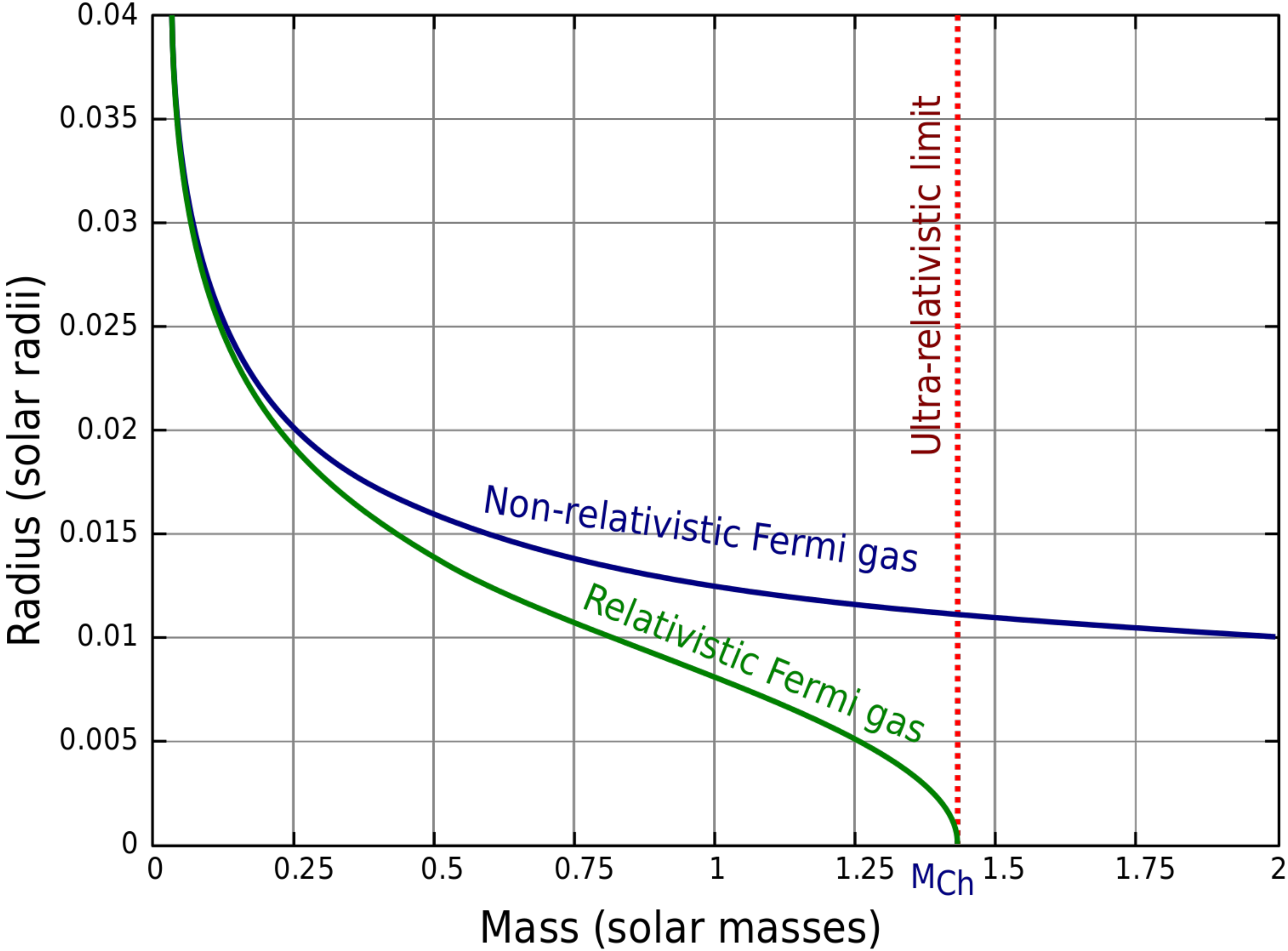
$$\left(\frac{R}{\xi}\right)^{3/2}=\frac{\left[\left(5/2\right)K\right]^{3/2}}{4\pi G}\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{-1/2}$$

$$\left(\frac{R}{\xi}\right)=\frac{\left[\left(5/2\right)K\right]}{\left(4\pi\right)^{2/3}G}\left[\frac{M}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{-1/3} \qquad K=1.00\times10^{13}\ Y_e^{5/3}$$

$$\xi_1=3.654$$

$$R=\frac{3.654\left[\left(2.5\right)\left(1.00\times10^{13}\right)\right]}{\left[\left(4\pi\right)^{2/3}\left(6.67\times10^{-8}\right)\right]}\left(2.714\right)^{1/3}\ Y_e^{5/3}M^{-1/3} \qquad -\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}=2.714$$

$$=8800\ km\left(\frac{Y_e}{0.5}\right)^{5/3}\left(\frac{M_\odot}{M}\right)^{1/3}=0.0127\ R_\odot\left(\frac{Y_e}{0.5}\right)^{5/3}\left(\frac{M_\odot}{M}\right)^{1/3}$$



http://en.wikipedia.org/wiki/White_dwarf

The Chandrasekhar Mass

$$\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n}=\frac{\left[\left(n+1\right)K\right]^n}{4\pi G}$$

If fully relativstic throughout, $P = K\rho^{4/3}$ and n = 3

$$\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{2}=\frac{\left[4K\right]^3}{4\pi G}$$

$$\frac{M}{2.01824}=\frac{\left(4K\right)^{3/2}}{\left(4\pi\right)^{1/2}G^{3/2}}$$

$$M_{Ch}=\frac{\left(2.01824\right)\left(\left(4\right)\left(1.2435\times10^{15}Y_e^{4/3}\right)\right)^{3/2}}{\left(4\pi\right)^{1/2}\left(6.67\times10^{-8}\right)^{3/2}}$$

$$= 1.456\left(\frac{Y_e}{0.5}\right)^2 M_\odot$$

Another useful expression is for the central pressure

$$P_c = K\rho_c^{(n+1)/n}$$

Use the previous equation for the mass-radius relation to solve for K as a function of R and M and put it in the equation for P_c

$$\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n} = \frac{\left[\left(n+1\right)K\right]^n}{4\pi G}$$

$$P_c = K\rho_c^{(n+1)/n} = \left[\frac{(4\pi G)^{1/n}}{(n+1)}\right]\left[\frac{GM}{-\xi_1^2\left(d\theta/d\xi\right)_{\xi_1}}\right]^{(n-1)/n}\left(\frac{R}{\xi_1}\right)^{(3-n)/n}\rho_c^{(n+1)/n}$$

But $\rho_c = \bar{\rho} D_n = \frac{3M}{4\pi R^3}\left(-\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1\right)^{-1}$ so we have R in terms of ρ_c and M

$$\left(\frac{R}{\xi_1}\right)^3 = \frac{3M}{4\pi\rho_c\xi_1^3\left[\frac{-3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1\right]} \qquad \left(\frac{R}{\xi_1}\right)^{(3-n)/n} = \left[\frac{3M}{4\pi\rho_c\xi_1^3\left[\frac{-3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1\right]}\right]^{(3-n)/3n}$$

So we can obtain P_c as a function of just M and ρ_c

$$P_c = \frac{1}{n+1}(4\pi)^{1/n-1/n+1/3} G^{1/n+1-1/n} M^{1-1/n+1/n-1/3} \xi_1^{-2+2/n-2/n+2/3} \left(\frac{d\theta}{d\xi}\right)_{\xi_1}^{-1+1/n-1/n+1/3} \rho_c^{1+1/n-1/n+1/3}$$

$$P_c = \frac{(4\pi)^{1/3} G}{n+1} M^{2/3} \rho_c^{4/3} \xi_1^{-4/3} \left(\frac{d\theta}{d\xi}\right)_{\xi_1}^{-2/3}$$

$$P_c = C_n G M^{2/3} \rho_c^{4/3}$$

$$C_n = \frac{(4\pi)^{1/3}}{n+1} \left[\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-2/3}$$

Pols. p 49 (4.18)

which is a slowly varying function of n

$C_{n=1} = 0.542$ $\quad$ $C_{n=2} = 0.431$

$C_{n=3} = 0.364$

$$P_c = C_n G M^{2/3} \rho_c^{4/3}$$

1) For a given polytropic index and mass the ratio $\dfrac{P_c^3}{\rho_c^4}$ is a constant as the star expands or contracts.
2) For a given polytropic index this ratio increases as M^2
3) For a given mass this ratio does not vary much across a reasonable range of polytropic indices $1.5 \leq n \leq 3$

Important example: Suppose P = $P_{ideal} = \dfrac{\rho N_A k T}{\mu}$

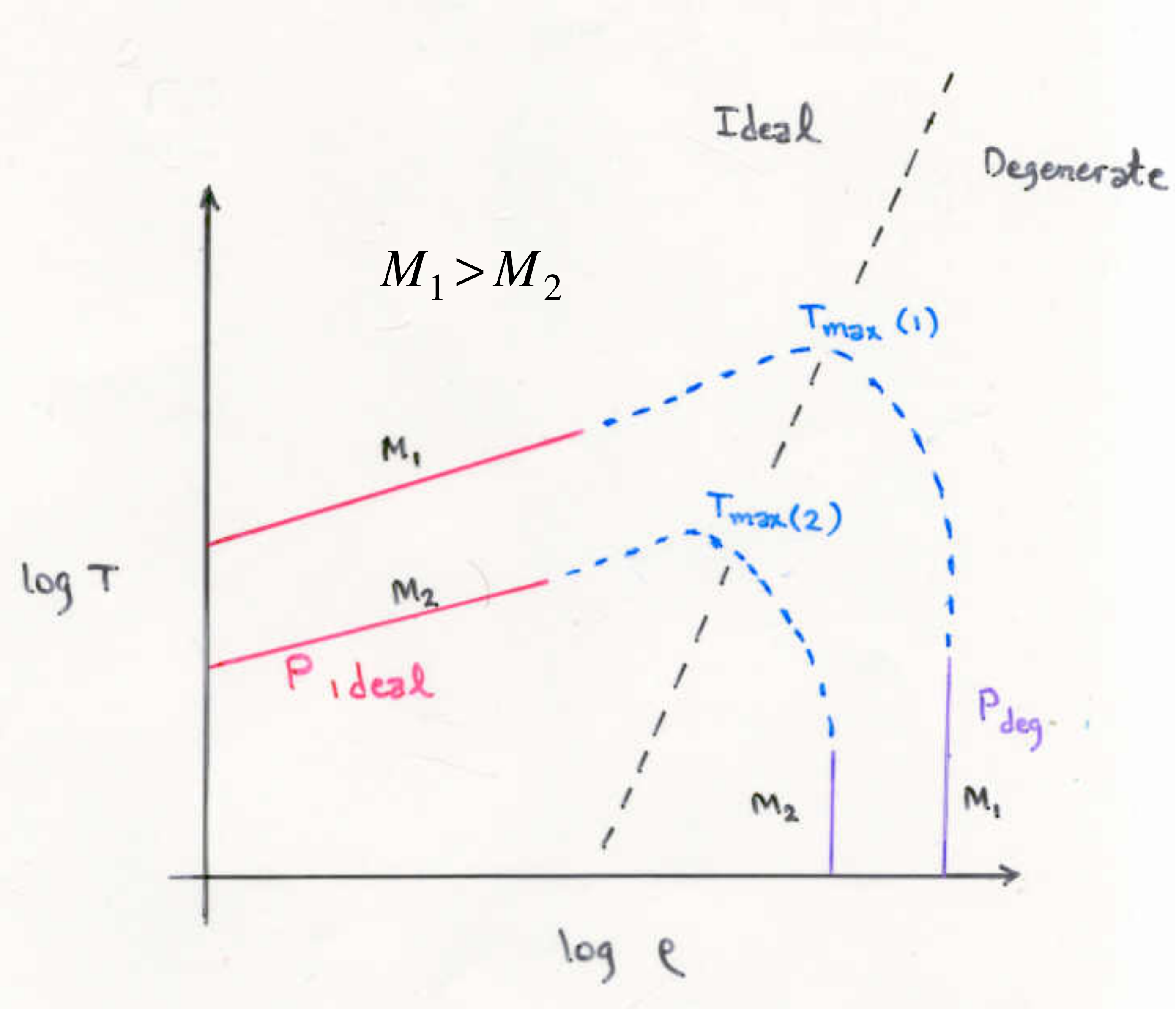
Then $\dfrac{T_c^3}{\rho_c}$ in a contracting polytrope is a constant (that increases with mass). Stellar cores will evolve trying to keep $\rho_c \propto T_c^3$ and higher mass stars will have a higher central temperature at a given central density

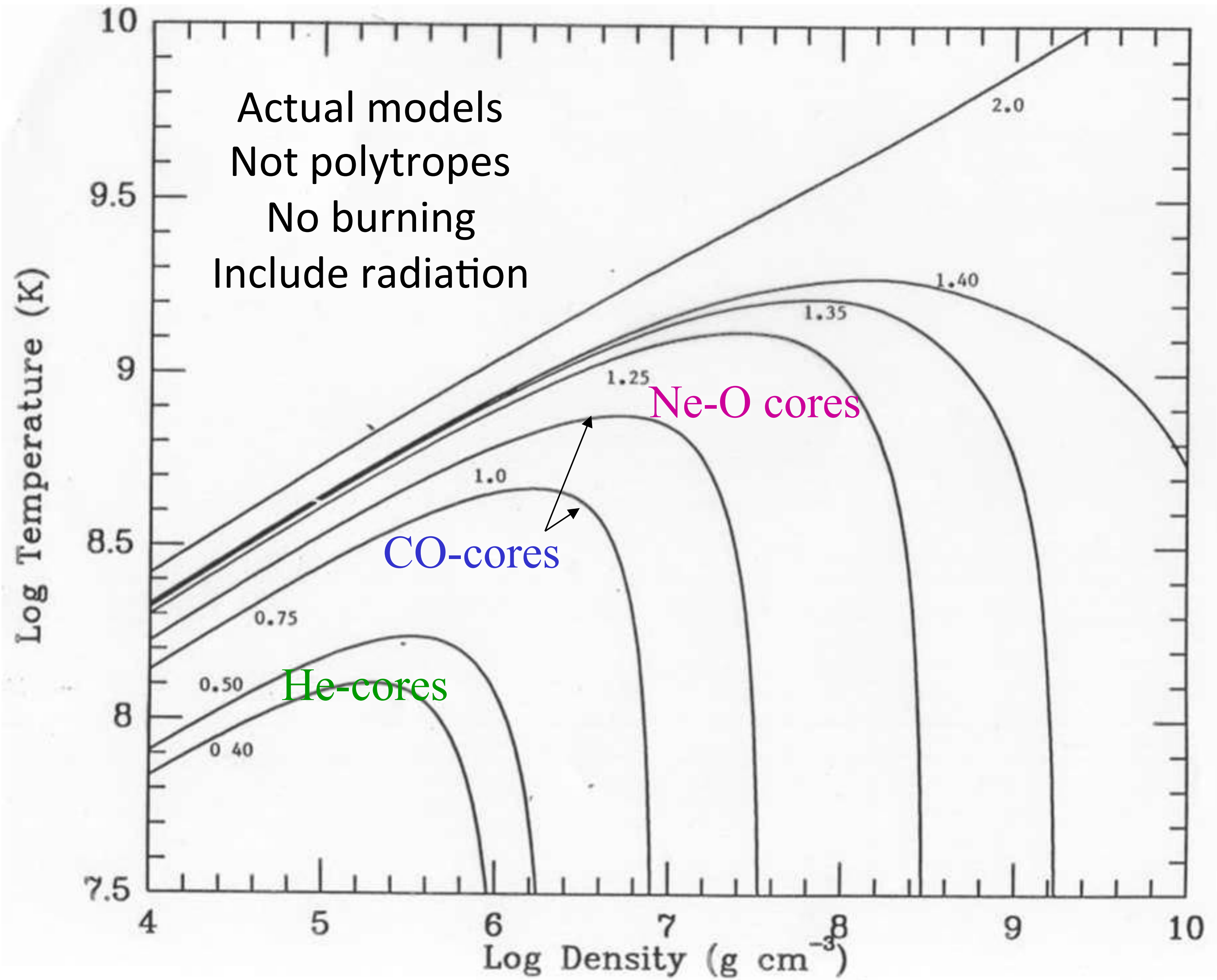
$$P_c = C_n G M^{2/3} \rho_c^{4/3}$$

$$P_c^3 = C_n^3 G^3 M^2 \rho_c^4$$

$$\left(\frac{N_A k T_c}{\mu C_n G} \right)^3 = M^2 \rho_c$$

$$T_c^3 \propto M^2 \rho_c$$


Ideal
Degenerate
$M_1 > M_2$
T_{max} (1)
M_1
T_{max}(2)
log T
M_2
P ideal
P_{deg}
M_2
M_1
log ρ



Actual models
Not polytropes
No burning
Include radiation
2.0
1.40
1.35
1.25
Ne-O cores
1.0
CO-cores
0.75
0.50
He-cores
0 40
10
9.5
9
8.5
8
7.5
4
5
6
7
8
9
10
Log Temperature (K)
Log Density (g cm^-3)

Critical Masses

0.08 $M_\odot$ Lower limit for hydrogen ignition

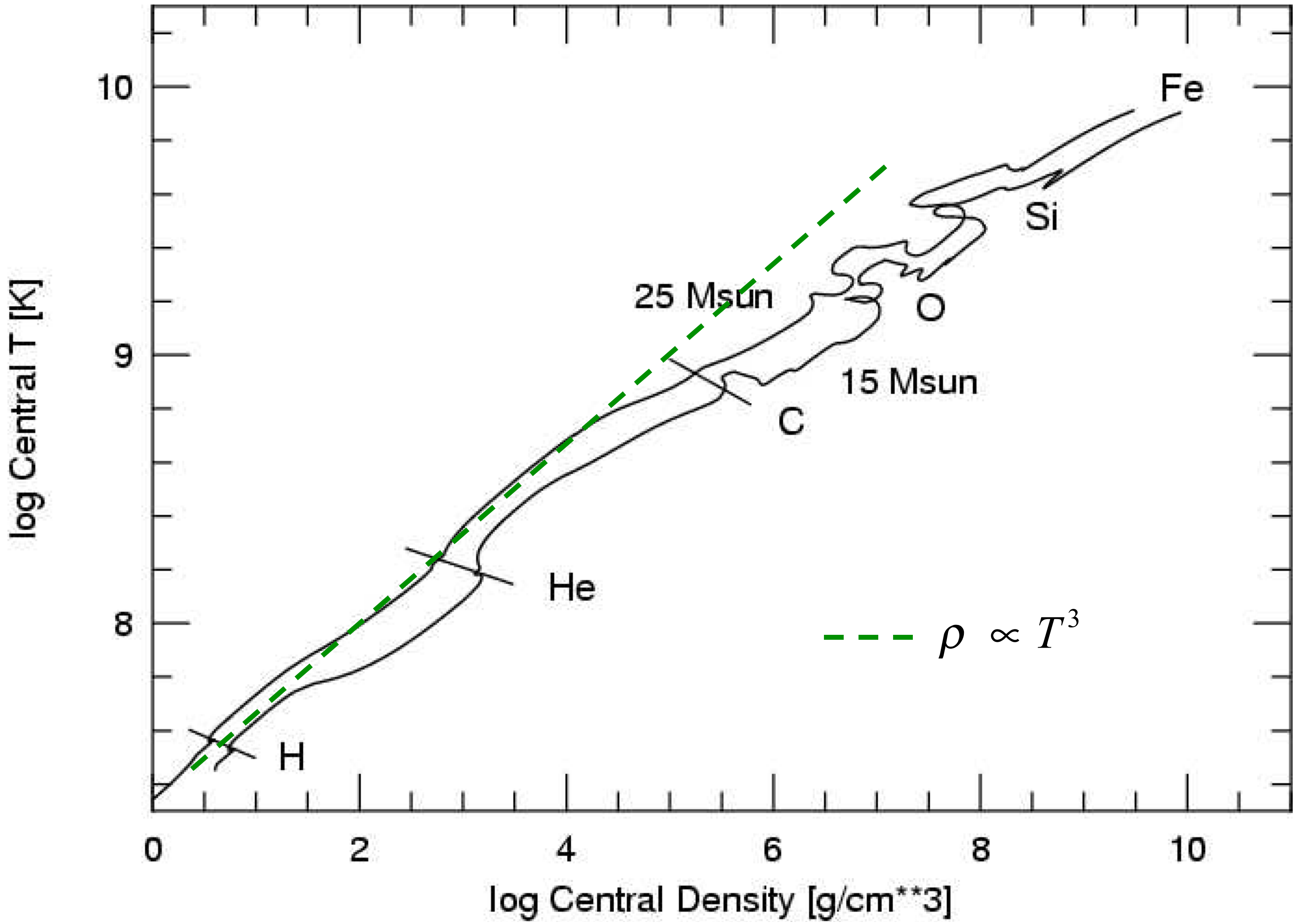
0.45 $M_\odot$ helium ignition

7.25 $M_\odot$ carbon ignition

9.25 $M_\odot$ neon, oxygen, silicon ignition (off center)

~11 $M_\odot$ ignite all stages at the stellar center

These are for models that ignore rotation. With rotation the numbers may be shifted to lower values. Low metallicity may raise the numbers slightly since less initial He means a smaller helium core.



Fe
Si
O
25 Msun
15 Msun
C
He
H
$\rho \propto T^3$
10
9
8
0
2
4
6
8
10
log Central T [K]
log Central Density [g/cm**3]

Temperature as functions of ρ_c

One can solve explicitly for the central temperature in terms of the density if the pressure is entirely due to ideal gas

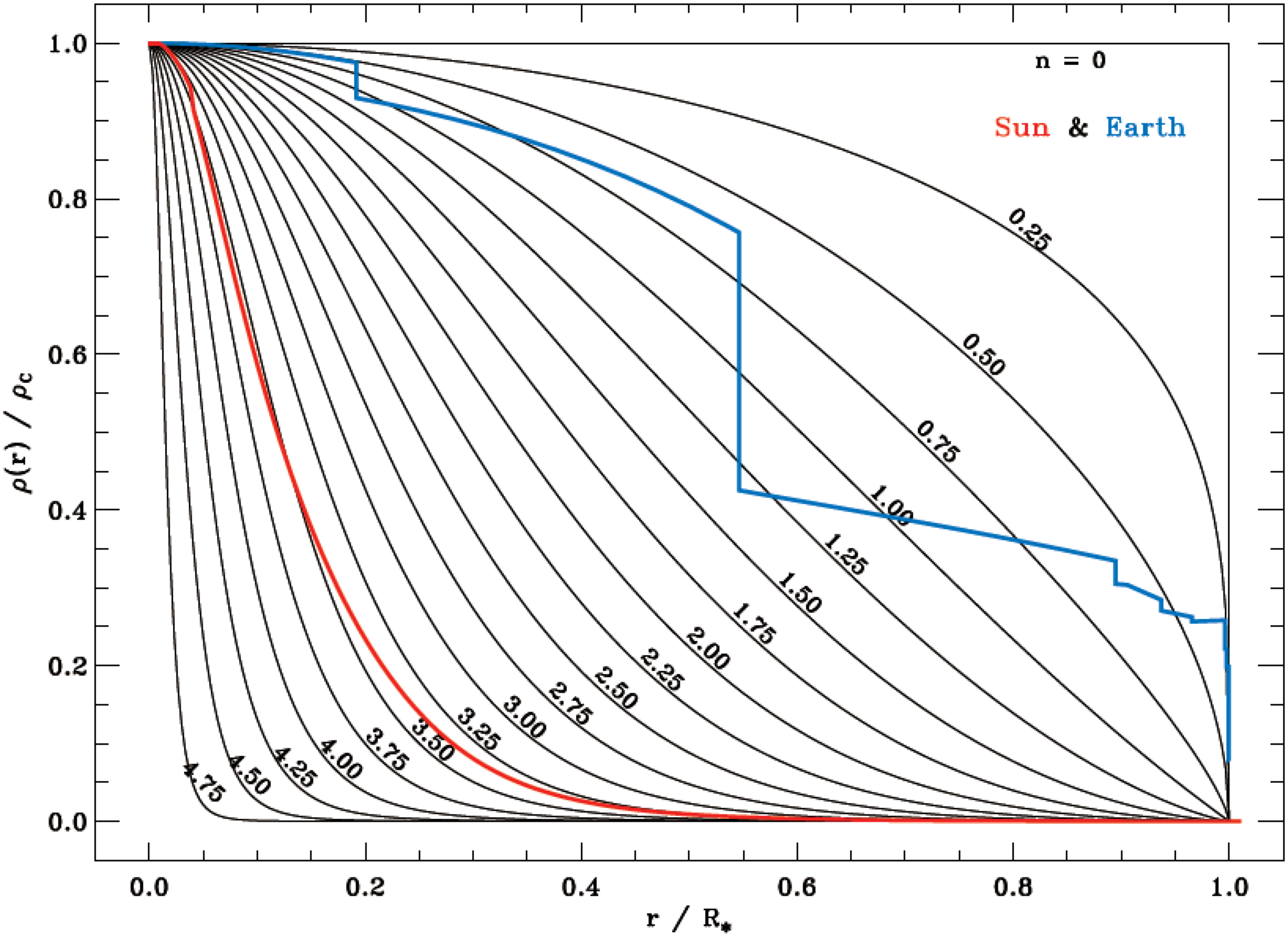
$$P_c = GM^{2/3}\rho_c^{4/3}\ C_n\ = P_{ideal} = \frac{\rho_c N_A k T_c}{\mu}$$

$$T_c = \frac{GM^{2/3}\rho_c^{1/3}\,\mu C_n}{N_A k} \qquad C_n = 0.36\ \text{ for n = 3}$$

$$T_c = 1.51\times 10^7\ \text{K}\ \left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{\mu}{0.61}\right)\left(\frac{C_n}{0.36}\right)\rho_c^{1/3}$$

if $\rho_c = 160\ \text{g cm}^{-3}$ for the sun

(actually 15.7 M K today from neutrinos)



The structure of the sun and earth compared with polytropes of various indices (from M. Zingale). The sun is about an n = 3 polytrope.

Useful Polytropic Indices

n	γ	Description
0	—	Incompressible gas; constant density
0.42857	10/3	Thomas-Fermi EOS
1	2	analytic solution to Lane-Emden equqation; constant R_*
1.5	5/3	ideal monatomic gas EOS; convective; non-relativistic degenerate
2	3/2	Holzer & Axford's maximum γ for an accelerating solar wind
2.5	7/5	ideal diatomic gas EOS
3	4/3	Eddington's standard model; ultra-relativistic degenerate; constant M_*
3.25	17/13	Chandrasekhar's constant-ϵ Kramers model
5	6/5	Schuster sphere of infinite radius
∞	1	Isothermal gas; Bonnor-Ebert sphere

$$n = 0 \qquad \theta(\xi) = 1 - \frac{\xi^2}{6} \qquad \xi_0 = \sqrt{6}$$

$$n = 1 \qquad \theta(\xi) = \frac{\sin \xi}{\xi} \qquad \xi_1 = \pi$$

$$n = 5 \qquad \theta(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \qquad \xi_5 = \infty$$

Lecture 7b

Polytropes:
Binding Energies and the "Standard Model" (n = 3)

Prialnik Chapter 5
Glatzmaier and Krumholz Chapter 10
Pols 4

The gravitational binding energy of polytropes

First a useful identity:

For a polytrope $P = K\rho^{\gamma} = K\rho^{\frac{n+1}{n}} = K\rho^{1+\frac{1}{n}}$

$$dP = K\frac{n+1}{n}\rho^{1/n}\, d\rho$$

but $$\frac{P}{\rho} = K\rho^{1/n}$$

multiply and divide by (n+1)

$$d\left(\frac{P}{\rho}\right) = \frac{K}{n}\rho^{(1-n)/n}\, d\rho = \left(K\frac{n+1}{n}\rho^{1/n}\, d\rho\right)\frac{1}{\rho}\frac{1}{(n+1)}$$

$$= \frac{1}{(n+1)}\frac{dP}{\rho}$$

The gravitational binding energy of polytropes

So now let's calculate the gravitational binding energy:

$$\Omega = -\int_0^R \frac{Gm}{r}\,dm = -\frac{1}{2}\int_{cent}^{surf} \frac{G}{r}\,d(m^2)$$

Integrate by parts

$$\int d(uv) = uv = \int u\,dv + \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

$$\Omega = -\left[\frac{Gm^2}{2r}\right]_{cent}^{surf} - \frac{1}{2}\int_{cent}^{surf} \frac{Gm^2}{r^2}\,dr \qquad \text{since } \frac{d}{dr}\left(\frac{G}{r}\right) = -\frac{G}{r^2}$$

$$= -\frac{GM^2}{2R} - \frac{1}{2}\int_{cent}^{surf} \frac{Gm^2}{r^2}\,dr$$

At the surface the first term is zero

Now use the equation of hydrostatic equilibrium

$$dP = -\left(-Gm/r\right)\left(\rho/r\right)dr$$

$$\Omega = -\frac{GM^2}{2R} - \frac{1}{2}\int_{cent}^{surf} \frac{Gm^2}{r^2} dr \quad \text{from previous page}$$

$$= -\frac{GM^2}{2R} + \frac{1}{2}\int_{cent}^{surf} m \frac{dP}{\rho} \quad \text{from hydrostatic equilibrium}$$

$$= -\frac{GM^2}{2R} + \frac{n+1}{2}\int_{cent}^{surf} m \, d\left(\frac{P}{\rho}\right)$$

since $\frac{dP}{\rho} = \left(n+1\right)d\left(\frac{P}{\rho}\right)$ from ID 2 pages ago

$$\Omega = -\frac{GM^2}{2R} + \frac{n+1}{2}\int_{cent}^{surf} m\, d\left(\frac{P}{\rho}\right)$$ from previous page

$$= -\frac{GM^2}{2R} + \left[\frac{n+1}{2} m \frac{P}{\rho}\right]_{cent}^{surf} - \frac{n+1}{2}\int_{cent}^{surf} \frac{P}{\rho} dm$$

at the center m = 0; at the surface P= 0; drop 2nd term

$$\Omega = -\frac{GM^2}{2R} - \frac{n+1}{2}\int_{cent}^{surf} \frac{P}{\rho} dm$$

but by the Virial Theorem

$$\int_{cent}^{surf} \frac{P}{\rho} dm = -\frac{\Omega}{3}$$

So $$\Omega = -\frac{GM^2}{2R} + \frac{n+1}{6}\Omega$$

$$\Omega\left(1 - \frac{n+1}{6}\right) = \Omega\left(\frac{6-n-1}{6}\right) = -\frac{GM^2}{2R}$$

$$\Omega = -\left(\frac{6}{5-n}\right)\frac{GM^2}{2R}$$

$$\Omega = -\left(\frac{3}{5-n}\right)\frac{GM^2}{R}$$

This is the gravitational binding energy of a polytrope of index n with mass M and radius R. Note the singularity at n = 5 and also the correct n = 0 limit. Note also that the polytrope of mass M and radius R is more tighly bound if it is more centrally condensed, i.e., n is larger

For example if the sun could be characterized by a polytropic index n = 3, its binding energy would be

$$\Omega_{\odot} = \frac{3}{2}\frac{GM_{\odot}^2}{R_{\odot}} = 1.5\frac{\left(6.67x10^{-8}\right)\left(1.99x10^{33}\right)^2}{\left(6.96x10^{10}\right)}$$

$$= 5.7\times10^{48} \text{ erg}$$

Accurate stellar models give 6.9×10^{48} erg

To the extent that any star is supported by ideal gas pressure, the Virial theorem holds and the total energy of the star

$$\mathrm{E}_{tot} = U + \Omega = \frac{\Omega}{2} = -\left(\frac{3}{10-2n}\right)\frac{GM^2}{R}$$

(note typo with "–" sign fixed from earlier version)

Kelvin Helmholz time scale (again)

The time scale required for an adjustment of stellar structure in the absence of energy sources other than gravitation is the Kelvin-Helmholtz time

$$\tau_{KH} = \frac{3}{10-2n}\frac{GM^2}{RL}$$

where L is the luminosity of the star (or region of the the star in light (or neutrinos).

For the sun (n = 3)

$$\tau_{KH} \approx \frac{3}{10-2n}\frac{GM^2}{L} = \frac{3}{4}\frac{GM_\odot^2}{R_\odot L_\odot}$$

$$= 7.4\times10^{14} \text{ sec} = 23 \text{ million years}$$

which is close to correct

Eddington's standard model (n=3)

Consider a star in which radiation pressure is important (though not necessarily dominant)

$$\frac{dP_{rad}}{dr} = \frac{d}{dr}\left(\frac{1}{3}aT^4\right) = \frac{4}{3}aT^3\frac{dT}{dr}$$

But for radiative diffusion, $\frac{dT}{dr} = \frac{3\kappa\rho}{16\pi acT^3}\frac{L(r)}{r^2}$ so

$$\frac{dP_{rad}}{dr} = -\frac{\kappa\rho}{4\pi c}\frac{L(r)}{r^2}$$

but hydrostatic equilibrium requires

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

Recall the definition of the Eddington lumnosity and divide 2 eqns

$$\mathrm{L}_{Ed} = \frac{4\pi GMc}{\kappa} \Rightarrow$$

$$\frac{dP_{rad}}{dP} = \frac{\kappa L(r)}{4\pi Gmc} = \frac{L(r)}{L_{Edd}}$$

Define $\beta = \frac{P_{gas}}{P} = 1 - \frac{P_{rad}}{P}$ where $P = P_{gas} + P_{rad}$,

$$\frac{dP_{rad}}{dP} = (1-\beta) = \frac{\kappa L(r)}{4\pi G m c} = \frac{L(r)}{L_{Edd}}$$

If, and it is a big IF, β (or 1-β) were a constant throughout the star, then one could write everywhere

$$L(r) = \left(1-\beta\right) L_{Ed}$$

$\beta =$ constant would imply that the star was an n=3 polytrope!

$$\mathrm{P} = \frac{P_{rad}}{(1-\beta)} = \frac{aT^4}{3(1-\beta)} \Rightarrow T = \left[\frac{3P(1-\beta)}{a}\right]^{1/4}$$

$$P = \frac{P_{gas}}{\beta} = \frac{N_A k}{\mu\beta}\rho T \Rightarrow P = \frac{N_A k}{\mu\beta}\rho\left[\frac{3P(1-\beta)}{a}\right]^{1/4}$$

hence $\mathrm{P}^{3/4} = \dfrac{N_A k}{\mu\beta}\rho\left[\dfrac{3(1-\beta)}{a}\right]^{1/4}$ and

$$\mathrm{P} = \left[\frac{3(N_A k)^4(1-\beta)}{a(\mu\beta)^4}\right]^{1/3} \rho^{4/3}$$

If β = constant thoughout the star, this would be the equation for an n = 3 polytrope and the multiplier of $\rho^{4/3}$ is K.

Recall $M = -\frac{(n+1)^{3/2}}{\sqrt{4\pi}} \xi_1^2 \frac{d\theta}{d\xi}\Big)_{\xi_1} \left(\frac{K}{G}\right)^{3/2} \rho_c^{\frac{3-n}{2n}}$

$$K = \left[\frac{3(N_A k)^4 (1-\beta)}{a(\mu\beta)^4}\right]^{1/3}$$

For n = 3, this becomes

$$M = -\frac{4}{\sqrt{\pi}} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1} \left(\frac{K}{G}\right)^{3/2} = 4.56 \left(\frac{K}{G}\right)^{3/2}$$

$$M = 4.56 \left[\frac{3(N_A k)^4 (1-\beta)}{a(\mu\beta)^4 G^3}\right]^{1/2}$$

$\lim_{\beta \to 0} M \to 0$

$\lim_{\beta \to 1} M \to \infty$

Eddington's quartic equation

$$M = \frac{18.1 M_\odot}{\mu^2} \left(\frac{1-\beta}{\beta^4}\right)^{1/2}$$

Alternatively

$$1-\beta = \left(\frac{aG^3}{3(N_A k)^4}\right)\frac{\pi}{16\xi_1^4\left(\frac{d\theta}{d\xi}\right)^2}\mu^4\beta^4 M^2$$

For each β there is a unique M. R drops out. Like a Chandrasekhar mass, but $M_{crit} = f(\beta)$

For stars of constant β, Eddington's quartic equation says how β varies with mass and composition, μ.

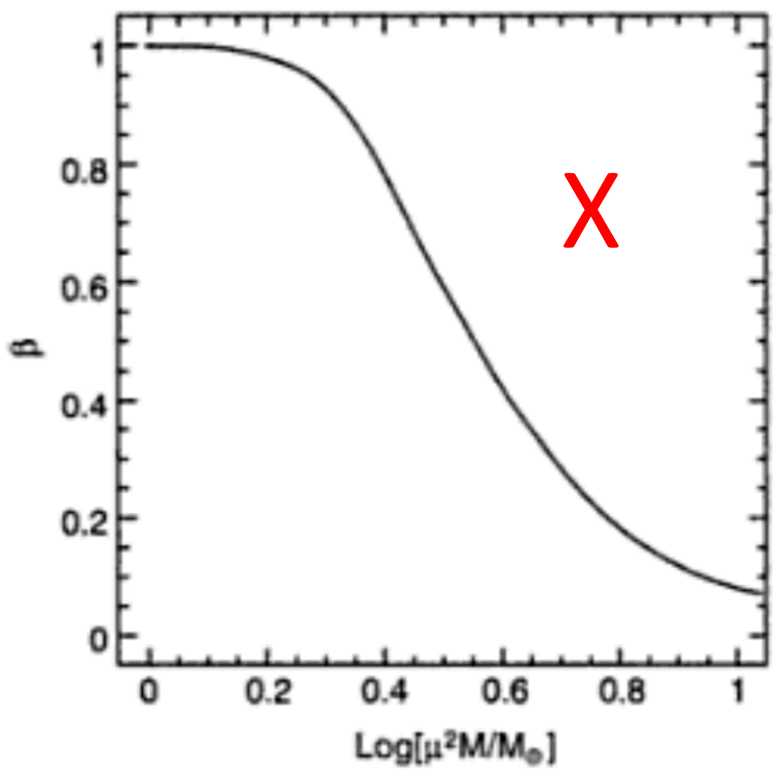
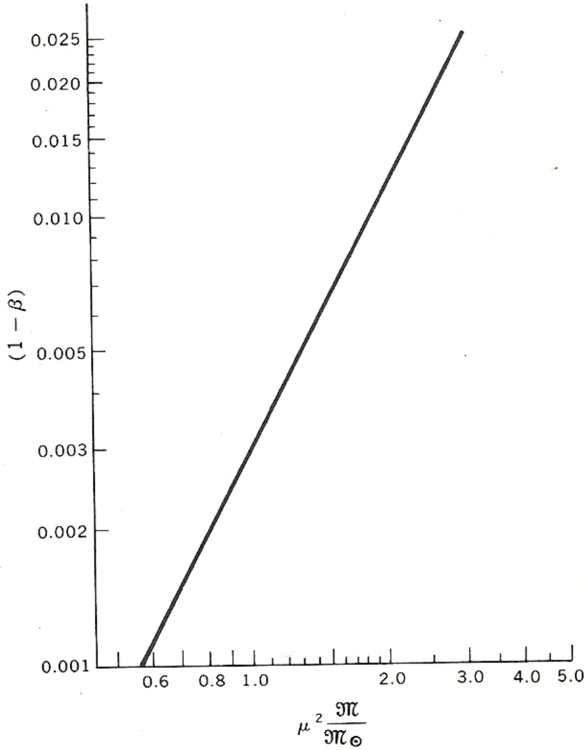


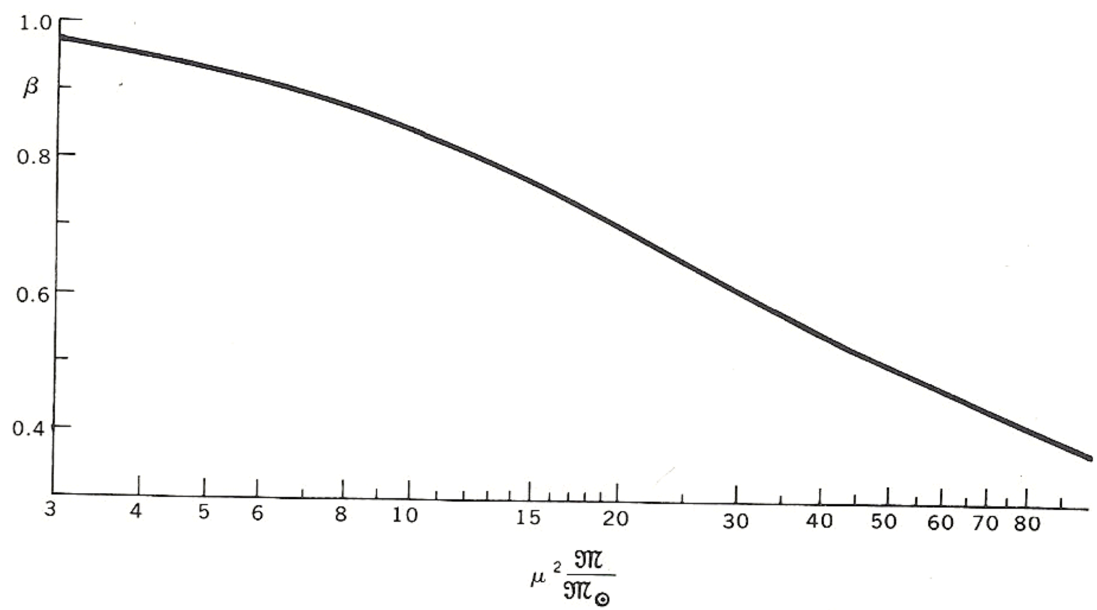
Figure 5.3 Solution of the Eddington quartic equation.

Prialnik 5.3
This is wrong!!!



Correct figures
from Clayton p. 163

$\mu^2 \approx 0.37$ for main sequence stars



$$1-\beta = 4.13\times 10^{-4}\left(\frac{M}{M_\odot}\right)^2\left(\frac{\mu}{0.61}\right)^4\beta^4$$ and since

L(r) = $(1-\beta)\,L_{Edd}$

$$L = \left(\frac{aG^3}{3(N_A k)^4}\right)\frac{\pi}{16\xi_1^4\left(d\theta/d\xi\right)^2_{\xi_1}}\mu^4\beta^4 M^2\frac{4\pi Gc}{\kappa}M$$

$$= \frac{\pi^2}{12\xi_1^4\left(d\theta/d\xi\right)^2_{\xi_1}}\left(\frac{acG^4}{(N_A k)^4}\right)\left(\frac{\mu^4\beta^4}{\kappa}\right)M^3$$

$$= 5.5\,\beta^4\left(\frac{\mu}{0.61}\right)^4\left(\frac{1\ \mathrm{cm^2\ g^{-1}}}{\kappa_{surf}}\right)\left(\frac{M}{M_\odot}\right)^3 L_\odot$$

Mass luminosty Relation

where κ_{surf} is the value of the opacity near the surface.

This was obtained with no mention of nuclear reactions.

For M not too far from $M_\odot$ β is close to 1 and L $\propto M^3$.

At higher masses however the mass dependence of β becomes important. $0.0004\left(\frac{M}{M_\odot}\right)^2\left(\frac{\mu}{0.61}\right)^4\beta^4$ eventually dominates and $\beta^4 \propto M^{-2}$ so that L$\propto$M. In fact, as we have discussed, the luminosity of very massive stars approaches the Eddington limit as $\beta \to 0$

This was all predicated on a very doubtful postulate however that β is a constant throughout the star (or $L(r)/L_{edd}$ is a constant). Why or when might this be approximately true?

Recall $\frac{dP_{rad}}{dP} = \frac{\kappa L(r)}{4\pi G m c} = \frac{L(r)}{L_{Edd}}$

and the energy equation $\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon(r)$ or $\frac{dL}{dm} = \varepsilon(m)$

where ε is the nuclear energy generation in erg $g^{-1} s^{-1}$

$$\langle \varepsilon(r) \rangle = \frac{\int_0^r \varepsilon(m)\,dm}{\int_0^r dm}$$ and clearly $\langle \varepsilon(R) \rangle = L / M$

define $\eta(r) = \frac{\langle \varepsilon(r) \rangle}{\langle \varepsilon(R) \rangle} = \frac{L(r)}{L_*} / \frac{m(r)}{M_*}$

$$\frac{dP_{rad}}{dP} = \frac{\kappa(r)L(r)}{4\pi Gm(r)c} = \frac{L}{4\pi GMc}\,\kappa(r)\eta(r)$$

Integrating from the surface, where both P_{rad} and P are assumed to be zero, inwards

$$P_{rad} = \frac{L}{4\pi GMc}\,\overline{\kappa\eta}(r)\,P \quad \text{where} \quad \overline{\kappa\eta}(r) = \frac{1}{P(r)}\int_{0(surf)}^{r} \kappa\eta\, dP'$$

which yields

$$1-\beta = \frac{L}{4\pi GMc}\,\overline{\kappa\eta}(r)$$

$$1-\beta = \frac{L}{4\pi GMc}\,\overline{\kappa\eta}(r)$$

The condition for the right hand side being a constant, which leads to β being a constant, is that, at each r, the pressure weighted average product of opacity times average energy generation interior to r be a constant. Since the energy generation is centrally concentrated .

$\frac{L(r)}{L_*} / \frac{m(r)}{M_*}$ is slowly *decreasing* with m(r) except at the center. κ(r) on the other hand is slowly *increasing* with r. Either it is a constant (electron scattering) or Kramer's like proportional to $\rho T^{-7/2}$. But roughly $\rho \propto T^3$. So the product is approximately constant especially where P is large. This was Eddington's reasoning.

For n = 3 one can also derive useful equations for the central conditions based upon the original polytropic equation for mass

$$M = -4\pi\alpha^3\rho_c\,\xi_1^2\,\frac{d\theta}{d\xi}\Bigg)_{\xi_1} = 2.01824\,(4\pi\alpha^3\rho_c)$$

and the definitions $\alpha = \left[\dfrac{P_c(n+1)}{4\pi G\rho_c^2}\right]^{1/2} = \left[\dfrac{P_c}{\pi G\rho_c^2}\right]^{1/2}$

and $P_c = \dfrac{P_{ideal}}{\beta} = \dfrac{\rho_c N_A k T_c}{\mu\beta}$ and $\dfrac{\rho_c}{\bar{\rho}} = \dfrac{4\pi R^3\rho_c}{3M} = 54.18$

$$P_c = 1.242\times10^{17}\left(\frac{\left(M/M_\odot\right)^2}{\left(R/R_\odot\right)^4}\right)$$

$$T_c = 19.57\times10^6\,\beta\mu\left(\frac{\left(M/M_\odot\right)}{\left(R/R_\odot\right)}\right)K$$

$$T_c = 4.62\times10^6\,\beta\mu\left(M/M_\odot\right)^{2/3}\rho_c^{1/3}\;\;K$$

For example,

$$T_c = 4.62 \times 10^6 \, \beta\mu \left(M / M_\odot \right)^{2/3} \rho_c^{1/3} \;\; K$$

The central density of the zero age sun (when its composition was constant) was 82 g cm^{-3} and its temperature was 13.6 MK (Bohm-Vitense, Sellar Structure and Evolution, Vol 3, p 156). Its radius was 0.884 times its present radius. The formula gives for $\beta \approx 1$, $\mu = 0.61$

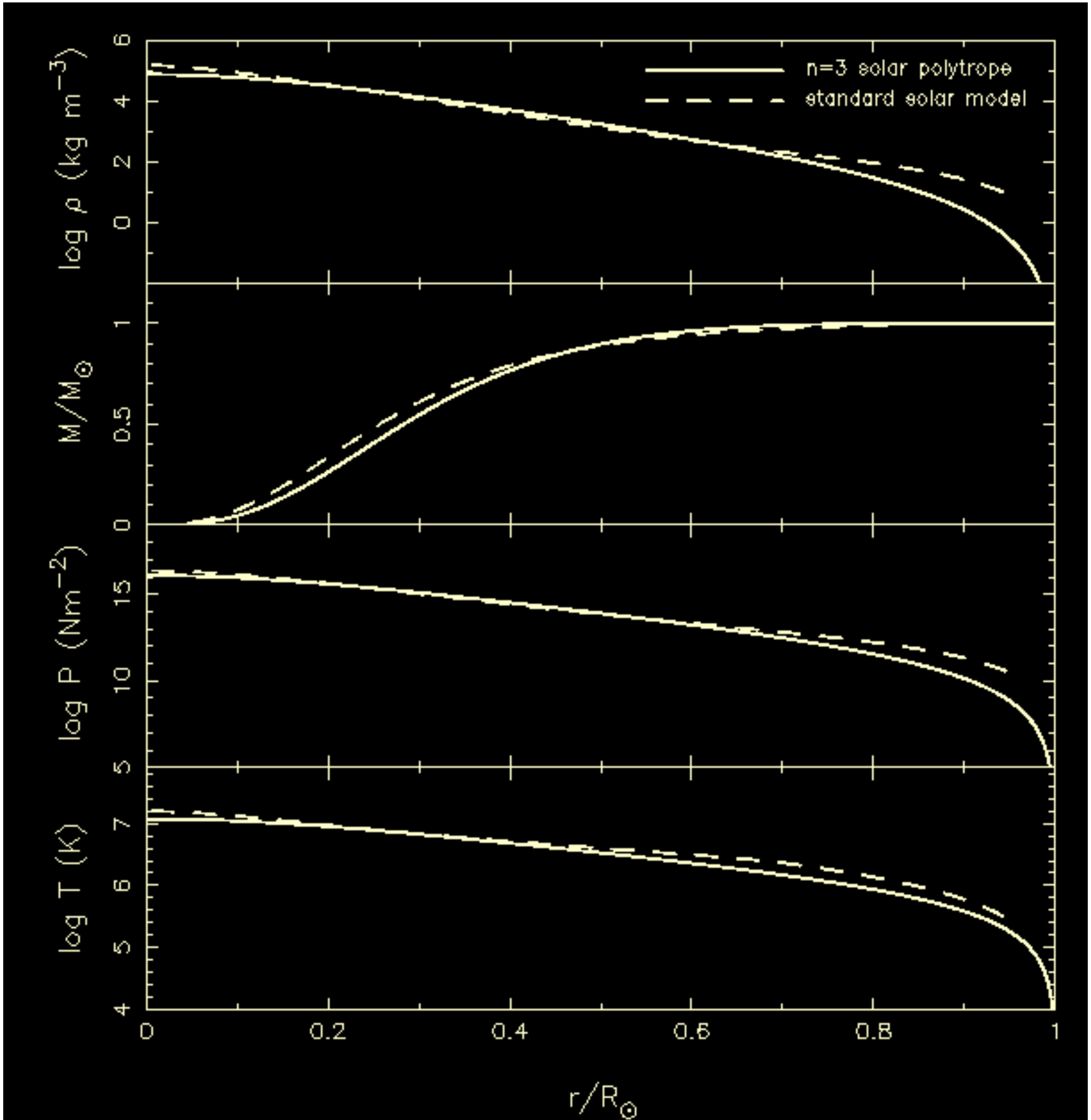
$$T_c = 19.57 \, (0.61)(1/0.884)$$

$$= 13.5 \text{ MK} \quad \text{in very good agreement}$$

In order to specfify a the radius (or equivalently the mean density)on the on the main sequence, it is necessary to specify and energy source, e.g., nuclear reactions, which we have not done so far. But taking the empirical (relation on the MS) that $\frac{R}{R_\odot} = \left(\frac{M}{M_{sun}}\right)^{2/3}$ one has for $\beta \approx 1$ (a better job could be done with the quartic equation) and the ZAMS value for the solar radius

$$T_c = 19.6 \times 10^6 \ (0.61) \left(\frac{(M / M_\odot)}{(R / R_\odot)} \right) \ = \ 13.5 \left(M / M_\odot \right)^{1/3} \ \text{MK}$$

These two relations for R and T predict correctly that more massive stars should have higher central temperatures and lower central densities on the main sequence.



Comparison of the n=3 polytrope of the Sun versus the Standard Solar Model.

The surface is not well fit because the surface of the sun is convective

- Trends:
  - Massive stars have more radiation pressure dominance
  - Massive stars have higher T
- Note: this is best for a ZAMS star—structure changes as the star evolves

Table 7.2. Eddington Standard Model

$\mu^2\mathcal{M}/\mathcal{M}_\odot$	β
1.0	0.9970
2.0	0.9885
5.0	0.9412
10.0	0.8463
50.0	0.5066