Lecture 7 – Part 1
Sources of Stellar Opacity

Although the theory of stellar opacity is complex in detail, the end results are easily summarized, with important consequences for the macroscopic properties of stars. The electron scattering contribution to the frequency-dependent opacity reads

$$\rho \kappa_v(\text{es}) = n_e \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2, \tag{7.1}$$

where n_e is the number density of free electrons. At temperatures higher than a few tens of million K, stellar interiors are virtually completely ionized, and electron scattering is the dominant opacity source. The Rosseland mean then behaves as κ = constant.
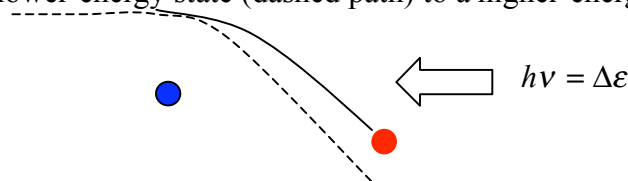
Kramers Law

At temperatures lower than a few times 10^7 K, the important thermal processes of emission and absorption are due to *free-free, bound-free*, and *bound-bound* transitions. Free-free radiation is the name astronomers give to the bremsstrahlung mechanism, when an electron in the presence of an ion makes a transition from one free (ionized) state to another free state. (Electron-electron scattering yields no radiation because the two electrons move in opposite ways as to cancel any wave contribution to the electric field – two free electrons have no dipole moment). In the absorption counterpart of the emission process, the electron makes a transition from a positive-energy state to a higher positive-energy state, absorbing a photon from the continuum of the radiation field (Figure 7.1).

When free-free or bound-free processes dominate, Henrik Kramers (1894-1952) showed that the Rosseland mean opacity follows what is now known as *Kramers law*:

$$\kappa \approx C\rho T^{-7/2}, \tag{7.2}$$

where C is a constant that depends on chemical composition. The dependence on density to the first power arises because the product $\rho\kappa$ must be proportional to ρ^2 in radiation processes that involve the participation of two material particles (e.g., an electron and an ion). The dependence on $T^{-7/2}$ means that, at a given density, free-free radiation loses importance relative to electron scattering at high temperatures.

Figure 7.1. Free-free absorption of radiation boosts the orbit of a free electron from a lower-energy state (dashed path) to a higher-energy state (solid path).

$$h\nu = \Delta\varepsilon$$

Derivation of Kramers Law for Free-Free Radiation

The time spent in the encounter of an electron and an ion in which the electron experiences significant acceleration can be shown in many astrophysical contexts to be short in comparison with the wave period of the radiation from the resulting brehmsstrahlung process (see Chapter 15 of Shu, Radiation). As a consequence, the electron experiences a sharp impulse of acceleration, whose Fourier decomposition leads to a flat spectrum of emitted radiation as a function of frequency v. (Recall that a Dirac delta function in time has a Fourier transform in frequency which is a constant.) However, physically, the emitted energy carried away by any photon hv from any electron cannot exceed its total initial kinetic energy ε, since the maximum that can be emitted (and still have the electron in a free state) is that which brings the electron to rest at infinity with respect to the ion. In a thermal distribution, the number of electrons that have kinetic energy ε is proportional to the Boltzmann factor $\exp(-\varepsilon/kT)$. As a consequence, the volume emissivity of free-free radiation must have a frequency dependence proportional to $\exp(-hv/kT)$. For technical reasons, there is an additional temperature dependence of $T^{-1/2}$, and since the basic process involves a two-body encounter between electron and ion, the volumetric rate must further be proportional to the product of the electron and ion number densities, $n_e n_i$. Putting everything together, we may write $\rho j_v \propto n_e n_i T^{-1/2} e^{-hv/kT}$. From charge neutrality, we require $n_e = \bar{Z}_i n_i$ where $\bar{Z}_i$ is the atomic number of the average ion. Since most of the mass is associated with ions, we can approximate the mass density as $\rho = \bar{m}_i n_i$, where $\bar{m}_i$ is the mean mass of ions. From Kirchhoff's law, $j_v = 4\pi\kappa_v^{\text{abs}} B_v(T)$, we now get the identification,

$$\kappa_v^{\text{abs}} \propto \rho T^{-1/2} v^{-3}[1 - e^{-hv/kT}],$$

where we recognize the term in the square bracket as the LTE correction for stimulated emission.

A more exact calculation for the frequency-dependent free-free opacity, corrected for stimulated emission, yields equation (15.29) of Shu, Radiation:

$$\rho\kappa_v(\text{ff}) = \sum_i n_{Z_i} n_e \left(\frac{2m_e}{3\pi kT}\right)^{1/2}\left[\frac{4\pi Z_i^2 e^6}{3m_e^2 chv^3}\right]g_{\text{ff}}(v)\left(1 - e^{-hv/kT}\right), \tag{7.3}$$

where $g_{\text{ff}}(v)$ is a quantum-mechanical correction called the Gaunt factor that varies slowly with v and is of order unity for most hv/kT of practical interest. (This statement is not true for applications in radio astronomy; see Problem Set 4 of Shu, Radiation). The expression (7.3) clearly depends on the chemical composition; however, until hydrogen and helium are exhausted, the primary contribution to the free-free opacity comes from these two, relatively invariant species, as long as we are in deep enough layers where both hydrogen and helium exist in ionized form.

If the free-free contribution were all that existed, the Rosseland

mean opacity would be given by

$$\frac{1}{\kappa} \propto \frac{1}{\rho T^{9/2}} \int_0^\infty \frac{e^{2h\nu/kT}\nu^7 d\nu}{[e^{h\nu/kT}-1]^3} \propto \frac{T^{7/2}}{\rho} \int_0^\infty \frac{e^{2x}x^7 dx}{[e^x-1]^3},$$

where the last integral is just a numerical constant equal to $4\pi^4/15$ times Strömgren's function $\varphi(x)$ evaluated at x equal to infinity, $\varphi(\infty) \cong 196$.

A numerical fit in cgs units when ionized hydrogen and helium, with mass fractions, respectively, of X and Y, provide the main scatterers of free electrons, yields Kramers' law in the form:

$$\kappa_R(\text{ff}) = 4 \times 10^{22}(X+Y)(1+X)\rho T^{-7/2}. \tag{7.4}$$

Bound-Free Opacity

In the bound-free (photo-ionization) process, the absorption of a continuum photon of sufficient energy ejects a bound electron from an incompletely ionized atom into a free state, with the difference in energy $h\nu$ of the photon and the ionization potential I going into the excess kinetic energy $m_e\nu^2/2$ of the freed electron (Figure 7.2). The cross-section for the bound-free process is zero below threshold, i.e., for photon energies lower than I; it generally reaches a maximum value at the ionization edge; and it declines typically as ν^{-3} at higher frequencies (Figure 7.3). In a hydrogenic atom of
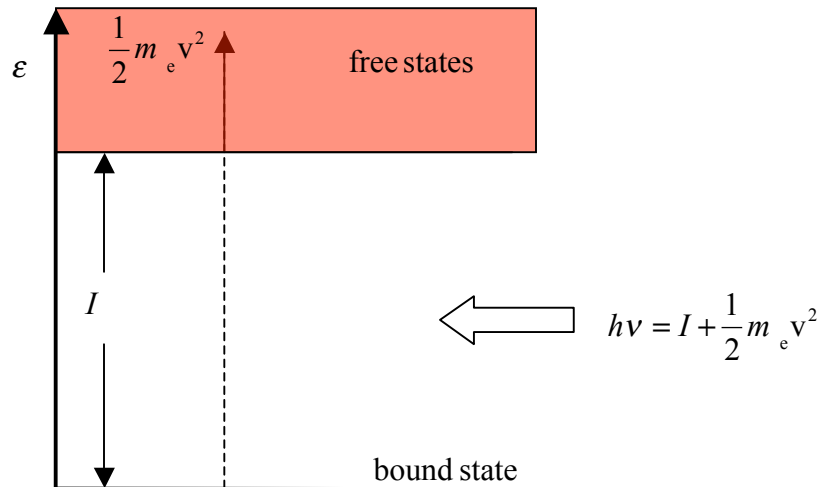
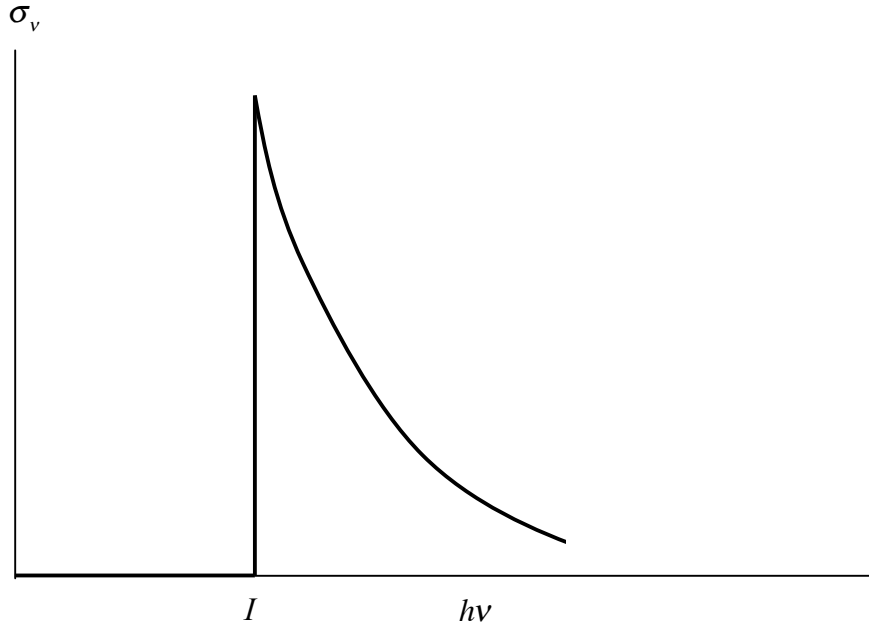**Figure 7.2.** Bound-free absorption.

Figure 7.3. Schematic plot of frequency dependence of typical photo-ionization cross-section.

nuclear charge Z and one bound electron in a level with principal quantum number n, the photo-ionization cross-section reads (Chapter 23 of Shu, Radiation):

$$\sigma_\nu(Z) = n^{-5} \frac{8\pi}{3\sqrt{3}} \frac{Z^4 m_e e^{10}}{c\hbar^3 (h\nu)^3} g_{bf}(\nu), \quad \text{for } h\nu \geq I = \frac{Ze^2}{2a_Z},$$

where $g_{bf}(\nu)$ is a slowly-varying, order-unity, bound-free Gaunt factor.

The total contribution to the frequency-dependent bound-free cross-section then reads

$$\rho\kappa_\nu(\text{bf}) = \sum_i n_{Z_i} \sigma_\nu(Z_i)\left(1 - e^{-h\nu/kT}\right) . \tag{7.5}$$

The ionization edges prevent the frequency-integrated Rosseland-mean associated with bound-free transitions from behaving exactly as $T^{-7/2}$. However, these frequency edges occur at higher values for partially-ionized elements of effective nuclear charge Z_i, and the cosmic abundance of high-Z elements is lower, typically, than for low-Z elements. A coincidence then occurs where the net behavior of $\rho\kappa_\nu(\text{bf})$ is similar, except for minor bumps, to $\rho\kappa_\nu(\text{ff})$. This results in a net temperature dependence that still goes as $T^{-7/2}$.

One might think that the density dependence of bound-free transitions in $\rho\kappa$ is linear rather than quadratic in ρ. In fact, the primary contributions come from heavy ions with only one bound electron that has number density n_{Z_i}, where most of the corresponding element of atomic number Z, atomic weight A, and fractional abundance

by mass X_Z is completely ionized with number density $n_{Z_{i+1}} = X_Z \rho / A m_H$. In this case, ionization equilibrium makes $n_i \propto n_{i+1} n_e \propto \rho^2$, with the free electrons coming from the ionization of hydrogen and helium, and therefore n_e like $n_{Z_{i+1}}$, is also proportional to ρ (see Part 2 of this Lecture).

Thus, $\kappa_\nu(\mathrm{bf})$ contributes approximately the same density and temperature dependence to the Rosseland mean κ_R as $\kappa_\nu(\mathrm{ff})$. A rough numerical fit for the Rosseland mean of the bound-free opacity of an elemental mix scaled to the solar composition is

$$\kappa_R(\mathrm{bf}) = 3 \times 10^{25} (1 - X - Y) \left(1 + X + \frac{3}{4}Y \right) \rho T^{-7/2}. \tag{7.6}$$

We recognize the factor $(1-X-Y)$ as the mass fraction of elements heavier than hydrogen and helium, which is taken to be completely ionized in stellar interiors with mass fractions, respectively of X and Y. The factor $(1+X+3Y/4)$ comes from the abundance of free electrons under these circumstances (see Problem Set 2).

A comparison of equations (7.4) and (7.6) shows that the larger peak cross-sections of bound-free processes imply that bound-free transitions typically dominate free-free transitions as long as heavy-element abundances exceed about 10^{-3} by mass, as true for a Population I star, but not an extreme Population II or a hypothetical Population III star. Thus, when one speaks of a Kramers law of the form of equation (7.2), one generally has in mind a mixture of bound-free and free-free processes.

Bound-Bound Opacity

The bound-bound process involves the absorption of a line photon which causes an electron in an incompletely ionized atom to make an upward transition from a lower atomic energy-level to a higher atomic energy-level. The bound-bound process involves the absorption of a line photon which causes an electron in an incompletely ionized atom to make an upward transition from a lower atomic energy-level to a higher atomic energy-level. Since bound-bound (bb) absorption involves a narrow band of frequencies surrounding a discrete value and since there are a relatively limited number of incompletely ionized states of atomic species in the deep interiors of stars, their contribution to the total opacity is usually less important than those from electron scattering (es) and the free-free (ff) and bound-free (bf) processes. For this reason, we shall generally disregard this contribution to the total opacity for stellar interiors work (but not for stellar atmospheres). Nevertheless, one of the recent major contributions of the OPACITY project in the revision of opacities in the solar envelope was the meticulous calculation of the contribution of bound-bound transitions from all the important elements in the periodic table.

Opacity Due to H⁻

When the temperatures drops below about $10^4 - 10^5$ K, hydrogen and helium

recombine into neutral atoms (helium before hydrogen), and the free electrons necessary to produce free-free absorption largely drop out of the matter mix. The bound-free contributions from more easily ionized, neutral, heavy atoms are offset by the fact that an exponentially small number of photons are available to produce the photo-ionization in a Planck function when the temperature T is low. Thus, the Sun's surface layers have no obviously important sources of opacity, yet when we try to peer into the deeper layers of the Sun from the outside, we quickly encounter an opaque yellow ball with a color temperature (characterizing the *quality* of radiation) close to the effective temperature of 5800 K(characterizing the *quantity* of radiation). The near equality of color and effective temperatures yields an astronomical puzzle as to what provides the opacity in the outer layers (the photosphere) of the Sun. Until the 1940s, astronomers had no answer to this question, i.e., no one had a fundamental explanation for why the Sun was yellow.

Rupert Wildt (1905-1976) provided the correct suggestion in 1939, and a few years later, Subrahmanyan Chandrasekhar (1910-1995) performed the crucial quantum-mechanical calculations that justified Wildt's suggestion. Wildt proposed that the opacity in the photosphere of the Sun, and other relatively cool stars, was supplied by the H^- ion (see below). In some sense, the neutral hydrogen atom is both an alkali metal and a halogen in that it both likes to give up its valence electron (to become H^+) and add a valence electron (to become H^-). The importance of this realization is that H^- as a two-electron system contains exactly one bound state and an infinity of "ionized" states (a free electron moving in the weakly attractive potential of a neutral hydrogen atom whose electronic configuration is polarized by the presence of the extra electron). The bound state is separated from the ionized states ($H + e^-$) by an ionization potential of only 0.754 eV, which means that optical photons in the photosphere of the Sun are readily capable of ionizing H^-, thereby producing a non-negligible contribution to the bound-free opacity. Moreover, the negative ion H^- itself can act as a partner to other free electrons to provide an unusual kind of free-free opacity.
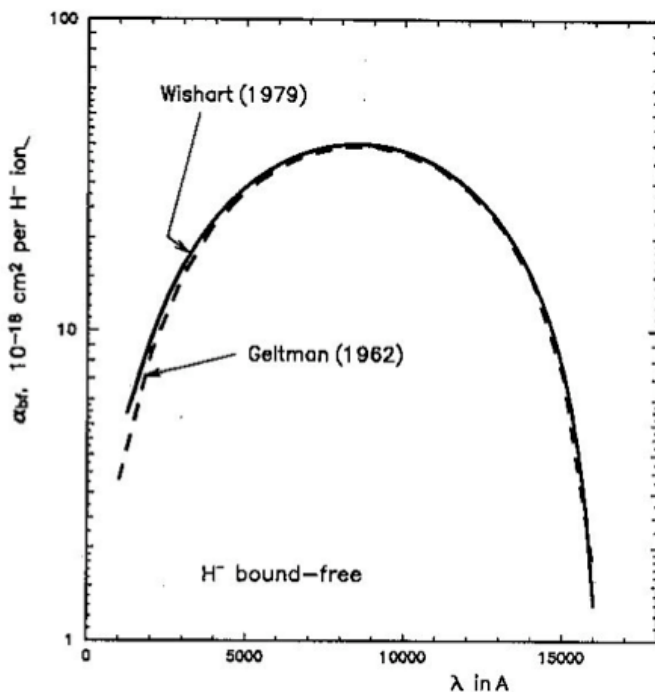
The bound-free and free-free contributions of H^- do not lead to the usual Kramers law, equation (7.2), because the H^- abundance in the solar photosphere is highly sensitive to other considerations. When the temperature drops below about $10^4 K$, the opacity curve (for given density) reaches a peak (with the partial ionization of atomic hydrogen) and then declines as a very steep function of decreasing temperature (as the opacity due to H^- takes over as the main source of continuum opacity). The steep functional dependence on temperature arises because the free electrons that are needed in H^- combine with atomic H in the chemical reaction,

$$H^- = H + e^-$$
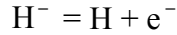
are supplied by the ionization of trace specifies like sodium (Na) or potassium (K), which have low enough ionization potentials to be ionized by the non-energetic radiation field of the surface of the Sun. This means that the abundances of H^- and free electrons decrease rapidly for decreasing temperatures T.

To compute the abundance of H⁻, We assume the concentration of all relevant species to be in LTE. General principles of chemistry then require the number densities of atomic hydrogen (the dominant constituent in the solar atmosphere), free electrons, and the H^- ion to satisfy the *law of mass action*:

$$\frac{n_{H^-}}{n_H n_e} = K_1(T),$$

where $K_1(T)$ is the chemical "constant" of the reaction. Part 2 of Lecture 7 gives Meghnad Saha's derivation of the expression for $K_1(T)$ for general ionization equilibria involving ideal gases. Here we only wish to note that free electrons at the relatively low temperatures of the solar atmosphere are quite rare. They mostly come from the ionization of t trace specifies like sodium (Na) or potassium (K) that have very low ionization potentials. In turn, the ionization equilibrium of such metals, $M^+ + e^- = M$, satisfies its own law of mass action:

$$\frac{n_{M^+} n_e}{n_M} = K_2(T).$$

If we suppose for simplicity that the ionization of M is the only source of free electrons, then charge neutrality requires $n_e = n_{M^+}$. Keeping track only the temperature dependences, we obtain for the free-free opacity due to H^-:

$$\rho \kappa_{H^-}^{ff} \propto n_{H^-} n_e T^{-7/2} = n_H n_M T^{-7/2} K_1(T) K_2(T). \tag{7.7}$$

The factor $K_1(T) K_2(T)$ gives the right-hand side of equation (7.7) a very steep (increasing) functional dependence on temperature T.

Total Opacity

In Lecture 5, we recounted how the Norwegian astronomer Svein Rosseland (1894-1985) showed how to combine and average all the contributions to obtain a mean value of the opacity that is appropriate for the problem of radiative transfer in stellar interiors. Figure 7.4 gives a schematic drawing of the net result when we take the Rosseland mean (5.28) of the sum of the individual contributions (5.27) from electron scattering, free-free absorption, bound-free absorption, and bound-bound transitions.

At high temperatures the expression (7.3) decreases relative to the expression (7.1). If there is not a compensating increase in the density, electron scattering becomes dominant over other forms of stellar opacity when the temperature in stellar interiors much exceeds about 2×10^7 K (as they do in stars appreciably more massive than the Sun). In the Sun, the change in temperature more than compensates for the change in density, and the opacity increases slowly as we move outwards from the center. This trend does not continue right to the surface, since realistic opacities cannot physically diverge as $T \rightarrow 0$. Thus, as the stellar gas becomes predominantly neutral, equation (7.4) obtains contributions only from those species that are still capable of free-free and bound-free transitions in the low photon-energy environment that characterizes cool stellar photospheres. In practice, this means H⁻, whose abundance is highly sensitive, as explained in the previous section, to the exact value of the ambient temperature. This sensitivity gives the precipitous decline of κ at low T apparent in Figure 7.4. An illuminating plot of the Rosseland opacity that shows both the density and temperature dependence appears in Chapter 3.3 in the textbook by Clayton.

Figure 7.4. Schematic plot of $\log \kappa$ versus $\log T$ for fixed ρ. Recall that power laws appear as straight lines in a log-log plot.