Algebras of Lorch analytic mappings defined in uniform algebras

Luiza A. Moraes

Instituto de Matemática
Universidade Federal do Rio de Janeiro

Conference on Non-Linear Functional Analysis Valencia 2017

All the results presented in this talk were obtained in collaboration with

Guilherme V.S.Mauro (Universidade Federal da Integração Latino-Americana)

These results are part of

Guilherme V.S.Mauro and L.A.M. , *Algebras of Lorch analytic mappings on uniform algebras.* , Pre-print.

and

Guilherme V.S.Mauro, *Espectros de Álgebras de Aplicações Analíticas no sentido de Lorch*, Tese de Doutorado, UFRJ (2016)

Definitions and Notation

$\mathcal{A} =$ complex commutative unital Fréchet algebra.

Definitions and Notation

$\mathcal{A} =$ complex commutative unital Fréchet algebra.

$\mathcal{M}(\mathcal{A}) =$ the set of the non-null complex valued continuous homomorphisms defined in $\mathcal{A}$ (= spectrum of $\mathcal{A}$).

Definitions and Notation

$\mathcal{A} =$ complex commutative unital Fréchet algebra.

$\mathcal{M}(\mathcal{A}) =$ the set of the non-null complex valued continuous homomorphisms defined in $\mathcal{A}$ (= spectrum of $\mathcal{A}$).

$\tau_G =$ topology induced in $\mathcal{M}(\mathcal{A})$ by the weak*-topology (Gelfand topology). $\mathcal{M}(\mathcal{A})$ is always considered endowed with τ_G.

Definitions and Notation

$\mathcal{A}=$ complex commutative unital Fréchet algebra.

$\mathcal{M}(\mathcal{A})=$ the set of the non-null complex valued continuous homomorphisms defined in $\mathcal{A}$ (= spectrum of $\mathcal{A}$).

$\tau_G=$ topology induced in $\mathcal{M}(\mathcal{A})$ by the weak*-topology (Gelfand topology). $\mathcal{M}(\mathcal{A})$ is always considered endowed with τ_G.

Well known: $\mathcal{M}(\mathcal{A})$ is a compact Hausdorff space whenever $\mathcal{A}$ is a (commutative) Banach algebra

Definitions and Notation

$\mathcal{A} =$ complex commutative unital Fréchet algebra.

$\mathcal{M}(\mathcal{A}) =$ the set of the non-null complex valued continuous homomorphisms defined in $\mathcal{A}$ (= spectrum of $\mathcal{A}$).

$\tau_G =$ topology induced in $\mathcal{M}(\mathcal{A})$ by the weak*-topology (Gelfand topology). $\mathcal{M}(\mathcal{A})$ is always considered endowed with τ_G.

Well known: $\mathcal{M}(\mathcal{A})$ is a compact Hausdorff space whenever $\mathcal{A}$ is a (commutative) Banach algebra

The **radical** $\mathcal{R}(\mathcal{A})$ of is the intersection of all maximal ideals in $\mathcal{A}$.

Definitions and Notation

$\mathcal{A}$ = complex commutative unital Fréchet algebra.

$\mathcal{M}(\mathcal{A})$ = the set of the non-null complex valued continuous homomorphisms defined in $\mathcal{A}$ (= spectrum of $\mathcal{A}$).

τ_G = topology induced in $\mathcal{M}(\mathcal{A})$ by the weak*-topology (Gelfand topology). $\mathcal{M}(\mathcal{A})$ is always considered endowed with τ_G.

Well known: $\mathcal{M}(\mathcal{A})$ is a compact Hausdorff space whenever $\mathcal{A}$ is a (commutative) Banach algebra

The **radical** $\mathcal{R}(\mathcal{A})$ of is the intersection of all maximal ideals in $\mathcal{A}$.

An algebra $\mathcal{A}$ is called **semi-simple** if $\mathcal{R}(\mathcal{A}) = \{0\}$.

Definitions and Notation

$\mathcal{A}$ = complex commutative unital Fréchet algebra.

$\mathcal{M}(\mathcal{A})$ = the set of the non-null complex valued continuous homomorphisms defined in $\mathcal{A}$ (= spectrum of $\mathcal{A}$).

τ_G = topology induced in $\mathcal{M}(\mathcal{A})$ by the weak*-topology (Gelfand topology). $\mathcal{M}(\mathcal{A})$ is always considered endowed with τ_G.

Well known: $\mathcal{M}(\mathcal{A})$ is a compact Hausdorff space whenever $\mathcal{A}$ is a (commutative) Banach algebra

The **radical** $\mathcal{R}(\mathcal{A})$ of is the intersection of all maximal ideals in $\mathcal{A}$.

An algebra $\mathcal{A}$ is called **semi-simple** if $\mathcal{R}(\mathcal{A}) = \{0\}$.

Well known:

- $\mathcal{M}(\mathcal{A})$ coincides with the set of the closed maximal ideals of $\mathcal{A}$.

Definitions and Notation

$\mathcal{A}$ = complex commutative unital Fréchet algebra.

$\mathcal{M}(\mathcal{A})$ = the set of the non-null complex valued continuous homomorphisms defined in $\mathcal{A}$ (= spectrum of $\mathcal{A}$).

τ_G = topology induced in $\mathcal{M}(\mathcal{A})$ by the weak*-topology (Gelfand topology). $\mathcal{M}(\mathcal{A})$ is always considered endowed with τ_G.

Well known: $\mathcal{M}(\mathcal{A})$ is a compact Hausdorff space whenever $\mathcal{A}$ is a (commutative) Banach algebra

The **radical** $\mathcal{R}(\mathcal{A})$ of is the intersection of all maximal ideals in $\mathcal{A}$.

An algebra $\mathcal{A}$ is called **semi-simple** if $\mathcal{R}(\mathcal{A}) = \{0\}$.

Well known:

- $\mathcal{M}(\mathcal{A})$ coincides with the set of the closed maximal ideals of $\mathcal{A}$.
- As a consequence:

$$\mathcal{R}(\mathcal{A}) = \bigcap_{\varphi \in \mathcal{M}(\mathcal{A})} \varphi^{-1}(0).$$

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

$U =$ open connected subset of E.

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

$U =$ open connected subset of E.

$B_r(z_0) = \{z \in E : \|z - z_0\| < r\}$ for all $z_0 \in E$ and $r > 0$.

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

$U =$ open connected subset of E.

$B_r(z_0) = \{z \in E : \|z - z_0\| < r\}$ for all $z_0 \in E$ and $r > 0$.

$B_1(0)$ will be denoted by B_E.

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

$U =$ open connected subset of E.

$B_r(z_0) = \{z \in E : \|z - z_0\| < r\}$ for all $z_0 \in E$ and $r > 0$.

$B_1(0)$ will be denoted by B_E.

$d_U(a) = \mathrm{dist}\,(a, \partial U) > 0$ for every $a \in U$.

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

$U =$ open connected subset of E.

$B_r(z_0) = \{z \in E : \|z - z_0\| < r\}$ for all $z_0 \in E$ and $r > 0$.

$B_1(0)$ will be denoted by B_E.

$d_U(a) = \mathrm{dist}\,(a, \partial U) > 0$ for every $a \in U$.

- $B \subset U$ is **U-bounded** if B is bounded and $\mathrm{dist(B, \partial U)} > 0$.

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

$U =$ open connected subset of E.

$B_r(z_0) = \{z \in E : \|z - z_0\| < r\}$ for all $z_0 \in E$ and $r > 0$.

$B_1(0)$ will be denoted by B_E.

$d_U(a) = \mathrm{dist}\,(a, \partial U) > 0$ for every $a \in U$.

- $B \subset U$ is **U-bounded** if B is bounded and $\mathrm{dist}(B, \partial U) > 0$.
- $\mathcal{H}_b(U, E)$ denotes the space of holomorphic mappings from U into E which are bounded on the U-bounded subsets of U endowed with the topology τ_b of uniform convergence on the U-bounded subsets of U.

Definitions and Notation

From now

$E =$ commutative complex Banach algebra with a unit e s.th. $\|e\| = 1$.

$U =$ open connected subset of E.

$B_r(z_0) = \{z \in E : \|z - z_0\| < r\}$ for all $z_0 \in E$ and $r > 0$.

$B_1(0)$ will be denoted by B_E.

$d_U(a) = \mathrm{dist}\,(a, \partial U) > 0$ for every $a \in U$.

- $B \subset U$ is **U-bounded** if B is bounded and $\mathrm{dist}(B, \partial U) > 0$.
- $\mathcal{H}_b(U, E)$ denotes the space of holomorphic mappings from U into E which are bounded on the U-bounded subsets of U endowed with the topology τ_b of uniform convergence on the U-bounded subsets of U.
- Well known: $\mathcal{H}_b(U, E)$ endowed with the usual pointwise operations is an algebra.

The Lorch analytic Mappings

Definition

If U is an open subset of commutative complex Banach algebra E, we say that $f : U \to E$ has a **Lorch-derivative** $f'(z_0) \in E$ at $z_0 \in U$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E$ satisfying $\|h\| < \delta$ we have

$$\|f(z_0 + h) - f(z_0) - h f'(z_0)\| < \varepsilon \|h\|.$$

Definition

If U is an open subset of commutative complex Banach algebra E, we say that $f : U \to E$ has a **Lorch-derivative** $f'(z_0) \in E$ at $z_0 \in U$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E$ satisfying $\|h\| < \delta$ we have

$$\|f(z_0 + h) - f(z_0) - h\,f'(z_0)\| < \varepsilon\|h\|.$$

We say that f is **Lorch analytic** in U if f has a Lorch-derivative at each point of U.

Definition

If U is an open subset of commutative complex Banach algebra E, we say that $f : U \to E$ has a **Lorch-derivative** $f'(z_0) \in E$ at $z_0 \in U$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E$ satisfying $\|h\| < \delta$ we have

$$\|f(z_0 + h) - f(z_0) - h f'(z_0)\| < \varepsilon \|h\|.$$

We say that f is **Lorch analytic** in U if f has a Lorch-derivative at each point of U.

- Easy to check: f is holomorphic in U in the standard sense in holomorphy whenever f is Lorch analytic in U .

The Lorch analytic Mappings

Definition

If U is an open subset of commutative complex Banach algebra E, we say that $f : U \to E$ has a **Lorch-derivative** $f'(z_0) \in E$ at $z_0 \in U$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E$ satisfying $\|h\| < \delta$ we have

$$\|f(z_0 + h) - f(z_0) - h f'(z_0)\| < \varepsilon \|h\|.$$

We say that f is **Lorch analytic** in U if f has a Lorch-derivative at each point of U.

- Easy to check: f is holomorphic in U in the standard sense in holomorphy whenever f is Lorch analytic in U .
- The converse is not true. Classical example: The linear mapping $f : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $f(z_1, z_2) = (z_2, z_1)$ is not Lorch analytic.

The Lorch analytic mappings

The Lorch analytic mappings

$\mathcal{H}_L(U)$ denotes the vector space of all Lorch analytic mappings in an open subset U of E.

The Lorch analytic mappings

$\mathcal{H}_L(U)$ denotes the vector space of all Lorch analytic mappings in an open subset U of E.

Proposition

Let U_1 and U_2 be connected open subsets of E. If $f \in \mathcal{H}_L(U_1)$ is such that $f(U_1) \subset U_2$ and $g \in \mathcal{H}_L(U_2)$, then $g \circ f \in \mathcal{H}_L(U_1)$ and

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

for all $z \in U_1$.

The Lorch analytic mappings

$\mathcal{H}_L(U)$ denotes the vector space of all Lorch analytic mappings in an open subset U of E.

Proposition

Let U_1 and U_2 be connected open subsets of E. If $f \in \mathcal{H}_L(U_1)$ is such that $f(U_1) \subset U_2$ and $g \in \mathcal{H}_L(U_2)$, then $g \circ f \in \mathcal{H}_L(U_1)$ and

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

for all $z \in U_1$.

Proposition

A mapping $f : U \to E$ is Lorch analytic in U if and only if given any $z_0 \in U$ there exists $r > 0$ and there exist (unique) elements $a_n \in E$, such that $B_r(z_0) \subset U$ and $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, for all $z \in B_r(z_0)$.

The spaces $\mathcal{H}_L(E)$ and $\mathcal{H}_L(U)$

Known:

- $\mathcal{H}_L(U) \subsetneq \mathcal{H}_b(U, E)$ in cases $U = E$ and $U = B_r(z_0)$.

The spaces $\mathcal{H}_L(E)$ and $\mathcal{H}_L(U)$

Known:

- $\mathcal{H}_L(U) \subsetneq \mathcal{H}_b(U, E)$ in cases $U = E$ and $U = B_r(z_0)$.
- Since $\mathcal{H}_L(E) \subset \mathcal{H}_b(E, E)$, we have that $f \in \mathcal{H}_L(E)$ if and only if there exists $(a_n)_n \subset E$ satisfying $\lim_{n\to\infty} \|a_n\|^{\frac{1}{n}} = 0$ and such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for all} \quad z \in E.$$

Known:

- $\mathcal{H}_L(U) \subsetneq \mathcal{H}_b(U, E)$ in cases $U = E$ and $U = B_r(z_0)$.
- Since $\mathcal{H}_L(E) \subset \mathcal{H}_b(E, E)$, we have that $f \in \mathcal{H}_L(E)$ if and only if there exists $(a_n)_n \subset E$ satisfying $\lim_{n\to\infty} \|a_n\|^{\frac{1}{n}} = 0$ and such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for all } \; z \in E.$$

- With the usual operations, $\mathcal{H}_L(E)$ and $\mathcal{H}_L(B_r(z_0))$ are **closed** subalgebras of $\mathcal{H}_b(E, E)$ and $\mathcal{H}_b(B_r(z_0), E)$ respectively, and so $(\mathcal{H}_L(E), \tau_b)$ and $(\mathcal{H}_b(B_r(z_0), E), \tau_b)$ are commutative Fréchet algebras (with identity).

The spaces $\mathcal{H}_L(E)$ and $\mathcal{H}_L(U)$

Known:

- $\mathcal{H}_L(U) \subsetneq \mathcal{H}_b(U, E)$ in cases $U = E$ and $U = B_r(z_0)$.
- Since $\mathcal{H}_L(E) \subset \mathcal{H}_b(E, E)$, we have that $f \in \mathcal{H}_L(E)$ if and only if there exists $(a_n)_n \subset E$ satisfying $\lim_{n\to\infty} \|a_n\|^{\frac{1}{n}} = 0$ and such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for all} \quad z \in E.$$

- With the usual operations, $\mathcal{H}_L(E)$ and $\mathcal{H}_L(B_r(z_0))$ are **closed** subalgebras of $\mathcal{H}_b(E, E)$ and $\mathcal{H}_b(B_r(z_0), E)$ respectively, and so $(\mathcal{H}_L(E), \tau_b)$ and $(\mathcal{H}_b(B_r(z_0), E), \tau_b)$ are commutative Fréchet algebras (with identity).
- For arbitrary U we may have $\mathcal{H}_L(U) \not\subset \mathcal{H}_b(U, E)$ and in this case we can not consider $\mathcal{H}_L(U)$ endowed with the topology τ_b.

The spaces $\mathcal{H}_L(E)$ and $\mathcal{H}_L(U)$

Known:

- $\mathcal{H}_L(U) \subsetneq \mathcal{H}_b(U, E)$ in cases $U = E$ and $U = B_r(z_0)$.
- Since $\mathcal{H}_L(E) \subset \mathcal{H}_b(E, E)$, we have that $f \in \mathcal{H}_L(E)$ if and only if there exists $(a_n)_n \subset E$ satisfying $\lim_{n\to\infty} \|a_n\|^{\frac{1}{n}} = 0$ and such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for all} \ \ z \in E.$$

- With the usual operations, $\mathcal{H}_L(E)$ and $\mathcal{H}_L(B_r(z_0))$ are **closed** subalgebras of $\mathcal{H}_b(E, E)$ and $\mathcal{H}_b(B_r(z_0), E)$ respectively, and so $(\mathcal{H}_L(E), \tau_b)$ and $(\mathcal{H}_b(B_r(z_0), E), \tau_b)$ are commutative Fréchet algebras (with identity).
- For arbitrary U we may have $\mathcal{H}_L(U) \not\subset \mathcal{H}_b(U, E)$ and in this case we can not consider $\mathcal{H}_L(U)$ endowed with the topology τ_b.

The spaces $\mathcal{H}_L(E)$ and $\mathcal{H}_L(U)$

Known:

- $\mathcal{H}_L(U) \subsetneq \mathcal{H}_b(U, E)$ in cases $U = E$ and $U = B_r(z_0)$.
- Since $\mathcal{H}_L(E) \subset \mathcal{H}_b(E, E)$, we have that $f \in \mathcal{H}_L(E)$ if and only if there exists $(a_n)_n \subset E$ satisfying $\lim_{n\to\infty} \|a_n\|^{\frac{1}{n}} = 0$ and such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for all } \; z \in E.$$

- With the usual operations, $\mathcal{H}_L(E)$ and $\mathcal{H}_L(B_r(z_0))$ are **closed** subalgebras of $\mathcal{H}_b(E, E)$ and $\mathcal{H}_b(B_r(z_0), E)$ respectively, and so $(\mathcal{H}_L(E), \tau_b)$ and $(\mathcal{H}_b(B_r(z_0), E), \tau_b)$ are commutative Fréchet algebras (with identity).
- For arbitrary U we may have $\mathcal{H}_L(U) \not\subset \mathcal{H}_b(U, E)$ and in this case we can not consider $\mathcal{H}_L(U)$ endowed with the topology τ_b.

We are going to introduce a convenient topology in $\mathcal{H}_L(U)$.

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

For $f : U \to F$ (where $F =$ complex Banach space) and $B_r(z) \subset U$, let

$$\|f\|_{B_r(z)} =: \sup\{\|f(w)\| \, ; \, w \in B_r(z)\}.$$

$$\mathcal{H}_d(U, F) =: \{f \in \mathcal{H}(U; F) : \|f\|_{B_r(z)} < \infty \ \forall z \in U \text{ and } \forall\ 0 < r < d_U(z)\}$$

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

For $f : U \to F$ (where $F =$ complex Banach space) and $B_r(z) \subset U$, let

$$\|f\|_{B_r(z)} =: \sup\{\|f(w)\| \, ; \, w \in B_r(z)\}.$$

$\mathcal{H}_d(U, F) =: \{f \in \mathcal{H}(U; F) : \|f\|_{B_r(z)} < \infty \; \forall z \in U \text{ and } \forall \; 0 < r < d_U(z)\}$

- $\mathcal{H}_b(U; F) \subset \mathcal{H}_d(U; F) \subset \mathcal{H}(U; F)$

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

For $f : U \to F$ (where $F =$ complex Banach space) and $B_r(z) \subset U$, let

$$\|f\|_{B_r(z)} =: \sup\{\|f(w)\| \, ; \, w \in B_r(z)\}.$$

$\mathcal{H}_d(U, F) =: \{f \in \mathcal{H}(U; F) : \|f\|_{B_r(z)} < \infty \; \forall z \in U \text{ and } \forall \; 0 < r < d_U(z)\}$

- $\mathcal{H}_b(U; F) \subset \mathcal{H}_d(U; F) \subset \mathcal{H}(U; F)$
- $\mathcal{H}_L(U) \subset \mathcal{H}_d(U; E)$.

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

For $f : U \to F$ (where $F =$ complex Banach space) and $B_r(z) \subset U$, let

$$\|f\|_{B_r(z)} =: \sup\{\|f(w)\| \, ; \, w \in B_r(z)\}.$$

$\mathcal{H}_d(U, F) =: \{f \in \mathcal{H}(U; F) : \|f\|_{B_r(z)} < \infty \; \forall z \in U \text{ and } \forall \, 0 < r < d_U(z)\}$

- $\mathcal{H}_b(U; F) \subset \mathcal{H}_d(U; F) \subset \mathcal{H}(U; F)$
- $\mathcal{H}_L(U) \subset \mathcal{H}_d(U; E)$.

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

For $f : U \to F$ (where $F =$ complex Banach space) and $B_r(z) \subset U$, let

$$\|f\|_{B_r(z)} =: \sup\{\|f(w)\| \, ; \, w \in B_r(z)\}.$$

$\mathcal{H}_d(U, F) =: \{f \in \mathcal{H}(U; F) : \|f\|_{B_r(z)} < \infty \; \forall z \in U \text{ and } \forall \; 0 < r < d_U(z)\}$

- $\mathcal{H}_b(U; F) \subset \mathcal{H}_d(U; F) \subset \mathcal{H}(U; F)$
- $\mathcal{H}_L(U) \subset \mathcal{H}_d(U; E)$.

Let $\tau_d =$ topology generated in $\mathcal{H}_d(U, F)$ by the family of semi-norms

$$\{\| \cdot \|_{B_r(z)} : 0 < r < d_U(z) \text{ and } z \in U\}.$$

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

For $f : U \to F$ (where $F =$ complex Banach space) and $B_r(z) \subset U$, let

$$\|f\|_{B_r(z)} =: \sup\{\|f(w)\| \,;\, w \in B_r(z)\}.$$

$\mathcal{H}_d(U, F) =: \{f \in \mathcal{H}(U; F) : \|f\|_{B_r(z)} < \infty \; \forall z \in U \text{ and } \forall\, 0 < r < d_U(z)\}$

- $\mathcal{H}_b(U; F) \subset \mathcal{H}_d(U; F) \subset \mathcal{H}(U; F)$
- $\mathcal{H}_L(U) \subset \mathcal{H}_d(U; E)$.

Let $\tau_d =$ topology generated in $\mathcal{H}_d(U, F)$ by the family of semi-norms

$$\{\|\cdot\|_{B_r(z)} : 0 < r < d_U(z) \text{ and } z \in U\}.$$

- If $U = B_r(z_0)$, where $z_0 \in E, r > 0$, $\mathcal{H}_L(B_r(z_0)) \subsetneqq \mathcal{H}_b(B_r(z_0), E) = \mathcal{H}_d(B_r(z_0), E)$ and $\tau_d = \tau_b$ in $\mathcal{H}_L(B_r(z_0))$.

The space $\mathcal{H}_L(U, E)$

- It is known that if $f \in \mathcal{H}_L(U)$ and $w_0 \in U$, then $r_b f(w_0) = d_U(w_0)$ where $r_b f(w_0) =$ radius of boundedness of f at w_0.

For $f : U \to F$ (where $F =$ complex Banach space) and $B_r(z) \subset U$, let

$$\|f\|_{B_r(z)} =: \sup\{\|f(w)\| \, ; \, w \in B_r(z)\}.$$

$\mathcal{H}_d(U, F) =: \{f \in \mathcal{H}(U; F) : \|f\|_{B_r(z)} < \infty \; \forall z \in U \text{ and } \forall \; 0 < r < d_U(z)\}$

- $\mathcal{H}_b(U; F) \subset \mathcal{H}_d(U; F) \subset \mathcal{H}(U; F)$
- $\mathcal{H}_L(U) \subset \mathcal{H}_d(U; E)$.

Let $\tau_d =$ topology generated in $\mathcal{H}_d(U, F)$ by the family of semi-norms

$$\{\| \cdot \|_{B_r(z)} : 0 < r < d_U(z) \text{ and } z \in U\}.$$

- If $U = B_r(z_0)$, where $z_0 \in E, r > 0$, $\mathcal{H}_L(B_r(z_0)) \subsetneqq \mathcal{H}_b(B_r(z_0), E) = \mathcal{H}_d(B_r(z_0), E)$ and $\tau_d = \tau_b$ in $\mathcal{H}_L(B_r(z_0))$.

- In particular $\mathcal{H}_L(B_E) \subsetneqq \mathcal{H}_b(B_E, E) = \mathcal{H}_d(B_E, E)$ and $\tau_b = \tau_d$.

The space $\mathcal{H}_L(U, E)$

- E separable $\Rightarrow (\mathcal{H}_d(U, F), \tau_d)$ is a Fréchet space (Dineen and Venkova, for $F = \mathbb{C}$)

The space $\mathcal{H}_L(U, E)$

- E separable $\Rightarrow (\mathcal{H}_d(U, F), \tau_d)$ is a Fréchet space (Dineen and Venkova, for $F = \mathbb{C}$)

The space $\mathcal{H}_L(U, E)$

- E separable $\Rightarrow (\mathcal{H}_d(U, F), \tau_d)$ is a Fréchet space (Dineen and Venkova, for $F = \mathbb{C}$)

Proposition

$\mathcal{H}_L(U)$ is a **closed** subspace of $(\mathcal{H}_d(U; E), \tau_d)$ whenever E is separable. Consequently, $(\mathcal{H}_L(U), \tau_d)$ is a Fréchet space whenever E is separable.

- E separable $\Rightarrow (\mathcal{H}_d(U, F), \tau_d)$ is a Fréchet space (Dineen and Venkova, for $F = \mathbb{C}$)

Proposition

$\mathcal{H}_L(U)$ is a **closed** subspace of $(\mathcal{H}_d(U; E), \tau_d)$ whenever E is separable. Consequently, $(\mathcal{H}_L(U), \tau_d)$ is a Fréchet space whenever E is separable.

- From now $\mathcal{H}_L(U)$ denotes $(\mathcal{H}_L(U), \tau_d)$.

The spectrum of the Fréchet algebra (with identity) $\mathcal{H}_L(U)$ has been described in cases $U = E$ and by L.A.M. and A.F. Pereira in case $U = E$ and by G.V.S. Mauro, L.A.M. and A.F. Pereira in case $U = B_r(z_0)$. In both cases $\tau_d = \tau_b$.

The symbolic calculus (or functional calculus)

Take any non empty open subset Ω of $\mathbb{C}$.

The symbolic calculus (or functional calculus)

Take any non empty open subset Ω of $\mathbb{C}$.

Let $E_\Omega = \{x \in E : \sigma(x) \subset \Omega\}$ (where $\sigma(x)$ denotes the spectrum of x).

The symbolic calculus (or functional calculus)

Take any non empty open subset Ω of $\mathbb{C}$.

Let $E_\Omega = \{x \in E : \sigma(x) \subset \Omega\}$ (where $\sigma(x)$ denotes the spectrum of x).

It is well known that E_Ω is open in E.

The symbolic calculus (or functional calculus)

Take any non empty open subset Ω of $\mathbb{C}$.

Let $E_\Omega = \{x \in E : \sigma(x) \subset \Omega\}$ (where $\sigma(x)$ denotes the spectrum of x).

It is well known that E_Ω is open in E.

The symbolic calculus associates to each analytic function $f : \Omega \longrightarrow \mathbb{C}$ the mapping $\tilde{f} : E_\Omega \longrightarrow E$ defined by

$$\tilde{f}(x) = \int_\Gamma f(\lambda)(\lambda e - x)^{-1} d\lambda.$$

Where Γ is any contour surrounding $\sigma(x)$ in Ω.

The symbolic calculus (or functional calculus)

Take any non empty open subset Ω of $\mathbb{C}$.

Let $E_\Omega = \{x \in E : \sigma(x) \subset \Omega\}$ (where $\sigma(x)$ denotes the spectrum of x).

It is well known that E_Ω is open in E.

The symbolic calculus associates to each analytic function $f : \Omega \longrightarrow \mathbb{C}$ the mapping $\tilde{f} : E_\Omega \longrightarrow E$ defined by

$$\tilde{f}(x) = \int_\Gamma f(\lambda)(\lambda e - x)^{-1} d\lambda.$$

Where Γ is any contour surrounding $\sigma(x)$ in Ω.

We are going to present a description of the spectrum of $(\mathcal{H}_L(E_\Omega), \tau_d)$ in case $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and E is a uniform algebra. By using this characterization we can show that in this case the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$ is semi-simple.

Notation and Definitions

As usual,
$\{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < r\}$ will be denoted by $\Delta_r(\lambda_0)$
and
$\Delta_1(0)$ will be denoted by Δ.

Notation and Definitions

As usual,
$\{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < r\}$ will be denoted by $\Delta_r(\lambda_0)$
and
$\Delta_1(0)$ will be denoted by Δ.

Easy to check:
$\phi(E_\Omega) = \Omega$ for every $\phi \in \mathcal{M}(E)$.

Notation and Definitions

As usual,
$\{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < r\}$ will be denoted by $\Delta_r(\lambda_0)$
and
$\Delta_1(0)$ will be denoted by Δ.

Easy to check:
$\phi(E_\Omega) = \Omega$ for every $\phi \in \mathcal{M}(E)$. In particular, $\phi(B_r(z_0)) = \Delta_r(\phi(z_0))$.

Notation and Definitions

As usual,
$\{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < r\}$ will be denoted by $\Delta_r(\lambda_0)$
and
$\Delta_1(0)$ will be denoted by Δ.

Easy to check:
$\phi(E_\Omega) = \Omega$ for every $\phi \in \mathcal{M}(E)$. In particular, $\phi(B_r(z_0)) = \Delta_r(\phi(z_0))$.

- Given any $a \in E$, and $n \in \mathbb{N}$ $\quad P_{a,n}(z) = az^n \;\; \forall z \in E$.

In particular we will write $\quad P_{e,1}(z) = z \quad$ and $\quad P_{a,0}(z) = a \;\; \forall z \in E$.

Notation and Definitions

As usual,
$\{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < r\}$ will be denoted by $\Delta_r(\lambda_0)$
and
$\Delta_1(0)$ will be denoted by Δ.

Easy to check:
$\phi(E_\Omega) = \Omega$ for every $\phi \in \mathcal{M}(E)$. In particular, $\phi(B_r(z_0)) = \Delta_r(\phi(z_0))$.

- Given any $a \in E$, and $n \in \mathbb{N}$ $\quad P_{a,n}(z) = az^n \ \ \forall z \in E$.

In particular we will write $\quad P_{e,1}(z) = z \quad$ and $\quad P_{a,0}(z) = a \ \ \forall z \in E$.

Notation and Definitions

As usual,
$\{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < r\}$ will be denoted by $\Delta_r(\lambda_0)$
and
$\Delta_1(0)$ will be denoted by Δ.

Easy to check:
$\phi(E_\Omega) = \Omega$ for every $\phi \in \mathcal{M}(E)$. In particular, $\phi(B_r(z_0)) = \Delta_r(\phi(z_0))$.

- Given any $a \in E$, and $n \in \mathbb{N}$ $\quad P_{a,n}(z) = az^n \;\; \forall z \in E$.

In particular we will write $\quad P_{e,1}(z) = z \quad$ and $\quad P_{a,0}(z) = a \;\; \forall z \in E$.

Using this notation we frequently write

$$f = \sum_{n=0}^{\infty} P_{a_n,0}(P_{e,1} - P_{z_0,0})^n \text{ in } U$$

instead of

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \text{ for all } z \in U.$$

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

It is easy to check

- $\psi_0 \in \mathcal{M}(E)$

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

It is easy to check

- $\psi_0 \in \mathcal{M}(E)$
- $\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ for every $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0)))$.

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

It is easy to check

- $\psi_0 \in \mathcal{M}(E)$
- $\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ for every $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0)))$.

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

It is easy to check

- $\psi_0 \in \mathcal{M}(E)$
- $\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ for every $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0)))$.

Definition (Glickfeld)

Let $f \in \mathcal{H}_L(U)$ and $\phi \in \mathcal{M}(E)$. If there is a (necessarily unique) $g \in \mathcal{H}(\phi(U))$ so that $g \circ \phi = \phi \circ f$ on U, we say that g is the **quotient function of** f with respect to ϕ, and we write $g = f_\phi$.

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

It is easy to check

- $\psi_0 \in \mathcal{M}(E)$
- $\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ for every $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0)))$.

Definition (Glickfeld)

Let $f \in \mathcal{H}_L(U)$ and $\phi \in \mathcal{M}(E)$. If there is a (necessarily unique) $g \in \mathcal{H}(\phi(U))$ so that $g \circ \phi = \phi \circ f$ on U, we say that g is the **quotient function of** f with respect to ϕ, and we write $g = f_\phi$.

$U = B_r(z_0)$ where $z_0 \in E$ and $r > 0 \Rightarrow f_\phi$ exists for all $\phi \in \mathcal{M}(E)$. (easy)

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

It is easy to check

- $\psi_0 \in \mathcal{M}(E)$
- $\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ for every $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0)))$.

Definition (Glickfeld)

Let $f \in \mathcal{H}_L(U)$ and $\phi \in \mathcal{M}(E)$. If there is a (necessarily unique) $g \in \mathcal{H}(\phi(U))$ so that $g \circ \phi = \phi \circ f$ on U, we say that g is the **quotient function of** f with respect to ϕ, and we write $g = f_\phi$.

$U = B_r(z_0)$ where $z_0 \in E$ and $r > 0 \Rightarrow f_\phi$ exists for all $\phi \in \mathcal{M}(E)$. (easy)

$U =$ star-shaped $\Rightarrow f_\phi$ exists for all $\phi \in \mathcal{M}(E)$.(Glickfeld)

Notation and Definitions

Given any $\psi \in \mathcal{M}(\mathcal{H}_L(U))$, define $\psi_0(a) = \psi(P_{a,0})$ for all $a \in U$.

It is easy to check

- $\psi_0 \in \mathcal{M}(E)$
- $\psi(P_{e,1}) \in \Delta_r(\psi_0(z_0))$ for every $\psi \in \mathcal{M}(\mathcal{H}_L(B_r(z_0)))$.

Definition (Glickfeld)

Let $f \in \mathcal{H}_L(U)$ and $\phi \in \mathcal{M}(E)$. If there is a (necessarily unique) $g \in \mathcal{H}(\phi(U))$ so that $g \circ \phi = \phi \circ f$ on U, we say that g is the **quotient function of** f with respect to ϕ, and we write $g = f_\phi$.

$U = B_r(z_0)$ where $z_0 \in E$ and $r > 0 \Rightarrow f_\phi$ exists for all $\phi \in \mathcal{M}(E)$. (easy)

$U =$ star-shaped $\Rightarrow f_\phi$ exists for all $\phi \in \mathcal{M}(E)$.(Glickfeld)

$U =$ simply connected $\not\Rightarrow f_\phi$ exists for all $\phi \in \mathcal{M}(E)$.(Glickfeld)

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let $\widetilde{\mathcal{H}}(\Omega) = \{\tilde{f} : f \in \mathcal{H}(\Omega)\}$.

It is not difficult to show that

- $\widetilde{\mathcal{H}}(\Omega)$ is a closed subalgebra of $(\mathcal{H}_L(E_\Omega), \tau_d)$.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let $\widetilde{\mathcal{H}}(\Omega) = \{\tilde{f} : f \in \mathcal{H}(\Omega)\}$.

It is not difficult to show that

- $\widetilde{\mathcal{H}}(\Omega)$ is a closed subalgebra of $(\mathcal{H}_L(E_\Omega), \tau_d)$.
- $(\widetilde{\mathcal{H}}(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let $\widetilde{\mathcal{H}}(\Omega) = \{\tilde{f} : f \in \mathcal{H}(\Omega)\}$.

It is not difficult to show that

- $\widetilde{\mathcal{H}}(\Omega)$ is a closed subalgebra of $(\mathcal{H}_L(E_\Omega), \tau_d)$.
- $(\widetilde{\mathcal{H}}(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let $\widetilde{\mathcal{H}}(\Omega) = \{\tilde{f} : f \in \mathcal{H}(\Omega)\}$.

It is not difficult to show that

- $\widetilde{\mathcal{H}}(\Omega)$ is a closed subalgebra of $(\mathcal{H}_L(E_\Omega), \tau_d)$.
- $(\widetilde{\mathcal{H}}(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

Proposition

Given any $f \in \mathcal{H}(\Omega)$ and $\phi \in \mathcal{M}(E)$ the quotient function $\widetilde{f}_\phi$ of $\tilde{f}$ with respect to ϕ exists and $\widetilde{f}_\phi = f$.

Let $\widetilde{\mathcal{H}}(\Omega) = \{\tilde{f} : f \in \mathcal{H}(\Omega)\}$.

It is not difficult to show that

- $\widetilde{\mathcal{H}}(\Omega)$ is a closed subalgebra of $(\mathcal{H}_L(E_\Omega), \tau_d)$.
- $(\widetilde{\mathcal{H}}(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

Proposition

Given any $f \in \mathcal{H}(\Omega)$ and $\phi \in \mathcal{M}(E)$ the quotient function $\widetilde{f}_\phi$ of $\tilde{f}$ with respect to ϕ exists and $\widetilde{f}_\phi = f$.

This guarantees the existence of F_ϕ for every $F \in \widetilde{\mathcal{H}}(\Omega) \subset \mathcal{H}_L(E_\Omega)$. We will show that this is true for all the elements of $\mathcal{H}_L(E_\Omega)$.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ s.th. h^{-1} is analytic.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ s.th. h^{-1} is analytic.

It seems natural to try to prove a Riemann Mapping Theorem in case of Banach algebras E in the context of the Lorch analytic mappings.

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ s.th. h^{-1} is analytic.
It seems natural to try to prove a Riemann Mapping Theorem in case of Banach algebras E in the context of the Lorch analytic mappings.

Definition

Let U, V be two domains in E. We say that an homeomorphism $f : U \to V$ is an L-homeomorphism between U and V if $f \in H_L(U)$ and $f^{-1} \in H_L(V)$. We say that U and V are L-homeomorphic if there exists an L-homeomorphism between U and V.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ s.th. h^{-1} is analytic.
It seems natural to try to prove a Riemann Mapping Theorem in case of Banach algebras E in the context of the Lorch analytic mappings.

Definition

Let U, V be two domains in E. We say that an homeomorphism $f : U \to V$ is an L-homeomorphism between U and V if $f \in H_L(U)$ and $f^{-1} \in H_L(V)$. We say that U and V are L-homeomorphic if there exists an L-homeomorphism between U and V.

A first attempt to show a Riemann Mapping Theorem in the context of the Lorch analytic mappings in E would be to show that the open unit ball B_E is L-homeomorphic to every bounded homeomorphic image of itself. But Glickfeld showed that this cannot be proved for general Banach algebras by presenting an example of a simply connected domain D contained in the algebra $E = C([0,1])$ such that D é homeomorphic to the ball B_E but D is not L-homeomorphic to B_E.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Warren characterized which domains are L-homeomorphic to B_E in case $E = C(X)$, where X is a compact Hausdorff space.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Warren characterized which domains are L-homeomorphic to B_E in case $E = C(X)$, where X is a compact Hausdorff space.

Recall that a simple closed curve in a Banach algebra E is, by definition, a continuous mapping $\Gamma : \partial\Delta \to E$ such that, for each $\phi \in \mathcal{M}(E)$, $\phi \circ \Gamma$ is a homeomorphism between $\partial\Delta$ and $\phi \circ \Gamma(\partial\Delta) \subset \mathbb{C}$.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Warren characterized which domains are L-homeomorphic to B_E in case $E = C(X)$, where X is a compact Hausdorff space.

Recall that a simple closed curve in a Banach algebra E is, by definition, a continuous mapping $\Gamma : \partial\Delta \to E$ such that, for each $\phi \in \mathcal{M}(E)$, $\phi \circ \Gamma$ is a homeomorphism between $\partial\Delta$ and $\phi \circ \Gamma(\partial\Delta) \subset \mathbb{C}$.
The image of each $\phi \circ \Gamma$ is always a Jordan curve in $\mathbb{C}$.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Warren characterized which domains are L-homeomorphic to B_E in case $E = C(X)$, where X is a compact Hausdorff space.

Recall that a simple closed curve in a Banach algebra E is, by definition, a continuous mapping $\Gamma : \partial\Delta \to E$ such that, for each $\phi \in \mathcal{M}(E)$, $\phi \circ \Gamma$ is a homeomorphism between $\partial\Delta$ and $\phi \circ \Gamma(\partial\Delta) \subset \mathbb{C}$.
The image of each $\phi \circ \Gamma$ is always a Jordan curve in $\mathbb{C}$.
By definition, the interior of the curve Γ is the set

$$int(\Gamma) = \{z \in E \,;\; \phi(z) \in int(\phi \circ \Gamma) \text{ for all } \phi \in \mathcal{M}(E)\},$$

where $int(\phi \circ \Gamma)$ is the interior of the Jordan curve $\phi \circ \Gamma$ in the usual sense.

Warren characterized which domains are L-homeomorphic to B_E in case $E = C(X)$, where X is a compact Hausdorff space.

Recall that a simple closed curve in a Banach algebra E is, by definition, a continuous mapping $\Gamma : \partial\Delta \to E$ such that, for each $\phi \in \mathcal{M}(E)$, $\phi \circ \Gamma$ is a homeomorphism between $\partial\Delta$ and $\phi \circ \Gamma(\partial\Delta) \subset \mathbb{C}$.
The image of each $\phi \circ \Gamma$ is always a Jordan curve in $\mathbb{C}$.
By definition, the interior of the curve Γ is the set

$$int(\Gamma) = \{z \in E \, ; \ \phi(z) \in int(\phi \circ \Gamma) \text{ for all } \phi \in \mathcal{M}(E)\} \, ,$$

where $int(\phi \circ \Gamma)$ is the interior of the Jordan curve $\phi \circ \Gamma$ in the usual sense.

Warren considered the case when E is a semi-simple Banach algebra and showed that in this case B_E is L-homeomorphic to $int(\Gamma)$ by a mapping $g : B_E \to int(\Gamma)$ whenever Γ satisfies the following condition: for all $T \in E^*$, $T \circ \Gamma$ has a continuous extension to $\overline{\Delta}$ that is holomorphic in Δ.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ. We may take $\tilde{h} \in \widetilde{\mathcal{H}}(E_\Omega)$ that corresponds to h via the functional calculus and get:

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ. We may take $\tilde{h} \in \widetilde{\mathcal{H}}(E_\Omega)$ that corresponds to h via the functional calculus and get:

Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and E be a Banach algebra. There is an L-homeomorphism between E_Ω and E_Δ given by $\widetilde{h} \in \widetilde{\mathcal{H}}(\Omega)$ where h is a conformal mapping between Ω and Δ.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ. We may take $\tilde{h} \in \widetilde{\mathcal{H}}(E_\Omega)$ that corresponds to h via the functional calculus and get:

Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and E be a Banach algebra. There is an L-homeomorphism between E_Ω and E_Δ given by $\widetilde{h} \in \widetilde{\mathcal{H}}(\Omega)$ where h is a conformal mapping between Ω and Δ.

Now we are able to show:

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ. We may take $\tilde{h} \in \widetilde{\mathcal{H}}(E_\Omega)$ that corresponds to h via the functional calculus and get:

Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and E be a Banach algebra. There is an L-homeomorphism between E_Ω and E_Δ given by $\widetilde{h} \in \widetilde{\mathcal{H}}(\Omega)$ where h is a conformal mapping between Ω and Δ.

Now we are able to show:

Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let E be an unitary commutative semi-simple Banach algebra. Then B_E and E_Ω are L-homeomorphic.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

If $\Omega \subsetneq \mathbb{C}$ is a simply-connected domain, by the Riemann Mapping Theorem there exists a one-to-one analytic mapping h from Ω onto Δ. We may take $\tilde{h} \in \widetilde{\mathcal{H}}(E_\Omega)$ that corresponds to h via the functional calculus and get:

Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and E be a Banach algebra. There is an L-homeomorphism between E_Ω and E_Δ given by $\widetilde{h} \in \widetilde{\mathcal{H}}(\Omega)$ where h is a conformal mapping between Ω and Δ.

Now we are able to show:

Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let E be an unitary commutative semi-simple Banach algebra. Then B_E and E_Ω are L-homeomorphic.

Idea of the proof:

Idea of the proof:
Consider the simple closed curve $\Gamma_0 : \partial\Delta \longrightarrow E$ given by $\Gamma_0(\lambda) = \lambda\mathbf{e}$ where $\mathbf{e}$ is the unit in E. It is easy to check that $int(\phi \circ \Gamma_0) = \Delta$ for all $\phi \in \mathcal{M}(E)$ and hence $int(\Gamma_0) = E_\Delta$.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Idea of the proof:
Consider the simple closed curve $\Gamma_0 : \partial\Delta \longrightarrow E$ given by $\Gamma_0(\lambda) = \lambda\mathbf{e}$ where $\mathbf{e}$ is the unit in E. It is easy to check that $int(\phi \circ \Gamma_0) = \Delta$ for all $\phi \in \mathcal{M}(E)$ and hence $int(\Gamma_0) = E_\Delta$.
Show that Warren's result applies and get the existence of an L-homeomorphism $g : B_E \to E_\Delta$.

Idea of the proof:

Consider the simple closed curve $\Gamma_0 : \partial\Delta \longrightarrow E$ given by $\Gamma_0(\lambda) = \lambda\mathbf{e}$ where **e** is the unit in E. It is easy to check that $int(\phi \circ \Gamma_0) = \Delta$ for all $\phi \in \mathcal{M}(E)$ and hence $int(\Gamma_0) = E_\Delta$.

Show that Warren's result applies and get the existence of an L-homeomorphism $g : B_E \to E_\Delta$.

By the previous theorem we have that $\widetilde{h^{-1}} : E_\Delta \to E_\Omega$ is an L-homeomorphism between E_Δ and E_Ω.

Idea of the proof:
Consider the simple closed curve $\Gamma_0 : \partial\Delta \longrightarrow E$ given by $\Gamma_0(\lambda) = \lambda\mathbf{e}$ where $\mathbf{e}$ is the unit in E. It is easy to check that $int(\phi \circ \Gamma_0) = \Delta$ for all $\phi \in \mathcal{M}(E)$ and hence $int(\Gamma_0) = E_\Delta$.
Show that Warren's result applies and get the existence of an L-homeomorphism $g : B_E \to E_\Delta$.
By the previous theorem we have that $\widetilde{h^{-1}} : E_\Delta \to E_\Omega$ is an L-homeomorphism between E_Δ and E_Ω.
Hence, $\widetilde{h^{-1}} \circ g : B_E \to E_\Omega$ is an L-homeomorphism between B_E and E_Ω.
This completes the proof.

The algebras $\widetilde{\mathcal{H}}(\Omega)$ and $(\mathcal{H}_L(E_\Omega), \tau_d)$

Idea of the proof:
Consider the simple closed curve $\Gamma_0 : \partial\Delta \longrightarrow E$ given by $\Gamma_0(\lambda) = \lambda\mathbf{e}$ where $\mathbf{e}$ is the unit in E. It is easy to check that $int(\phi \circ \Gamma_0) = \Delta$ for all $\phi \in \mathcal{M}(E)$ and hence $int(\Gamma_0) = E_\Delta$.
Show that Warren's result applies and get the existence of an L-homeomorphism $g : B_E \to E_\Delta$.
By the previous theorem we have that $\widetilde{h^{-1}} : E_\Delta \to E_\Omega$ is an L-homeomorphism between E_Δ and E_Ω.
Hence, $\widetilde{h^{-1}} \circ g : B_E \to E_\Omega$ is an L-homeomorphism between B_E and E_Ω.
This completes the proof.

Theorem 1 (G.V.S. Mauro, L.A.M. and A.F. Pereira)

Let E be a commutative Banach algebra with a unit element e. The spectrum $\mathcal{M}(\mathcal{H}_L(B_E))$ is homeomorphic to $\mathcal{M}(E) \times \Delta$ by the mapping

$$\delta : \mathcal{M}(E) \times \Delta \longrightarrow \mathcal{M}(\mathcal{H}_L(B_E))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(B_E)$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Recall that a **uniform algebra** is a closed subalgebra of $\mathcal{C}(X)$, where X is a compact Hausdorff space, which contains the constants and separates the points of X.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Recall that a **uniform algebra** is a closed subalgebra of $\mathcal{C}(X)$, where X is a compact Hausdorff space, which contains the constants and separates the points of X.

Clearly our uniform algebras are always unitary commutative Banach algebras. Moreover, it is well known that every uniform algebra E is semi-simple and satisfy the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z. Consequently, $E_\Delta = B_E$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Recall that a **uniform algebra** is a closed subalgebra of $\mathcal{C}(X)$, where X is a compact Hausdorff space, which contains the constants and separates the points of X.

Clearly our uniform algebras are always unitary commutative Banach algebras. Moreover, it is well known that every uniform algebra E is semi-simple and satisfy the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z. Consequently, $E_\Delta = B_E$.

Our next goal is to describe the spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$ when E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain. We recall that $(\mathcal{H}_L(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Recall that a **uniform algebra** is a closed subalgebra of $\mathcal{C}(X)$, where X is a compact Hausdorff space, which contains the constants and separates the points of X.

Clearly our uniform algebras are always unitary commutative Banach algebras. Moreover, it is well known that every uniform algebra E is semi-simple and satisfy the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z. Consequently, $E_\Delta = B_E$.

Our next goal is to describe the spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$ when E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain. We recall that $(\mathcal{H}_L(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

From now h denotes always a one-to-one analytic mapping h from Ω onto Δ.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Recall that a **uniform algebra** is a closed subalgebra of $\mathcal{C}(X)$, where X is a compact Hausdorff space, which contains the constants and separates the points of X.

Clearly our uniform algebras are always unitary commutative Banach algebras. Moreover, it is well known that every uniform algebra E is semi-simple and satisfy the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z. Consequently, $E_\Delta = B_E$.

Our next goal is to describe the spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$ when E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain. We recall that $(\mathcal{H}_L(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

From now h denotes always a one-to-one analytic mapping h from Ω onto Δ.

Note that $\widetilde{h} : E_\Omega \to E_\Delta, \;\; \widetilde{h}^{-1} : E_\Delta \to E_\Omega$ and it is easy to show that $\widetilde{h}^{-1} = \widetilde{h^{-1}}$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Recall that a **uniform algebra** is a closed subalgebra of $\mathcal{C}(X)$, where X is a compact Hausdorff space, which contains the constants and separates the points of X.

Clearly our uniform algebras are always unitary commutative Banach algebras. Moreover, it is well known that every uniform algebra E is semi-simple and satisfy the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z. Consequently, $E_\Delta = B_E$.

Our next goal is to describe the spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$ when E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain. We recall that $(\mathcal{H}_L(E_\Omega), \tau_d)$ is a Fréchet algebra whenever E is separable.

From now h denotes always a one-to-one analytic mapping h from Ω onto Δ.

Note that $\widetilde{h} : E_\Omega \to E_\Delta, \;\; \widetilde{h}^{-1} : E_\Delta \to E_\Omega$ and it is easy to show that $\widetilde{h}^{-1} = \widetilde{h^{-1}}$.

Clearly $E_\Delta = B_E \Rightarrow \widetilde{h}(z) \in B_E$ for all $z \in E_\Omega$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Proposition

Let E be a uniform algebra. If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $f \in \mathcal{H}_L(E_\Omega)$, then for every $\phi \in \mathcal{M}(E)$ there exists a function $f_\phi \in \mathcal{H}(\Omega)$ satisfying $f_\phi(\phi(z)) = \phi(f(z))$ for all $z \in E_\Omega$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Proposition

Let E be a uniform algebra. If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $f \in \mathcal{H}_L(E_\Omega)$, then for every $\phi \in \mathcal{M}(E)$ there exists a function $f_\phi \in \mathcal{H}(\Omega)$ satisfying $f_\phi(\phi(z)) = \phi(f(z))$ for all $z \in E_\Omega$.

Idea of the proof:

Proposition

Let E be a uniform algebra. If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $f \in \mathcal{H}_L(E_\Omega)$, then for every $\phi \in \mathcal{M}(E)$ there exists a function $f_\phi \in \mathcal{H}(\Omega)$ satisfying $f_\phi(\phi(z)) = \phi(f(z))$ for all $z \in E_\Omega$.

Idea of the proof: $f \in \mathcal{H}_L(E_\Omega) \Rightarrow f \circ \widetilde{h^{-1}} \in \mathcal{H}_L(B_E)$.

Proposition

Let E be a uniform algebra. If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $f \in \mathcal{H}_L(E_\Omega)$, then for every $\phi \in \mathcal{M}(E)$ there exists a function $f_\phi \in \mathcal{H}(\Omega)$ satisfying $f_\phi(\phi(z)) = \phi(f(z))$ for all $z \in E_\Omega$.

Idea of the proof: $f \in \mathcal{H}_L(E_\Omega) \Rightarrow f \circ \widetilde{h^{-1}} \in \mathcal{H}_L(B_E)$. Take

$$f \circ \widetilde{h^{-1}}(w) = \sum_{n=0}^{\infty} a_n w^n \text{ for all } w \in B_E.$$

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Proposition

Let E be a uniform algebra. If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $f \in \mathcal{H}_L(E_\Omega)$, then for every $\phi \in \mathcal{M}(E)$ there exists a function $f_\phi \in \mathcal{H}(\Omega)$ satisfying $f_\phi(\phi(z)) = \phi(f(z))$ for all $z \in E_\Omega$.

Idea of the proof: $f \in \mathcal{H}_L(E_\Omega) \Rightarrow f \circ \widetilde{h^{-1}} \in \mathcal{H}_L(B_E)$. Take

$$f \circ \widetilde{h^{-1}}(w) = \sum_{n=0}^{\infty} a_n w^n \text{ for all } w \in B_E.$$

Clearly $f(z) = (f \circ \widetilde{h^{-1}})(\widetilde{h}(z)) = \sum_{n=0}^{\infty} a_n(\widetilde{h}(z))^n$ for every $z \in E_\Omega$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Proposition

Let E be a uniform algebra. If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $f \in \mathcal{H}_L(E_\Omega)$, then for every $\phi \in \mathcal{M}(E)$ there exists a function $f_\phi \in \mathcal{H}(\Omega)$ satisfying $f_\phi(\phi(z)) = \phi(f(z))$ for all $z \in E_\Omega$.

Idea of the proof: $f \in \mathcal{H}_L(E_\Omega) \Rightarrow f \circ \widetilde{h^{-1}} \in \mathcal{H}_L(B_E)$. Take

$$f \circ \widetilde{h^{-1}}(w) = \sum_{n=0}^{\infty} a_n w^n \text{ for all } w \in B_E.$$

Clearly $f(z) = (f \circ \widetilde{h^{-1}})(\widetilde{h}(z)) = \sum_{n=0}^{\infty} a_n(\widetilde{h}(z))^n$ for every $z \in E_\Omega$.

Since $\phi(E_\Omega) = \Omega$, by using the above equality show that $f_\phi : \phi(E_\Omega) \to \mathbb{C}$ defined by

$$f_\phi(\lambda) = \sum_{n=0}^{\infty} \phi(a_n)(h(\lambda))^n \text{ for all } \lambda \in \Omega$$

satisfies $f_\phi \circ \phi = \phi \circ f$. This completes the proof.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let τ_π denote the restriction to $\mathcal{M}(E) \times \Omega$ of the product topology in $\mathcal{M}(E) \times \mathbb{C}$.

Theorem 2

If E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, the mapping

$$\delta : \mathcal{M}(E) \times \Omega \longrightarrow \mathcal{M}(\mathcal{H}_L(E_\Omega))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$ is a homeomorphism between $(\mathcal{M}(E) \times \Omega, \tau_\pi)$ and $(\mathcal{M}(\mathcal{H}_L(E_\Omega)), \tau_G)$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let τ_π denote the restriction to $\mathcal{M}(E) \times \Omega$ of the product topology in $\mathcal{M}(E) \times \mathbb{C}$.

Theorem 2

If E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, the mapping

$$\delta : \mathcal{M}(E) \times \Omega \longrightarrow \mathcal{M}(\mathcal{H}_L(E_\Omega))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$ is a homeomorphism between $(\mathcal{M}(E) \times \Omega, \tau_\pi)$ and $(\mathcal{M}(\mathcal{H}_L(E_\Omega)), \tau_G)$.

We noted already that given if $\psi \in \mathcal{M}(\mathcal{H}_L(B_E))$ then it can be proved that $\psi(P_{e,1}) \in \Delta$. Moreover we can get $\psi_0 \in \mathcal{M}(E)$ by defining $\psi_0(a) = \psi(P_{a,0})$ for all $a \in E$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let τ_π denote the restriction to $\mathcal{M}(E) \times \Omega$ of the product topology in $\mathcal{M}(E) \times \mathbb{C}$.

Theorem 2

If E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, the mapping

$$\delta : \mathcal{M}(E) \times \Omega \longrightarrow \mathcal{M}(\mathcal{H}_L(E_\Omega))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$ is a homeomorphism between $(\mathcal{M}(E) \times \Omega, \tau_\pi)$ and $(\mathcal{M}(\mathcal{H}_L(E_\Omega)), \tau_G)$.

We noted already that given if $\psi \in \mathcal{M}(\mathcal{H}_L(B_E))$ then it can be proved that $\psi(P_{e,1}) \in \Delta$. Moreover we can get $\psi_0 \in \mathcal{M}(E)$ by defining $\psi_0(a) = \psi(P_{a,0})$ for all $a \in E$.

Theorem 1 states that to each $\psi \in \mathcal{M}(\mathcal{H}_L(B_E))$ we can associate $(\phi, \lambda_0) \in \mathcal{M}(E) \times \Delta$ such that $\psi(f) = \delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(B_E)$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let τ_π denote the restriction to $\mathcal{M}(E) \times \Omega$ of the product topology in $\mathcal{M}(E) \times \mathbb{C}$.

Theorem 2

If E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, the mapping

$$\delta : \mathcal{M}(E) \times \Omega \longrightarrow \mathcal{M}(\mathcal{H}_L(E_\Omega))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$ is a homeomorphism between $(\mathcal{M}(E) \times \Omega, \tau_\pi)$ and $(\mathcal{M}(\mathcal{H}_L(E_\Omega)), \tau_G)$.

We noted already that given if $\psi \in \mathcal{M}(\mathcal{H}_L(B_E))$ then it can be proved that $\psi(P_{e,1}) \in \Delta$. Moreover we can get $\psi_0 \in \mathcal{M}(E)$ by defining $\psi_0(a) = \psi(P_{a,0})$ for all $a \in E$.

Theorem 1 states that to each $\psi \in \mathcal{M}(\mathcal{H}_L(B_E))$ we can associate $(\phi, \lambda_0) \in \mathcal{M}(E) \times \Delta$ such that $\psi(f) = \delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(B_E)$. This can be proved by choosing $\phi = \psi_0 \in \mathcal{M}(E)$ and $\lambda_0 = \psi(P_{e,1}) \in \Delta$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Let τ_π denote the restriction to $\mathcal{M}(E) \times \Omega$ of the product topology in $\mathcal{M}(E) \times \mathbb{C}$.

Theorem 2

If E is a uniform algebra and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, the mapping

$$\delta : \mathcal{M}(E) \times \Omega \longrightarrow \mathcal{M}(\mathcal{H}_L(E_\Omega))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$ is a homeomorphism between $(\mathcal{M}(E) \times \Omega, \tau_\pi)$ and $(\mathcal{M}(\mathcal{H}_L(E_\Omega)), \tau_G)$.

We noted already that given if $\psi \in \mathcal{M}(\mathcal{H}_L(B_E))$ then it can be proved that $\psi(P_{e,1}) \in \Delta$. Moreover we can get $\psi_0 \in \mathcal{M}(E)$ by defining $\psi_0(a) = \psi(P_{a,0})$ for all $a \in E$.

Theorem 1 states that to each $\psi \in \mathcal{M}(\mathcal{H}_L(B_E))$ we can associate $(\phi, \lambda_0) \in \mathcal{M}(E) \times \Delta$ such that $\psi(f) = \delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(B_E)$. This can be proved by choosing $\phi = \psi_0 \in \mathcal{M}(E)$ and $\lambda_0 = \psi(P_{e,1}) \in \Delta$.

A similar argument did not work when we tried to prove that δ is surjective in Theorem 2.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Next we are going to show that δ is surjective in Theorem 2. Our proof uses the following result:

Next we are going to show that δ is surjective in Theorem 2. Our proof uses the following result:

Lemma

Let E be a uniform algebra, $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and h be a conformal mapping between Ω and Δ. If $z_0 \in E_\Omega$ and $0 < r < d_{E_\Omega}(z_0) = \inf_{w \in E_\Omega} \|z_0 - w\|$, then $\widetilde{h}(B_r(z_0))$ is a B_E-bounded subset of B_E.

Next we are going to show that δ is surjective in Theorem 2. Our proof uses the following result:

Lemma

Let E be a uniform algebra, $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and h be a conformal mapping between Ω and Δ. If $z_0 \in E_\Omega$ and $0 < r < d_{E_\Omega}(z_0) = \inf_{w \in E_\Omega} \|z_0 - w\|$, then $\widetilde{h}(B_r(z_0))$ is a B_E-bounded subset of B_E.

The proof of this lemma uses strongly the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z.

Next we are going to show that δ is surjective in Theorem 2. Our proof uses the following result:

Lemma

Let E be a uniform algebra, $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and h be a conformal mapping between Ω and Δ. If $z_0 \in E_\Omega$ and $0 < r < d_{E_\Omega}(z_0) = \inf_{w \in E_\Omega} \|z_0 - w\|$, then $\widetilde{h}(B_r(z_0))$ is a B_E-bounded subset of B_E.

The proof of this lemma uses strongly the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z.

Let us show that δ is surjective.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Next we are going to show that δ is surjective in Theorem 2. Our proof uses the following result:

Lemma

Let E be a uniform algebra, $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and h be a conformal mapping between Ω and Δ. If $z_0 \in E_\Omega$ and $0 < r < d_{E_\Omega}(z_0) = \inf_{w \in E_\Omega} \|z_0 - w\|$, then $\widetilde{h}(B_r(z_0))$ is a B_E-bounded subset of B_E.

The proof of this lemma uses strongly the equality $\|z\| = \|\hat{z}\|_\infty$ for every $z \in E$ where $\hat{z}$ denotes the Gelfand transform of z.

Let us show that δ is surjective.

Take $\psi \in \mathcal{M}(\mathcal{H}_L(E_\Omega))$. We are going to get $(\phi, \lambda_0) \in \mathcal{M}(E) \times \Omega$ such that $\psi(f) = \delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Easy: $f \in \mathcal{H}_L(B_E) \Rightarrow f \circ \widetilde{h} \in \mathcal{H}_L(E_\Omega)$.

Easy: $f \in \mathcal{H}_L(B_E) \Rightarrow f \circ \widetilde{h} \in \mathcal{H}_L(E_\Omega)$.

Now, define

$$\begin{aligned} \psi_h : \mathcal{H}_L(B_E) &\longrightarrow \mathbb{C} \\ f &\longmapsto \psi(f \circ \widetilde{h}) \end{aligned}$$

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Easy: $f \in \mathcal{H}_L(B_E) \Rightarrow f \circ \widetilde{h} \in \mathcal{H}_L(E_\Omega)$.

Now, define

$$\begin{aligned} \psi_h : \mathcal{H}_L(B_E) &\longrightarrow \mathbb{C} \\ f &\longmapsto \psi(f \circ \widetilde{h}) \end{aligned}$$

- $\psi_h \in \mathcal{M}(\mathcal{H}_L(B_E))$. (The Lemma is used to show that ψ_h is continuous).

Easy: $f \in \mathcal{H}_L(B_E) \Rightarrow f \circ \widetilde{h} \in \mathcal{H}_L(E_\Omega)$.

Now, define

$$\begin{aligned}\psi_h : \mathcal{H}_L(B_E) &\longrightarrow \mathbb{C}\\ f &\longmapsto \psi(f \circ \widetilde{h})\end{aligned}$$

- $\psi_h \in \mathcal{M}(\mathcal{H}_L(B_E))$. (The Lemma is used to show that ψ_h is continuous).

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Easy: $f \in \mathcal{H}_L(B_E) \Rightarrow f \circ \widetilde{h} \in \mathcal{H}_L(E_\Omega)$.

Now, define

$$\begin{aligned} \psi_h : \mathcal{H}_L(B_E) &\longrightarrow \mathbb{C} \\ f &\longmapsto \psi(f \circ \widetilde{h}) \end{aligned}$$

- $\psi_h \in \mathcal{M}(\mathcal{H}_L(B_E))$. (The Lemma is used to show that ψ_h is continuous).

For every $a \in E$,

$$(\psi_h)_0(a) = \psi_h(P_{a,0}) = \psi(P_{a,0} \circ \widetilde{h}) = \psi(P_{a,0}) = \psi_0(a).$$

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Easy: $f \in \mathcal{H}_L(B_E) \Rightarrow f \circ \widetilde{h} \in \mathcal{H}_L(E_\Omega)$.

Now, define

$$\begin{aligned} \psi_h : \mathcal{H}_L(B_E) &\longrightarrow \mathbb{C} \\ f &\longmapsto \psi(f \circ \widetilde{h}) \end{aligned}$$

- $\psi_h \in \mathcal{M}(\mathcal{H}_L(B_E))$. (The Lemma is used to show that ψ_h is continuous).

For every $a \in E$,

$$(\psi_h)_0(a) = \psi_h(P_{a,0}) = \psi(P_{a,0} \circ \widetilde{h}) = \psi(P_{a,0}) = \psi_0(a).$$

Now: $f \in \mathcal{H}_L(E_\Omega) \Rightarrow f \circ \widetilde{h^{-1}} \in \mathcal{H}_L(B_E)$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Easy: $f \in \mathcal{H}_L(B_E) \Rightarrow f \circ \widetilde{h} \in \mathcal{H}_L(E_\Omega)$.

Now, define

$$\begin{aligned} \psi_h : \mathcal{H}_L(B_E) &\longrightarrow \mathbb{C} \\ f &\longmapsto \psi(f \circ \widetilde{h}) \end{aligned}$$

- $\psi_h \in \mathcal{M}(\mathcal{H}_L(B_E))$. (The Lemma is used to show that ψ_h is continuous).

For every $a \in E$,

$$(\psi_h)_0(a) = \psi_h(P_{a,0}) = \psi(P_{a,0} \circ \widetilde{h}) = \psi(P_{a,0}) = \psi_0(a).$$

Now: $f \in \mathcal{H}_L(E_\Omega) \Rightarrow f \circ \widetilde{h^{-1}} \in \mathcal{H}_L(B_E)$. And by Theorem 1, $\psi_h = \delta((\psi_h)_0, \mu_0)$ where $\mu_0 = \psi_h(P_{e,1}) \in \Delta$.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Since $(\psi_h)_0 = \psi_0$, for every $f \in \mathcal{H}_L(E_\Omega)$ we have

$$\psi(f) = \psi(f \circ \widetilde{h^{-1}} \circ \widetilde{h}) = \psi_h(f \circ \widetilde{h^{-1}}) = (f \circ \widetilde{h^{-1}})_{\psi_0}(\mu_0)$$

Since $(\psi_h)_0 = \psi_0$, for every $f \in \mathcal{H}_L(E_\Omega)$ we have

$$\psi(f) = \psi(f \circ \widetilde{h^{-1}} \circ \widetilde{h}) = \psi_h(f \circ \widetilde{h^{-1}}) = (f \circ \widetilde{h^{-1}})_{\psi_0}(\mu_0)$$

As h is a bijection between Ω and Δ, there exists $\lambda_0 \in \Omega$ such that $h(\lambda_0) = \mu_0$. So,

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Since $(\psi_h)_0 = \psi_0$, for every $f \in \mathcal{H}_L(E_\Omega)$ we have

$$\psi(f) = \psi(f \circ \widetilde{h^{-1}} \circ \widetilde{h}) = \psi_h(f \circ \widetilde{h^{-1}}) = (f \circ \widetilde{h^{-1}})_{\psi_0}(\mu_0)$$

As h is a bijection between Ω and Δ, there exists $\lambda_0 \in \Omega$ such that $h(\lambda_0) = \mu_0$. So,

$$\psi(f) = (f \circ \widetilde{h^{-1}})_{\psi_0}(\mu_0) = (f \circ \widetilde{h^{-1}})_{\psi_0}(h(\lambda_0)) = f_{\psi_0} \circ (\widetilde{h^{-1}})_{\psi_0} \circ h(\lambda_0)$$
$$= f_{\psi_0} \circ h^{-1} \circ h(\lambda_0) = f_{\psi_0}(\lambda_0) = \delta(\psi_0, \lambda_0)(f).$$

Hence, $\psi(f) = f_{\psi_0}(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$.
This shows that δ is surjective.

The spectrum of the algebra $(\mathcal{H}_L(E_\Omega), \tau_d)$

Since $(\psi_h)_0 = \psi_0$, for every $f \in \mathcal{H}_L(E_\Omega)$ we have

$$\psi(f) = \psi(f \circ \widetilde{h^{-1}} \circ \widetilde{h}) = \psi_h(f \circ \widetilde{h^{-1}}) = (f \circ \widetilde{h^{-1}})_{\psi_0}(\mu_0)$$

As h is a bijection between Ω and Δ, there exists $\lambda_0 \in \Omega$ such that $h(\lambda_0) = \mu_0$. So,

$$\psi(f) = (f \circ \widetilde{h^{-1}})_{\psi_0}(\mu_0) = (f \circ \widetilde{h^{-1}})_{\psi_0}(h(\lambda_0)) = f_{\psi_0} \circ (\widetilde{h^{-1}})_{\psi_0} \circ h(\lambda_0)$$
$$= f_{\psi_0} \circ h^{-1} \circ h(\lambda_0) = f_{\psi_0}(\lambda_0) = \delta(\psi_0, \lambda_0)(f).$$

Hence, $\psi(f) = f_{\psi_0}(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$.
This shows that δ is surjective.

As a consequence of Theorem 2 we get:

Proposition

Let E be a uniform algebra and $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Then, if E is a separable space, $\mathcal{H}_L(E_\Omega)$ is a semi-simple algebra.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.
- Every separable C^*-álgebra $\mathcal{A}$ (not necessarily unitary) has an approximation of the identity, i.e., a sequence $(\mathbf{e}_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \|\mathbf{e}_n z - z\| = 0$ for every $z \in \mathcal{A}$ and $\|\mathbf{e}_n\| \leq 1$ for every $n \in \mathbb{N}$.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.
- Every separable C^*-álgebra $\mathcal{A}$ (not necessarily unitary) has an approximation of the identity, i.e., a sequence $(\mathbf{e}_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \|\mathbf{e}_n z - z\| = 0$ for every $z \in \mathcal{A}$ and $\|\mathbf{e}_n\| \leq 1$ for every $n \in \mathbb{N}$.
- If $\mathcal{A}$ is a separable C^*-algebra and $\phi \in \mathcal{M}(\mathcal{A})$, then $ker(\phi)$ is a separable C^*-algebra as well.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.
- Every separable C^*-álgebra $\mathcal{A}$ (not necessarily unitary) has an approximation of the identity, i.e., a sequence $(\mathbf{e}_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \|\mathbf{e}_n z - z\| = 0$ for every $z \in \mathcal{A}$ and $\|\mathbf{e}_n\| \leq 1$ for every $n \in \mathbb{N}$.
- If $\mathcal{A}$ is a separable C^*-algebra and $\phi \in \mathcal{M}(\mathcal{A})$, then $ker(\phi)$ is a separable C^*-algebra as well.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.
- Every separable C^*-álgebra $\mathcal{A}$ (not necessarily unitary) has an approximation of the identity, i.e., a sequence $(\mathbf{e}_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \|\mathbf{e}_n z - z\| = 0$ for every $z \in \mathcal{A}$ and $\|\mathbf{e}_n\| \leq 1$ for every $n \in \mathbb{N}$.
- If $\mathcal{A}$ is a separable C^*-algebra and $\phi \in \mathcal{M}(\mathcal{A})$, then $ker(\phi)$ is a separable C^*-algebra as well.

Let $\mathcal{M}_0(\mathcal{H}_L(E_\Omega))$ denote the spectrum of $(\mathcal{H}_L(E_\Omega), \tau_0)$ endowed with the Gelfand topology τ_G.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.
- Every separable C^*-álgebra $\mathcal{A}$ (not necessarily unitary) has an approximation of the identity, i.e., a sequence $(\mathbf{e}_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \|\mathbf{e}_n z - z\| = 0$ for every $z \in \mathcal{A}$ and $\|\mathbf{e}_n\| \leq 1$ for every $n \in \mathbb{N}$.
- If $\mathcal{A}$ is a separable C^*-algebra and $\phi \in \mathcal{M}(\mathcal{A})$, then $ker(\phi)$ is a separable C^*-algebra as well.

Let $\mathcal{M}_0(\mathcal{H}_L(E_\Omega))$ denote the spectrum of $(\mathcal{H}_L(E_\Omega), \tau_0)$ endowed with the Gelfand topology τ_G.

Well known: If θ is a complex homomorphism defined in $\mathcal{H}(\Omega)$, then $\theta(f) = f(\lambda_\theta)$ for every $f \in \mathcal{H}(\Omega)$ where $\lambda_\theta = \theta(Id) \in \Omega$

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Let us consider the space locally convex space $(\mathcal{H}_L(E_\Omega), \tau_0)$. It is known:

- $(\mathcal{H}_L(E_\Omega), \tau_0)$ is complete.
- Every separable C^*-álgebra $\mathcal{A}$ (not necessarily unitary) has an approximation of the identity, i.e., a sequence $(\mathbf{e}_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \|\mathbf{e}_n z - z\| = 0$ for every $z \in \mathcal{A}$ and $\|\mathbf{e}_n\| \leq 1$ for every $n \in \mathbb{N}$.
- If $\mathcal{A}$ is a separable C^*-algebra and $\phi \in \mathcal{M}(\mathcal{A})$, then $ker(\phi)$ is a separable C^*-algebra as well.

Let $\mathcal{M}_0(\mathcal{H}_L(E_\Omega))$ denote the spectrum of $(\mathcal{H}_L(E_\Omega), \tau_0)$ endowed with the Gelfand topology τ_G.

Well known: If θ is a complex homomorphism defined in $\mathcal{H}(\Omega)$, then $\theta(f) = f(\lambda_\theta)$ for every $f \in \mathcal{H}(\Omega)$ where $\lambda_\theta = \theta(Id) \in \Omega$

Lemma 1

If ψ is a complex homomorphism defined in $\mathcal{H}_L(E_\Omega)$, then $\psi(\widetilde{f}) = f(\lambda_0)$ for every $f \in \mathcal{H}(\Omega)$, where $\lambda_0 = \psi(P_{\mathbf{e},1})$.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Lemma 2

Let E be a separable C^*-álgebra, $f \in \mathcal{H}_L(E_\Omega)$ and $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$. If $f(z) \in \ker(\psi_0)$ for every $z \in E_\Omega$, then $f \in \ker(\psi)$.

Lemma 2

Let E be a separable C^*-álgebra, $f \in \mathcal{H}_L(E_\Omega)$ and $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$. If $f(z) \in \ker(\psi_0)$ for every $z \in E_\Omega$, then $f \in \ker(\psi)$.

Idea of proof:
$\psi_0 \in \mathcal{M}(E) \Rightarrow \ker(\psi_0)$ is a separable C^*-álgebra $\Rightarrow$ there exists an approximation of the identity $(\mathbf{e}_n)_{n\in\mathbb{N}}$ in $ker(\psi_0)$.
Show that:

- $\{P_{e_n,1} \circ f; n \in \mathbb{N}\} \subset \mathcal{C}(E_\Omega, E)$ is equicontinuous.

Lemma 2

Let E be a separable C^*-álgebra, $f \in \mathcal{H}_L(E_\Omega)$ and $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$. If $f(z) \in \ker(\psi_0)$ for every $z \in E_\Omega$, then $f \in \ker(\psi)$.

Idea of proof:
$\psi_0 \in \mathcal{M}(E) \Rightarrow \ker(\psi_0)$ is a separable C^*-álgebra $\Rightarrow$ there exists an approximation of the identity $(\mathbf{e}_n)_{n\in\mathbb{N}}$ in $ker(\psi_0)$.
Show that:

- $\{P_{e_n,1} \circ f; n \in \mathbb{N}\} \subset \mathcal{C}(E_\Omega, E)$ is equicontinuous.

Lemma 2

Let E be a separable C^*−álgebra, $f \in \mathcal{H}_L(E_\Omega)$ and $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$. If $f(z) \in \ker(\psi_0)$ for every $z \in E_\Omega$, then $f \in \ker(\psi)$.

Idea of proof:
$\psi_0 \in \mathcal{M}(E) \Rightarrow \ker(\psi_0)$ is a separable C^*−álgebra $\Rightarrow$ there exists an approximation of the identity $(\mathbf{e}_n)_{n\in\mathbb{N}}$ in $ker(\psi_0)$.
Show that:

- $\{P_{e_n,1} \circ f; n \in \mathbb{N}\} \subset \mathcal{C}(E_\Omega, E)$ is equicontinuous.
- $\overline{\{(P_{e_n,1} \circ f)(z); n \in \mathbb{N}\}}$ is a compact set for every $z \in E_\Omega$.

Lemma 2

Let E be a separable C^*-álgebra, $f \in \mathcal{H}_L(E_\Omega)$ and $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$. If $f(z) \in \ker(\psi_0)$ for every $z \in E_\Omega$, then $f \in \ker(\psi)$.

Idea of proof:
$\psi_0 \in \mathcal{M}(E) \Rightarrow \ker(\psi_0)$ is a separable C^*-álgebra $\Rightarrow$ there exists an approximation of the identity $(\mathbf{e}_n)_{n\in\mathbb{N}}$ in $ker(\psi_0)$.
Show that:

- $\{P_{e_n,1} \circ f; n \in \mathbb{N}\} \subset \mathcal{C}(E_\Omega, E)$ is equicontinuous.
- $\overline{\{(P_{e_n,1} \circ f)(z); n \in \mathbb{N}\}}$ is a compact set for every $z \in E_\Omega$.

Lemma 2

Let E be a separable C^*-álgebra, $f \in \mathcal{H}_L(E_\Omega)$ and $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$. If $f(z) \in \ker(\psi_0)$ for every $z \in E_\Omega$, then $f \in \ker(\psi)$.

Idea of proof:
$\psi_0 \in \mathcal{M}(E) \Rightarrow \ker(\psi_0)$ is a separable C^*-álgebra $\Rightarrow$ there exists an approximation of the identity $(\mathbf{e}_n)_{n\in\mathbb{N}}$ in $ker(\psi_0)$.
Show that:

- $\{P_{e_n,1} \circ f; n \in \mathbb{N}\} \subset \mathcal{C}(E_\Omega, E)$ is equicontinuous.
- $\overline{\{(P_{e_n,1} \circ f)(z); n \in \mathbb{N}\}}$ is a compact set for every $z \in E_\Omega$.

Arzela-Ascoli Theorem $\Rightarrow$ there exists $g \in \mathcal{C}(E_\Omega, E)$ such that $P_{e_n,1} \circ f \overset{\tau_0}{\to} g$ (go to a subsequence, if necessary) .

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Lemma 2

Let E be a separable C^*-álgebra, $f \in \mathcal{H}_L(E_\Omega)$ and $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$. If $f(z) \in \ker(\psi_0)$ for every $z \in E_\Omega$, then $f \in \ker(\psi)$.

Idea of proof:
$\psi_0 \in \mathcal{M}(E) \Rightarrow \ker(\psi_0)$ is a separable C^*-álgebra $\Rightarrow$ there exists an approximation of the identity $(\mathbf{e}_n)_{n\in\mathbb{N}}$ in $ker(\psi_0)$.
Show that:

- $\{P_{e_n,1} \circ f; n \in \mathbb{N}\} \subset \mathcal{C}(E_\Omega, E)$ is equicontinuous.
- $\overline{\{(P_{e_n,1} \circ f)(z); n \in \mathbb{N}\}}$ is a compact set for every $z \in E_\Omega$.

Arzela-Ascoli Theorem $\Rightarrow$ there exists $g \in \mathcal{C}(E_\Omega, E)$ such that $P_{e_n,1} \circ f \overset{\tau_0}{\to} g$ (go to a subsequence, if necessary) .
In particular,

$$g(z) = \lim_{n\to\infty} P_{e_n,1}(f(z)) = \lim_{n\to\infty} e_n f(z) = f(z)$$

for all $z \in E_\Omega$, and hence $P_{e_n,1} \circ f \overset{\tau_0}{\to} f$.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Note that $P_{e_n,1} \circ f = P_{e_n,0} \cdot f$ since $P_{e_n,1}(f(z)) = e_n f(z) = P_{e_n,0}(z) f(z)$ for all $z \in E_\Omega$.
Now, $P_{e_n,1} \cdot f \stackrel{\tau_0}{\to} f$ and $\psi(P_{e_n,0}) = \psi_0(e_n) = 0$ for all $n \in \mathbb{N}$ implies

$$\psi(f) = \psi(\lim_{n\to\infty} P_{e_n,0} \cdot f) = \lim_{n\to\infty} \psi(P_{e_n,0} \cdot f) = \lim_{n\to\infty} \psi_0(e_n)\psi(f) = 0.$$

Hence $f \in \ker(\psi)$.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Note that $P_{e_n,1} \circ f = P_{e_n,0} \cdot f$ since $P_{e_n,1}(f(z)) = e_n f(z) = P_{e_n,0}(z) f(z)$ for all $z \in E_\Omega$.
Now, $P_{e_n,1} \cdot f \overset{\tau_0}{\to} f$ and $\psi(P_{e_n,0}) = \psi_0(e_n) = 0$ for all $n \in \mathbb{N}$ implies

$$\psi(f) = \psi(\lim_{n\to\infty} P_{e_n,0} \cdot f) = \lim_{n\to\infty} \psi(P_{e_n,0} \cdot f) = \lim_{n\to\infty} \psi_0(e_n)\psi(f) = 0.$$

Hence $f \in \ker(\psi)$.

Theorem

If E is a separable commutative C^*-álgebra with a unit element e and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, the mapping

$$\delta : \mathcal{M}(E) \times \Omega \longrightarrow \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$ is a homeomorphism between $(\mathcal{M}(E) \times \Omega, \tau_\pi)$ and $(\mathcal{M}_0(\mathcal{H}_L(E_\Omega)), \tau_G)$.

Note that $P_{e_n,1} \circ f = P_{e_n,0} \cdot f$ since $P_{e_n,1}(f(z)) = e_n f(z) = P_{e_n,0}(z) f(z)$ for all $z \in E_\Omega$.
Now, $P_{e_n,1} \cdot f \overset{\tau_0}{\to} f$ and $\psi(P_{e_n,0}) = \psi_0(e_n) = 0$ for all $n \in \mathbb{N}$ implies

$$\psi(f) = \psi(\lim_{n\to\infty} P_{e_n,0} \cdot f) = \lim_{n\to\infty} \psi(P_{e_n,0} \cdot f) = \lim_{n\to\infty} \psi_0(e_n)\psi(f) = 0.$$

Hence $f \in \ker(\psi)$.

Theorem

If E is a separable commutative C^*-álgebra with a unit element e and $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, the mapping

$$\delta : \mathcal{M}(E) \times \Omega \longrightarrow \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$$

defined by $\delta(\phi, \lambda_0)(f) = f_\phi(\lambda_0)$ for every $f \in \mathcal{H}_L(E_\Omega)$ is a homeomorphism between $(\mathcal{M}(E) \times \Omega, \tau_\pi)$ and $(\mathcal{M}_0(\mathcal{H}_L(E_\Omega)), \tau_G)$.

We are going to show that δ is surjective.

Given any $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$, take $\psi_0 \in \mathcal{M}(E)$ defined as in the proof of Theorem 2 and $\lambda_0 = \psi(P_{\mathbf{e},1}) \in \Omega$.

$f \in \mathcal{H}_L(E_\Omega) \Rightarrow f_{\psi_0} \in \mathcal{H}(\psi_0(E_\Omega)) = \mathcal{H}(\Omega) \Rightarrow \widetilde{f_{\psi_0}} \in \widetilde{\mathcal{H}}(\Omega) \subset \mathcal{H}_L(E_\Omega)$.

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Given any $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$, take $\psi_0 \in \mathcal{M}(E)$ defined as in the proof of Theorem 2 and $\lambda_0 = \psi(P_{\mathbf{e},1}) \in \Omega$.
$f \in \mathcal{H}_L(E_\Omega) \Rightarrow f_{\psi_0} \in \mathcal{H}(\psi_0(E_\Omega)) = \mathcal{H}(\Omega) \Rightarrow \widetilde{f_{\psi_0}} \in \widetilde{\mathcal{H}}(\Omega) \subset \mathcal{H}_L(E_\Omega)$.

Consider the mapping $f - \widetilde{f_{\psi_0}}$. For every $z \in E_\Omega$,

$$\psi_0(f(z) - \widetilde{f_{\psi_0}}(z)) = f_{\psi_0}(\psi_0(z)) - f_{\psi_0}(\psi_0(z)) = 0.$$

The algebra $(\mathcal{H}_L(E_\Omega), \tau_0)$

Given any $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$, take $\psi_0 \in \mathcal{M}(E)$ defined as in the proof of Theorem 2 and $\lambda_0 = \psi(P_{\mathbf{e},1}) \in \Omega$.

$f \in \mathcal{H}_L(E_\Omega) \Rightarrow f_{\psi_0} \in \mathcal{H}(\psi_0(E_\Omega)) = \mathcal{H}(\Omega) \Rightarrow \widetilde{f_{\psi_0}} \in \widetilde{\mathcal{H}}(\Omega) \subset \mathcal{H}_L(E_\Omega)$.

Consider the mapping $f - \widetilde{f_{\psi_0}}$. For every $z \in E_\Omega$,

$$\psi_0(f(z) - \widetilde{f_{\psi_0}}(z)) = f_{\psi_0}(\psi_0(z)) - f_{\psi_0}(\psi_0(z)) = 0.$$

Hence, $\left(f - \widetilde{f_{\psi_0}}\right)(z) \in \ker(\psi_0)$ for all $z \in E_\Omega$.

Given any $\psi \in \mathcal{M}_0(\mathcal{H}_L(E_\Omega))$, take $\psi_0 \in \mathcal{M}(E)$ defined as in the proof of Theorem 2 and $\lambda_0 = \psi(P_{\mathbf{e},1}) \in \Omega$.
$f \in \mathcal{H}_L(E_\Omega) \Rightarrow f_{\psi_0} \in \mathcal{H}(\psi_0(E_\Omega)) = \mathcal{H}(\Omega) \Rightarrow \widetilde{f_{\psi_0}} \in \widetilde{\mathcal{H}}(\Omega) \subset \mathcal{H}_L(E_\Omega)$.

Consider the mapping $f - \widetilde{f_{\psi_0}}$. For every $z \in E_\Omega$,

$$\psi_0(f(z) - \widetilde{f_{\psi_0}}(z)) = f_{\psi_0}(\psi_0(z)) - f_{\psi_0}(\psi_0(z)) = 0.$$

Hence, $\left(f - \widetilde{f_{\psi_0}}\right)(z) \in \ker(\psi_0)$ for all $z \in E_\Omega$.
By the Lemma2 , $f - \widetilde{f_{\psi_0}} \in \ker(\psi)$ and, by Lemma 1, $\psi(\widetilde{f_{\psi_0}}) = f_{\psi_0}(\lambda_0)$. Consequently,

$$0 = \psi(f - \widetilde{f_{\psi_0}}) = \psi(f) - \psi(\widetilde{f_{\psi_0}}) = \psi(f) - f_{\psi_0}(\lambda_0).$$

From this we get $\psi(f) = f_{\psi_0}(\lambda_0)(f)$ for every $f \in \mathcal{H}_L(E_\Omega)$ and so $\psi = \delta(\psi_0, \lambda_0)$.

Some References

1. B.W. Glickfeld, *The Riemann sphere of a commutative Banach algebras*, Trans. Amer. Math. Soc. **38** (1960), 414-425.

2. B.W. Glickfeld, *Contributions to The Theory of Holomorphic Function in Commutative Banach Algebras*, Thesis, Columbia University, New York, 1964.

3. B.W. Glickfeld, *On the inverse function theorem in commutative Banach algebras*, Illinois J. Math. 15 (1971) 212-221.

4. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

5. E. Hille and R.S. Phillips, *Functional Analysis and Semi-groups*, American Mathematical Society, Colloquium Publications XXXI, Baltimore, 1957.

6. E.R. Lorch, *The theory of analytic functions in normed abelian vector rings*, Trans. Amer. Math. Soc. **54** (1943), 414–425.

7. D.H. Luecking and L.A. Rubel, *Complex analysis. A Functional Analysis Approach*, Springer-Verlag New York, 1984.

Some References

8. G.V.S. Mauro, *Espectros de Álgebras de Aplicações Analíticas no sentido de Lorch*, Tese de Doutorado, UFRJ (2016)

9. G.V.S. Mauro, L.A. Moraes and A.F. Pereira, *Topological and algebraic properties of spaces of Lorch analytic mappings*, Math. Nachr. **289** (2015), 845 - 853.

10. G.V.S. Mauro and L.A. Moraes, *Algebras of Lorch analytic mappings on uniform algebras.* , Pre-print.

11. L.A. Moraes and A.F. Pereira, *The spectra of algebras of Lorch analytic mappings*, Topology **48** (2009), 91–99.

12. L.A. Moraes and A.F. Pereira, *The algebra of Lorch analytic mappings with the Hadamard product*, Publ. RIMS Kyoto Univ. 49 (2013) 111-122.

13. H.E. Warren, *A Riemann mapping theorem for* $C(X)$, Proc. Amer. Math. Soc. 28 (1971), 147-154.

14. H.E. Warren, *Analytic equivalence among simply connected domains in C(X)*, Trans. Amer. Math. Soc. **196** (1974), 265-288.