Acceleration radiation, transition probabilities and trans-Planckian physics

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Abstract. An important question in the derivation of the acceleration radiation, which also arises in Hawking's derivation of black hole radiance, is the need to invoke trans-Planckian physics in describing the creation of quanta. We point out that this issue can be further clarified by reconsidering the analysis in terms of particle detectors, transition probabilities and local two-point functions. By writing down separate expressions for the spontaneous- and induced-transition probabilities of a uniformly accelerated detector, we show that the bulk of the effect comes from the natural (non-trans-Planckian) scale of the problem, which largely diminishes the importance of the trans-Planckian sector. This is so, at least, when trans-Planckian physics is defined in a Lorentz-invariant way. This analysis also suggests how one can define and estimate the role of trans-Planckian physics in the Hawking effect itself.

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1. Introduction

After formulating general relativity, Einstein returned to the microscopic world. He introduced the concept of transition probabilities between stationary states in the context of the interaction of atoms with radiation. He established a link between black-body radiation and the theory of atomic spectra. In short, Einstein considered transitions between two states, an upper excited state 2 and a lower state 1, with energies $E_2 > E_1$. The probability per atom and per unit time of a jump from state 1 to state 2 induced by the environment radiation is

$$\dot{P}_{1\to 2} = B u_w, \tag{1.1}$$

where u_w is the energy density of radiation at frequency $w = (E_2 - E_1)/\hbar$ and B is one of the so-called Einstein coefficients. In addition, the probability per atom and per unit time of the decay of state 2 to state 1 is assumed to be

$$\dot{P}_{2\to 1} = B u_w + A, \tag{1.2}$$

where A represents the probability of spontaneous emission and $B u_w$ gives also the probability of induced emission. Thermal equilibrium is achieved if

$$N_1 \dot{P}_{1\to 2} = N_2 \dot{P}_{2\to 1}, \tag{1.3}$$

when the state population quotient N_2/N_1 obeys the Boltzmann distribution for probabilities $N_2/N_1 = e^{-\Delta E/k_B T}$ at the equilibrium temperature T (k_B is the Boltzmann constant and $\Delta E \equiv E_2 - E_1$). Einstein realized that thermal equilibrium implies that u_w turns out to be the Planck law for the energy density, provided that the quotient A/B is just $A/B = 2\hbar w^3/\pi c^3$. This analysis can be used to infer the thermal character of the environment radiation by analyzing only the transition probabilities of the atomic system; the environment radiation is thermal provided that the transition probabilities between the energy levels of the atomic system satisfy, at equilibrium, the so-called detailed balance relation $\dot{P}_{1\to 2}/\dot{P}_{2\to 1} = e^{-\Delta E/k_B T}$.

Many years later, physicists working in the theory of quantum fields in curved space realized that an atomic system following a uniformly accelerated worldline in Minkowski

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spacetime, with acceleration a, feels itself immersed in a thermal bath at the temperature $T = \hbar a/2\pi c k_B$, when the quantum state of the field is the ordinary Minkowski vacuum. The acceleration radiation effect can be analyzed from two different points of view. It can be derived by computing the expectation value of the number operator in the Minkowski vacuum state by using the formalism of Bogolubov transformations [1] in Rindler space [2, 3]. The Bogolubov coefficient approach is also the basis of Hawking's original derivation of black hole radiance [4] (see also [5]). On the other hand, the acceleration radiation effect can also be derived by studying the transition rate probabilities of uniformly accelerated particle detectors in Minkowski spacetime [6] (see also the review [7]). In this approach, the transition probabilities are often written in terms of the two-point function of the Minkowski vacuum state. In this form, the derivation is somewhat closer to the derivation of black hole radiance carried out by Fredenhagen and Haag [8].

When dealing with the acceleration radiation or the Hawking effect, an important question arises. To what extent are these thermal effects sensitive to trans-Planckian physics? In Hawking's original derivation, this issue emerges naturally because emitted quanta reaching future null infinity at sufficiently late times suffer an arbitrarily large blueshift when propagated backwards in time to past null infinity. In fact, the precursors of the Hawking quanta can have trans-Planckian frequencies in the vicinity of the horizon (see for instance [9]–[11])[5]. The same question arises in the derivation of the acceleration radiation. This is because, in any given inertial frame, the uniformly accelerated detector acquires an arbitrarily large velocity after sufficient proper time τ and, correspondingly, the thermal quanta it is observing at such times correspond to modes with arbitrarily large frequencies $w' \sim we^{a\tau}$ relative to the given inertial frame [13]. This fact is manifest in the derivation in terms of Bogolubov coefficients, which requires an unbounded integral in frequencies in the intermediate steps of the derivation. However, these modes are not detected by an inertial observer and their physical relevance is not clear. On the other hand, in the derivation of the acceleration radiation in terms of two-point functions, trans-Planckian physics seems to appear because ultrashort lapses of proper time are apparently important in obtaining the final result. However, this inference depends on the distributional character of the two-point function.

In this paper, we reanalyze this problem by studying the transition probabilities of uniformly accelerated particle detectors. We parallel Einstein's analysis by computing separately the induced and spontaneous transition probabilities of the detector and we obtain the thermal character of the radiation by means of the detailed balance relation. The splitting of the different contributions has the advantage of providing suitable mathematical expressions that allow us to define and evaluate the contribution of trans-Planckian physics in a Lorentz-invariant way. We find that the thermal outcome arises from scales of the same order as the acceleration a itself, which strongly suggests that the effect is indeed a low-energy phenomenon.

In section 2, we review the standard analysis of the acceleration radiation in Rindler spacetime in terms of Bogolubov coefficients. In section 3, we compute the spontaneous and induced transition probability rates of a uniformly accelerated particle detector in Minkowski spacetime, and we use the results to show that the radiation felt by the detector is thermal. In section 4, we repeat this analysis using the two-point function of the quantum field. Finally, in section 5, we use the results presented in sections 3 and 4 to analyze the role of trans-Planckian

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[5] This issue has been traditionally addressed by explicit modification of the standard relativistic dispersion relations [12]. Here, we follow a different approach that preserves the relativistic invariance.

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physics in the computation of the acceleration radiation. There, we also make some comments regarding the same topic in Hawking radiation. In section 6, we give our conclusions[6].

2. Acceleration radiation and Bogolubov transformations

The acceleration radiation was first derived in the context of the formalism of Bogolubov transformations relating inertial and accelerated modes. In this section, we will quickly review this derivation.

A uniformly accelerated (Rindler) observer has a natural coordinate system (τ, ξ, y, z) related to the inertial coordinates (t, x, y, z) by

$$t = \frac{e^{a\xi}}{a}\sinh a\tau, \quad x = \frac{e^{a\xi}}{a}\cosh a\tau, \quad y = y, \quad z = z. \tag{2.1}$$

The curve $\xi = 0$ represents a uniformly accelerated trajectory with proper acceleration a. The wave equation for a massless scalar field $\Box\phi(x) = 0$ in the coordinates of the accelerated observer becomes

$$(e^{-2a\xi}(-\partial_\tau^2 + \partial_\xi^2) + \partial_y^2 + \partial_z^2)\phi(\tau, \xi, y, z) = 0. \tag{2.2}$$

The y, z dependence can be trivially integrated using plane waves $\phi(t, \xi, y, z) = \phi(t, \xi)e^{ik_y y}e^{ik_z z}$. Introducing this ansatz in the equation, we find

$$[(-\partial_\tau^2 + \partial_\xi^2) - e^{2a\xi}(k_y^2 + k_z^2)]\phi(\tau, \xi) = 0. \tag{2.3}$$

This equation indicates that the free scalar field observed by the Minkowski observer appears to the uniformly accelerated observer like a scalar field in a repulsive potential $V(\xi) \propto e^{2a\xi}\vec{k}_\perp^2$, where $\vec{k}_\perp^2 = k_y^2 + k_z^2$. The exact form of the normalized modes, with natural support on the accessible region for the accelerated observer (right-hand Rindler wedge), can be expressed as

$$u_{w,\vec{k}_\perp}^R = \frac{e^{-iw\tau}}{2\pi^2\sqrt{a}}\sinh^{1/2}\left(\frac{\pi w}{a}\right)K_{iw/a}\left(\frac{|\vec{k}_\perp|}{a}e^{a\xi}\right)e^{i\vec{k}_\perp \cdot \vec{x}_\perp}, \tag{2.4}$$

where $\vec{k}_\perp \cdot \vec{x}_\perp = k_y y + k_z z$. The important point is that the above positive frequency modes cannot be expanded in terms of the standard purely positive frequency modes of the inertial observer,

$$u_{k_x,\vec{k}_\perp}^M = \frac{1}{\sqrt{2(2\pi)^3 k_0}}e^{-ik_0 t + i(k_x x + \vec{k}_\perp \cdot \vec{x}_\perp)}, \tag{2.5}$$

where $k_0 = \sqrt{k_x^2 + \vec{k}_\perp^2}$. The detailed analysis requires one to compute the corresponding Bogolubov coefficients. They are found to be [2, 7]

$$\beta_{w\vec{k}_\perp, k_x'\vec{k}_\perp'} = -\left[2\pi a k_0'(e^{2\pi w/a} - 1)\right]^{-1/2}\left(\frac{k_0' + k_x'}{k_0' - k_x'}\right)^{-iw/2a}\delta(\vec{k}_\perp - \vec{k}_\perp'). \tag{2.6}$$

With this result one can compute the important physical result that follows. The mean number n_w of Rindler particles in the Minkowski vacuum is directly tied to the integral

$$n_w = \int_{-\infty}^{+\infty}d\vec{k}' \beta_{w_1\vec{k}_\perp, \vec{k}'}\beta_{w_2\vec{k}_\perp, \vec{k}'}^*. \tag{2.7}$$

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[6] In the rest of the paper, we use units such that $\hbar = c = 1$ and $k_B = 1$.

The integration in $\vec{k}_\perp$ is trivial and the integration in k'_x reduces to

$$\int_{-\infty}^{+\infty} \mathrm{d}k'_x (2\pi a k'_0)^{-1} \left(\frac{k'_0 + k'_x}{k'_0 - k'_x} \right)^{-\mathrm{i}(w_1 - w_2)/2a}, \tag{2.8}$$

which, changing the integration variable to the rapidity $y = \tanh^{-1}(k'_x/k'_0)$, leads to

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{2\pi a} e^{-\mathrm{i}(w_1 - w_2)y/a} = \delta(w_1 - w_2). \tag{2.9}$$

Taking into account the remaining terms, one easily gets

$$\int_{-\infty}^{+\infty} \mathrm{d}\vec{k}' \beta_{w_1 \vec{k}_{\perp 1}, \vec{k}'} \beta^*_{w_2 \vec{k}_{\perp 2}, \vec{k}'} = \frac{1}{e^{2\pi w_1/a} - 1} \delta(w_1 - w_2) \delta(\vec{k}_{\perp 1} - \vec{k}_{\perp 2}). \tag{2.10}$$

The final outcome then becomes extremely simple and important, and parallels the Hawking effect on black hole radiance. The Minkowski vacuum can be described, in the spacetime region accessible for a uniformly accelerated observer (the Rindler wedge), as a thermal bath of Rindler quanta at temperature $T = a/2\pi$. This result [2, 3] was strongly reinforced by Unruh's interpretation in terms of particle detectors [6]. A uniformly accelerated particle detector is excited by the absorption of a Rindler quantum from the thermal bath. An inertial observer describes this process in a different way, as the emission of a Minkowski particle as the result of the interaction of the detector with the quantum field [14], as explicitly worked out in the next section.

3. Transition probabilities of an accelerated particle detector

In this section, we review the particle detector approach to the acceleration radiation effect. We compute separately the spontaneous and induced emission and absorption processes. The thermal character of the Minkowskian vacuum with respect to an accelerated observer is derived via the detailed balance relation.

3.1. Spontaneous emission of a uniformly accelerated detector

Let us consider a quantum mechanical system coupled to a free massless scalar quantum field $\Phi(x)$ in Minkowski spacetime. For simplicity, the field is assumed to be massless. The quantum mechanical system modeling our particle detector [6, 15] will have some internal energy states $|E\rangle$, which are eigenstates of the corresponding free Hamiltonian H_q. We will consider here two of those states, $|E_2\rangle$ and $|E_1\rangle$, with energies $E_2 > E_1$. The detector can interact with the quantum field by absorbing or emitting quanta. The interaction can be modeled in a simple way by coupling the field $\Phi(x)$ along the detector's trajectory $x = x(\tau)$ (τ is the detector's proper time) to a monopole moment operator $m(\tau)$ acting on the internal detector eigenstates through the Lagrangian

$$L_\mathrm{I} = g m(\tau) \Phi(x(\tau)), \tag{3.1}$$

where g is the strength of the coupling. In the interaction picture, the detector's operator $m(\tau)$ has the standard unitary time evolution $m(\tau) = e^{\mathrm{i}H_q\tau} m(0) e^{-\mathrm{i}H_q\tau}$.

Before analyzing the accelerated detector, it is useful to consider the simple example of an inertial detector. The spontaneous emission of an inertial detector can be studied by considering

the transition amplitude for the process $|E_2\rangle|0_M\rangle \rightarrow |E_1\rangle|\psi\rangle$, where $|0_M\rangle$ is the usual Minkowski vacuum state and $|\psi\rangle$ is the final state of the field. The field $\Phi(x)$ can be quantized by expanding it in standard plane-wave modes,

$$\Phi(x) = \int d^3k \left(u_{\vec{k}}^M a_{\vec{k}} + u_{\vec{k}}^{M*} a_{\vec{k}}^{\dagger} \right), \tag{3.2}$$

with

$$u_{\vec{k}}^M = \frac{1}{\sqrt{2(2\pi)^3 w}} e^{-i(wt - \vec{k}\vec{x})}, \tag{3.3}$$

where t and x are inertial coordinates and $w = |\vec{k}|$. The amplitude of the process is given, to first order in time-dependent perturbation theory, by

$$ig\langle E_1|m(0)|E_2\rangle \int d\tau \, e^{i(E_1 - E_2)\tau} \langle \psi|\Phi(x(\tau))|0_M\rangle. \tag{3.4}$$

Because of the monopolar interaction, this transition can only take place to one-particle (Minkowski) states. Taking $|\psi\rangle = |\vec{k}\rangle$, the corresponding amplitude is then[7]

$$ig\langle E_1|m(0)|E_2\rangle \int d\tau \, e^{i(E_1 - E_2)\tau} \frac{1}{\sqrt{2(2\pi)^3 w}} e^{i(wt(\tau) - \vec{k}\vec{x}(\tau))}, \tag{3.5}$$

where $(t(\tau), \vec{x}(\tau))$ is the trajectory of the detector. For the inertial detector, we have $t = \tau$, $\vec{x} = 0$. The transition probability of the final state $|E_1\rangle|\vec{k}\rangle$ is then given by squaring the above expression,

$$P_{2\rightarrow 1, \vec{k}} = g^2 |\langle E_1|m(0)|E_2\rangle|^2 \frac{1}{2(2\pi)^3 w} \int d\tau_1 \, d\tau_2 \, e^{i(E_1 - E_2 + w)(\tau_1 - \tau_2)}, \tag{3.6}$$

where $w = |\vec{k}|$. Therefore, the corresponding transition probability per unit time is then given by ($\Delta\tau \equiv \tau_1 - \tau_2$),

$$\dot{P}_{2\rightarrow 1, \vec{k}} = g^2 |\langle E_1|m(0)|E_2\rangle|^2 \frac{1}{2(2\pi)^3 w} \int d\Delta\tau \, e^{-i(\Delta E - w)\Delta\tau}$$

$$= g^2 |\langle E_1|m(0)|E_2\rangle|^2 \frac{1}{2(2\pi)^2 w} \delta(\Delta E - w), \tag{3.7}$$

where the delta function reflects the energy conservation of the process, with $\Delta E \equiv E_2 - E_1 > 0$. The transition $E_2 \rightarrow E_1$ is accompanied by the emission of a quantum of the field with energy $w = \Delta E$. Finally, the total transition probability rate for the detector is obtained by summing over all possible one-particle final states[8],

$$\dot{P}_{2\rightarrow 1} \text{ (spontaneous)} = g^2 |\langle E_1|m(0)|E_2\rangle|^2 \int d\Omega_{\vec{k}} w^2 \, dw \frac{1}{2(2\pi)^2 w} \delta(\Delta E - w)$$

$$= g^2 |\langle E_1|m(0)|E_2\rangle|^2 \frac{\Delta E}{2\pi}. \tag{3.8}$$

[7] Note that one could choose instead of $|\psi\rangle = |\vec{k}\rangle$ a superposition of one-particle states. However, since at the end we sum over all possible final states, the outcome will be independent of the particular basis chosen. Our choice is thus made on the grounds of technical and notational simplicity.

[8] Had we chosen a non-static inertial observer with $\vec{x} = \vec{v}t$, the delta function would take the form $\delta(\Delta E - \gamma(w - \vec{k} \cdot \vec{v}))$, but the final result is the same as for $\vec{v} = 0$.

The spontaneous emission rate is an intrinsic property of the detector and its interaction with the quantum field. Therefore, it is *insensitive* to the trajectory of the detector and is given by the previous expression. We can check this by computing explicitly the spontaneous emission rate of a detector following a uniformly accelerated trajectory (see appendix for explicit derivation of the same result for a freely falling detector in de Sitter spacetime).

Let us then consider that the detector follows a uniformly accelerated trajectory with proper acceleration a,

$$t = \frac{1}{a} \sinh a\tau, \quad x = \frac{1}{a} \cosh a\tau, \quad y = 0, \quad z = 0. \tag{3.9}$$

One can easily repeat the above calculation for the process $|E_2\rangle|0_R\rangle \rightarrow |E_1\rangle|\psi\rangle$, where now the initial state of the quantum field, $|0_R\rangle$, is taken as vacuum associated to the uniformly accelerated observer (usually called the Rindler vacuum) and $|\psi\rangle$ stands for the associated one-particle (Rindler) state. Using the coordinates (τ, ξ, y, z) associated with the accelerated observer, the modes defining the quantization are those given in (2.4). On the accelerated trajectory, we have $\xi = 0$ and, for simplicity, we take $\vec{x}_\perp = (0, 0)$. Then

$$u^R_{w,\vec{k}_\perp}(\tau) = \frac{e^{-iw\tau}}{2\pi^2\sqrt{a}} \sinh^{1/2}\left(\frac{\pi w}{a}\right) K_{(iw/a)}\left(\frac{|\vec{k}_\perp|}{a}\right). \tag{3.10}$$

Using the same arguments as for the inertial detector, we can express the transition probability rate for all possible one-particle (Rindler) final states as

$$\dot{P}_{2\rightarrow 1}(\text{spontaneous}) = g^2|\langle E_1|m(0)|E_2\rangle|^2$$

$$\times \int_0^{+\infty} |\vec{k}_\perp| d|\vec{k}_\perp| dw \left| K_{(iw/a)}\left(\frac{|\vec{k}_\perp|}{a}\right) \right|^2 \frac{(2\pi)^2}{(2\pi^2\sqrt{a})^2} \sinh\left(\frac{\pi w}{a}\right) \delta(E_1 - E_2 + w), \tag{3.11}$$

where a factor 2π comes from the one-dimensional (1D) angular integration of the transverse momentum. Performing the integral in $|\vec{k}_\perp|$, we have

$$\dot{P}_{2\rightarrow 1}(\text{spontaneous}) = g^2|\langle E_1|m(0)|E_2\rangle|^2 \frac{\Delta E}{2\pi}, \tag{3.12}$$

which, as expected, exactly coincides with the result (3.8).

3.2. Induced emission of a uniformly accelerated detector

We now study the process of induced emission of a uniformly accelerated detector in Minkowski spacetime when the quantum field is in the usual Minkowski vacuum state. Let us consider then the process $|E_2\rangle|0_M\rangle \rightarrow |E_1\rangle|\psi\rangle$ for a uniformly accelerated trajectory. Since the initial state $|0_M\rangle$ is not the vacuum state for an accelerated observer, one would expect the transition rate for this process to be enhanced by induced emission. We will obtain the probability rate for the process of induced emission by computing the total emission probability rate and subtracting from it the spontaneous emission rate. As before, the only non-vanishing contribution will be for one-particle Minkowski states, so we consider $|\psi\rangle = |\vec{k}\rangle$. The corresponding amplitude is then

$$ig\langle E_1|m(0)|E_2\rangle \int d\tau \, e^{i(E_1 - E_2)\tau} \frac{1}{\sqrt{2(2\pi)^3 w}} e^{iw(t(\tau) - \cos\theta x(\tau))}, \tag{3.13}$$

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where $t = t(\tau)$, $x = x(\tau)$ is the accelerated trajectory (3.9) and θ is the angle between $\vec{k}$ and the x-axis. Taking into account the form of the trajectory (3.9), this amplitude can be rewritten as

$$\frac{ig\langle E_1|m(0)|E_2\rangle}{\sqrt{2(2\pi)^3 w}} \int d\tau e^{i(E_1-E_2)\tau} e^{iw/2a(e^{a\tau}-e^{-a\tau}-\cos\theta (e^{a\tau}+e^{-a\tau}))}. \tag{3.14}$$

Squaring the modulus of the above amplitude, we obtain the transition probability

$$P_{2\to 1,\vec{k}} = \frac{g^2|\langle E_1|m(0)|E_2\rangle|^2}{2(2\pi)^3 w} \int d\tau_1 d\tau_2 e^{i(E_1-E_2)(\tau_1-\tau_2)}$$
$$\times e^{iw/2a(e^{a\tau_1}-e^{-a\tau_1}-e^{a\tau_2}+e^{-a\tau_2}-\cos\theta (e^{a\tau_1}+e^{-a\tau_1}-e^{a\tau_2}-e^{-a\tau_2}))}. \tag{3.15}$$

Defining $\Delta\tau = \tau_1 - \tau_2$ and $\Delta\tau^+ = (\tau_2 + \tau_1)/2$, we can rewrite $P_{2\to 1,\vec{k}}$ as

$$P_{2\to 1,\vec{k}} = \frac{g^2|\langle E_1|m(0)|E_2\rangle|^2}{2(2\pi)^3 w} \int d\Delta\tau^+ d\Delta\tau \; e^{i(E_1-E_2)\Delta\tau}$$
$$\times e^{i2wa^{-1}\sinh(a\Delta\tau/2)[\cosh(a\Delta\tau^+)-\cos\theta \sinh(a\Delta\tau^+)]}. \tag{3.16}$$

To work out the integral in $\Delta\tau$, it is convenient to perform the change of variable $U \equiv a^{-1}e^{-a\Delta\tau/2}$ and capture the dependence on $\Delta\tau^+$ in the positive-definite variable $\alpha = \cosh(a\Delta\tau^+) - \cos\theta \sinh(a\Delta\tau^+)$. We then obtain

$$P_{2\to 1,\vec{k}} = \frac{g^2|\langle E_1|m(0)|E_2\rangle|^2}{2(2\pi)^3 w} \int d\Delta\tau^+ 2a^{i2\Delta E/a-1} \int_0^\infty dU U^{i2\Delta E/a-1} e^{-i\alpha w(U-(a^2 U)^{-1})}. \tag{3.17}$$

The integral in U does not converge absolutely. This is because we are working with plane waves, instead of wave packets, to represent the states $|\psi\rangle$. An integration over frequencies using wave packets makes the integral convergent. Nonetheless, one can still work with plane waves by inserting an infinitesimal negative real part to make the integral convergent. Therefore, one must add the appropriate $\pm i\epsilon$ terms to w in the exponent. Using now the identity

$$\int_0^\infty dx x^c e^{a/x+bx} = 2(-a)^{(1+c)/2}(-b)^{-(1+c)/2}K_{-1-c}(2\sqrt{ab}), \tag{3.18}$$

for $Re[a] < 0$, $Re[b] < 0$, where K is a modified Bessel function, we obtain

$$P_{2\to 1,\vec{k}} = \frac{g^2|\langle E_1|m(0)|E_2\rangle|^2}{2(2\pi)^3 w} \int d\Delta\tau^+ \frac{4}{a}(iw-\epsilon)^{i\Delta E/a}(-iw-\epsilon)^{-i\Delta E/a} K_{-i2\Delta E/a}(2w\alpha/a). \tag{3.19}$$

Taking into account that, in the limit $\epsilon \to 0^+$, $\ln(-w+i\epsilon) = i\pi + \ln w$, we obtain

$$P_{2\to 1,\vec{k}} = \frac{g^2|\langle E_1|m(0)|E_2\rangle|^2 4e^{\pi\Delta E/a}}{2(2\pi)^3 wa} \int d\Delta\tau^+ K_{-i2\Delta E/a}(2w\alpha/a). \tag{3.20}$$

The transition probability rate for this process is then given by

$$\dot{P}_{2\to 1,\vec{k}} = \frac{g^2|\langle E_1|m(0)|E_2\rangle|^2 4e^{\pi\Delta E/a}}{2(2\pi)^3 wa} K_{-i2\Delta E/a}(2w\alpha/a). \tag{3.21}$$

Summing now over all possible energies for the one-particle final states, we obtain

$$\int_0^\infty dw\, w^2 \dot{P}_{2\to 1,\vec{k}} = \frac{g^2 |\langle E_1|m(0)|E_2\rangle|^2 4 e^{\pi \Delta E/a}}{2(2\pi)^3 a} \int_0^\infty dw\, w\, K_{-i2\Delta E/a}(2w\alpha/a)$$

$$= \frac{g^2 |\langle E_1|m(0)|E_2\rangle|^2}{4\pi} \frac{\Delta E}{2\pi} \frac{e^{2\pi \Delta E/a}}{(e^{2\pi \Delta E/a} - 1)} \frac{1}{\alpha^2}, \qquad (3.22)$$

where we have made use of the following identity,

$$\int_0^\infty dx\, x\, K_{-ia}(bx) = \frac{a\pi}{2b^2 \sinh(a\pi/2)}, \qquad (3.23)$$

where a and $b > 0$ are real numbers. We still have to perform the angular integration. Using

$$\int d\Omega_{\vec{k}} \alpha^{-2} = 2\pi \int_{-1}^1 d(\cos\theta) \frac{1}{(\cosh(a\Delta\tau^+) - \cos\theta \sinh(a\Delta\tau^+))^2} = 4\pi, \qquad (3.24)$$

we finally obtain

$$\dot{P}_{2\to 1} \equiv \int_0^\infty d\Omega_{\vec{k}}\, dw\, w^2 \dot{P}_{2\to 1,\vec{k}} = g^2 |\langle E_1|m(0)|E_2\rangle|^2 \frac{\Delta E}{2\pi} \frac{e^{2\pi \Delta E/a}}{e^{2\pi \Delta E/a} - 1}. \qquad (3.25)$$

Note that, in the limit $a \to 0$, we recover the expression for the spontaneous emission, which indicates that contribution is already contained in (3.25). Since the probability rate $\dot{P}_{2\to 1}$ is the sum of the probability rate for the spontaneous process $\dot{P}_{2\to 1}$ (spontaneous) plus that of the stimulated process $\dot{P}_{2\to 1}$ (induced), by subtracting $\dot{P}_{2\to 1}$ (spontaneous) from (3.25) we obtain

$$\dot{P}_{2\to 1} \text{ (induced)} = g^2 |\langle E_1|m(0)|E_2\rangle|^2 \frac{\Delta E}{2\pi} \frac{1}{e^{2\pi \Delta E/a} - 1}. \qquad (3.26)$$

3.3. Absorption of a uniformly accelerated detector

We can also consider the probability rate for the excitation of the detector $|E_1\rangle|0_M\rangle \to |E_2\rangle|\vec{k}\rangle$, accompanied by the emission of a Minkowski particle. This can easily be extracted from (3.21), and one obtains

$$\dot{P}_{1\to 2,\vec{k}} = \frac{g^2 |\langle E_1|m(0)|E_2\rangle|^2 4 e^{-\pi \Delta E/a}}{2(2\pi)^3 wa} K_{i2\Delta E/a}(2w\alpha/a). \qquad (3.27)$$

Summing on all possible final states, one obtains the excitation probability rate $\dot{P}_{1\to 2}$,

$$\dot{P}_{1\to 2} = g^2 |\langle E_1|m(0)|E_2\rangle|^2 \frac{\Delta E}{2\pi} \frac{1}{e^{2\pi \Delta E/a} - 1}, \qquad (3.28)$$

which, as expected (see (1.1) and (1.2)), coincides with the above induced emission rate $\dot{P}_{2\to 1}$ (induced).

3.4. Thermality

From the previous calculations we find

$$\frac{\dot{P}_{1\to 2}}{\dot{P}_{2\to 1}} = \frac{\dot{P}_{2\to 1} \text{ (induced)}}{\dot{P}_{2\to 1} \text{ (induced)} + \dot{P}_{2\to 1} \text{ (spontaneous)}} = \frac{\frac{\Delta E}{2\pi} \frac{1}{e^{2\pi \Delta E/a} - 1}}{\frac{\Delta E}{2\pi} \frac{1}{e^{2\pi \Delta E/a} - 1} + \frac{\Delta E}{2\pi}} = e^{-2\pi \Delta E/a}. \qquad (3.29)$$

According to Einstein's detailed balance relation for systems in thermal equilibrium,

$$N_2/N_1 = e^{-\Delta E/T} = \frac{\dot{P}_{1\to 2}}{\dot{P}_{2\to 1}}, \tag{3.30}$$

the result (3.29) shows that the detector internal energy states are populated as if they were immersed in a thermal bath at temperature $T = a/2\pi$. Therefore, following Einstein, the mean particle number per mode of the scalar radiation field should obey Planck's law,

$$n_w = \frac{1}{e^{w/T} - 1}, \tag{3.31}$$

in agreement with the result obtained from the Bogolubov coefficient approach.

An important comment is now in order. If one considers the detector's emission and absorption rates for a final state having momentum $\vec{k}$ of the emitted scalar particle, the thermal condition (3.30) is still satisfied for each individual mode $\vec{k}$. This can be seen from equations (3.21) and (3.27). Since the Bessel function $K_{i2\Delta E/a}(x)$ is real and invariant under the change $\Delta E \to -\Delta E$, the ratios $\dot{P}_{1\to 2,\vec{k}}/\dot{P}_{2\to 1,\vec{k}}$ lead to the Boltzmann thermal factor $e^{-2\pi\Delta E/a}$ for every $\vec{k}$. From this observation, the result (3.29) can be easily understood, since $\dot{P}_{2\to 1} = e^{\pi\Delta E/a}M(\Delta E/a)$ and $\dot{P}_{1\to 2} = e^{-\pi\Delta E/a}M(\Delta E/a)$, where $M(\Delta E/a)$ represents the integral over momenta $\vec{k}$ of the Bessel function times the (momentum independent) common factor $g^2|\langle E_1|m(0)|E_2\rangle|^2/(4\pi^3 a)$. The thermal result (3.29), therefore, stems from the thermal properties of the transition rates to individual $\vec{k}$-modes and is not the result of integrating over all the momenta $\vec{k}$. Indeed, one may also relate the absorption and emission probability rates by considering a shift in the variable $\Delta\tau$ in equation (3.16) of the form $\Delta\tau \to \Delta\tau + 2\pi i/a$, which immediately leads to $P_{1\to 2,\vec{k}} = e^{-2\pi\Delta E/a}P_{2\to 1,\vec{k}}$. We will come back to this point in section 5.

4. Transition probabilities and two-point functions

It is common in the literature (see, for instance, [16]) to express the transition probabilities computed in section 3 in terms of the two-point correlation function of the field. The sum over all possible one-particle states needed to obtain the transition probabilities in the previous section leads to a sum in modes $\sum_{\vec{k}}(u)_{\vec{k}}^M(x_1)u_{\vec{k}}^{M*}(x_2)$ that gives rise to the two-point function for the Minkowski vacuum. Then, if we perform the integration over all the final states in the expressions for the transition probabilities in the previous section prior to the integration in time, we obtain

$$P_{i\to f} = g^2|\langle E_f|m(0)|E_i\rangle|^2 F_{i\to f}(\Delta E), \tag{4.1}$$

where $F_{i\to f}(\Delta E)$ is the so-called response function,

$$F_{i\to f}(\Delta E) = \int_{-\infty}^{+\infty} d\tau_1\, d\tau_2\, e^{i(E_i - E_f)\Delta\tau} G_M(\Delta\tau - i\epsilon), \tag{4.2}$$

and $\Delta\tau = \tau_1 - \tau_2$ (from now on the limit $\epsilon \to 0^+$ is understood). In the previous expressions we can have $i = 1$, $f = 2$ or $i = 2$, $f = 1$, and ΔE is positive by definition. The quantity (4.2) is essentially given by the Fourier transform of the Wightman two-point function $G_M(\Delta\tau - i\epsilon)$ evaluated on the detector's trajectory. For a massless field, the Wightman two-point function for the Minkowski vacuum $|0_M\rangle$ in (4.2) is given by

$$G_M(x_1, x_2) \equiv \langle 0_M|\Phi(x_1)\Phi(x_2)|0_M\rangle = -\frac{1}{4\pi^2[(t_1 - t_2 - i\epsilon)^2 - (\vec{x}_1 - \vec{x}_2)^2]}, \tag{4.3}$$

with $(t, \vec{x})$ inertial coordinates. The projection on the accelerated trajectory (3.9) gives

$$G_M(\Delta\tau - i\epsilon) = \frac{-(a/2)^2}{4\pi^2 \sinh^2[(a/2)(\Delta\tau - i\epsilon)]}. \tag{4.4}$$

The transition probability per unit proper time is then given by

$$\dot{P}_{i\to f} = g^2 |\langle E_f | m(0) | E_i \rangle|^2 \dot{F}_{i\to f}(\Delta E), \tag{4.5}$$

where

$$\dot{F}_{i\to f}(\Delta E) = \int_{-\infty}^{+\infty} d\Delta\tau \, e^{i(E_i - E_f)\Delta\tau} G_M(\Delta\tau - i\epsilon). \tag{4.6}$$

Paralleling the previous section, we want to obtain separate expressions for the different processes. For the induced emission, we can obtain an expression simply by subtracting the spontaneous emission rate (3.12) from the total probability rate $\dot{P}_{2\to1}$ given by the above expression (4.5),

$$\dot{P}_{2\to1} \text{ (induced)} = g^2 |\langle E_1 | m(0) | E_2 \rangle|^2 \left(\dot{F}_{2\to1}(\Delta E) - \frac{\Delta E}{2\pi} \right). \tag{4.7}$$

If we now take into account the identity,

$$\int_{-\infty}^{+\infty} d\Delta\tau \, e^{i\Delta E\Delta\tau} G_R(\Delta\tau - i\epsilon) = \frac{\Delta E}{2\pi}, \tag{4.8}$$

where G_R is the vacuum two-point function of the accelerated observer (we recall that $|0_R\rangle$ is the Rindler vacuum),

$$G_R(\Delta\tau - i\epsilon) \equiv \langle 0_R | \Phi(x_1)\Phi(x_2) | 0_R \rangle = \frac{-1}{4\pi^2(\Delta\tau - i\epsilon)^2}, \tag{4.9}$$

then expression (4.7) can easily be rewritten in terms of an integral as[9]

$$\dot{P}_{2\to1} \text{ (induced)} = g^2 |\langle E_1 | m(0) | E_2 \rangle|^2 \int_{-\infty}^{+\infty} d\Delta\tau \, e^{i\Delta E\Delta\tau} [G_M(\Delta\tau - i\epsilon) - G_R(\Delta\tau - i\epsilon)]. \tag{4.10}$$

An advantage of this expression for the purely induced emission rate is that the integrand is now a smooth function over the real axis, even in the absence of the $i\epsilon$. This is so because of the universal short-distance behavior of the two-point functions for any physical state [the so-called Hadamard condition (see, for instance, [18])] that makes the divergences of both two-point functions cancel out. Therefore, the $i\epsilon$ prescription in the integrand of (4.10) is redundant and can be omitted,

$$\dot{P}_{2\to1} \text{ (induced)} = g^2 |\langle E_1 | m(0) | E_2 \rangle|^2 \int_{-\infty}^{+\infty} d\Delta\tau e^{i\Delta E\Delta\tau} [G_M(\Delta\tau) - G_R(\Delta\tau)]. \tag{4.11}$$

The result of this integral is

$$\int_{-\infty}^{+\infty} d\Delta\tau e^{i\Delta E\Delta\tau} \left[\frac{-(a/2)^2}{4\pi^2 \sinh^2[(a/2)\Delta\tau]} + \frac{1}{4\pi^2(\Delta\tau)^2} \right] = \frac{\Delta E}{2\pi} \frac{1}{e^{2\pi\Delta E/a} - 1}, \tag{4.12}$$

and, therefore, the result for $\dot{P}_{2\to1}$ (induced) coincides with (3.26).

---

[9] Note that $\dot{P}_{2\to1}$ (induced) $\neq \dot{P}_{2\to1}$. This fact was overlooked in [17], leading to a misunderstanding of the role of the subtraction term $G_R(\Delta\tau - i\epsilon)$ in (4.10).

The absorption probability rate $\dot{P}_{1\to 2}$ can be calculated in the same way, but with $\Delta E \to -\Delta E$ (recall that $\Delta E > 0$, by definition). Therefore, taking into account that the integral in (4.12) is real, one obtains

$$\dot{P}_{1\to 2} = g^2 |\langle E_1|m(0)|E_2\rangle|^2 \int_{-\infty}^{+\infty} \mathrm{d}\Delta\tau \, \mathrm{e}^{-\mathrm{i}\Delta E \Delta\tau}[G_{\mathrm{M}}(\Delta\tau) - G_{\mathrm{A}}(\Delta\tau)]$$

$$= g^2 |\langle E_f|m(0)|E_i\rangle|^2 \frac{\Delta E}{2\pi} \frac{1}{\mathrm{e}^{2\pi\Delta E/a} - 1} = \dot{P}_{2\to 1} \text{ (induced)}. \qquad (4.13)$$

Thus, as expected, the absorption rate coincides with the rate of induced emission. We reproduce in this way the same results as in the previous section.

It is interesting to note how the $\mathrm{i}\epsilon$ prescription, which provided a well-defined distributional sense to the two-point functions, allows one to write a single expression (4.5) to account for both the total emission and the absorption probability rates. This is so because, on the one hand, the spontaneous emission probability rate can be computed as the residue of the pole of the two-point function $G_{\mathrm{M}}(\Delta\tau)$ at $\Delta\tau = 0$ and, on the other hand, the stimulated emission and the absorption probability rate can be computed as the sum of the residues of the infinite number of poles of $\mathrm{e}^{\mathrm{i}(E_i - E_f)\Delta\tau} G_{\mathrm{M}}(\Delta\tau)$ in the positive and negative imaginary axes, respectively, excluding the one at $\Delta\tau = 0$. The $\mathrm{i}\epsilon$ displaces the real pole of $G_{\mathrm{M}}(\Delta\tau)$ from $\Delta\tau = 0$ to $\Delta\tau = \mathrm{i}\epsilon$ (recall that $\epsilon > 0$). Therefore, including the $\mathrm{i}\epsilon$ and using the Cauchy theorem, the integral (4.6) includes the induced and spontaneous contribution when $E_i - E_f > 0$ (emission) and only the induced one when $E_i - E_f < 0$ (absorption), allowing one to obtain the well-known compact integral representation for both the total emission and the absorption probability rates. We will come back to this point in section 5, where we show that there is an advantage in writing down the separate expressions above for the different processes when one studies the influence of trans-Planckian physics.

5. The role of trans-Planckian physics

The thermal spectrum obtained in the analysis in terms of Bogolubov coefficients (section 2) seems to depend crucially on the exact validity of relativistic field theory on all scales. The intermediate integral (2.8) involves an unbound integration in arbitrarily large Minkowskian momentum k'_x. If one introduces an ultraviolet cutoff Λ for $|k'_x|$ in that integral, which particularizes a given Lorentz frame, the resulting spectrum is dramatically modified. This can be implemented, for instance, by introducing a damping factor such as $\mathrm{e}^{-(y/2R)^2}$ [with $R \approx \ln(\Lambda/a)$] in (2.9) while keeping the integration limits up to infinity. Then the delta function turns into $\delta_\sigma(w_1 - w_2) = \mathrm{e}^{-(w_1 - w_2)^2/\sigma^2}/\sigma\sqrt{\pi}$, where $\sigma = 1/R$. The number of particles described by wave packets of the (standard) form $u_{w_j n\vec{k}_\perp} = \epsilon^{-1/2} \int_{w_j - \epsilon/2}^{w_j + \epsilon/2} \mathrm{d}w \, \mathrm{e}^{\mathrm{i}2\pi nw/\epsilon}(u)^R_{w,\vec{k}_\perp}$, which are localized at the instant $t_n = 2\pi n/\epsilon$ and are peaked at the frequency $w_j = (j + 1/2)\epsilon$ with width ϵ, then becomes $\delta(\vec{k}_{\perp_1} - \vec{k}_{\perp_2})(\mathrm{e}^{2\pi w_j/a} - 1)^{-1}\mathrm{e}^{-(t_n\sigma/2)^2}$, which decays exponentially with time as $t_n \to \infty$, showing that thermality becomes a transient process even if Λ is at the Planck scale.[10]

This apparent sensitivity of the thermal distribution of Rindler quanta to the high-frequency band of the spectrum of fluctuations of the field in the Minkowski vacuum contrasts with the

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[10] As we will show later in this section, a different conclusion is reached when we introduce the cutoff in the Bogolubov transformation method by means of a Lorentz-invariant procedure.

derivation in terms of transition probabilities of the detector of section 3. To show that the detector energy levels are thermally populated (as if it were immersed in a thermal bath) does not require one to integrate over all the Minkowskian momenta. This follows from the fact that the individual transition rates for each (Minkowskian) mode of momentum $\vec{k}$ satisfy the detailed balance relation, as discussed in the last paragraph of section 3. In addition, it can be shown that the relative contribution of trans-Planckian Minkowski modes to the integral (3.22) is negligible, even at late times. This indicates that the thermal properties of the radiation bath that excite the detector are not crucially linked to the integration over large frequencies/momenta in the spectrum of (real) Minkowskian modes.

Since the sum over momenta in section 3 is not the reason for the existence of the thermal properties, the only place where trans-Planckian physics could play a role is in the integration in $\Delta\tau$. In order to study this integral, it is convenient to first integrate in $\vec{k}$, which leads to the derivation of the acceleration radiation in terms of the two-point functions presented in section 4. Using (4.5) and (4.6), we have the following integral expression for the emission probability rate,

$$
\begin{aligned}
\dot{P}_{2\to1} &= g^2|\langle E_1|m(0)|E_2\rangle|^2 \int_{-\infty}^{+\infty} \mathrm{d}\Delta\tau\, \mathrm{e}^{\mathrm{i}\Delta E\Delta\tau/\hbar}[G_{\mathrm{M}}(\Delta\tau-\mathrm{i}\epsilon)] \\
&= g^2|\langle E_1|m(0)|E_2\rangle|^2 \int_{-\infty}^{+\infty} \mathrm{d}\Delta\tau\, \mathrm{e}^{\mathrm{i}\Delta E\Delta\tau/\hbar}\left[-\frac{\hbar(a/2)^2}{4\pi^2\sinh^2[(a/2)(\Delta\tau-\mathrm{i}\epsilon)]}\right] \\
&= g^2|\langle E_1|m(0)|E_2\rangle|^2 \frac{\Delta E}{2\pi}\frac{\mathrm{e}^{2\pi\Delta E/a}}{\mathrm{e}^{2\pi\Delta E/a}-1},
\end{aligned}
\tag{5.1}
$$

and for the absorption probability rate (see (4.13)),

$$
\begin{aligned}
\dot{P}_{1\to2} &= g^2|\langle E_1|m(0)|E_2\rangle|^2 \int_{-\infty}^{+\infty} \mathrm{d}\Delta\tau\, \mathrm{e}^{-\mathrm{i}\Delta E\Delta\tau/\hbar}[G_{\mathrm{M}}(\Delta\tau-\mathrm{i}\epsilon)] \\
&= g^2|\langle E_1|m(0)|E_2\rangle|^2 \int_{-\infty}^{+\infty} \mathrm{d}\Delta\tau\, \mathrm{e}^{-\mathrm{i}\Delta E\Delta\tau/\hbar}\left[-\frac{\hbar(a/2)^2}{4\pi^2\sinh^2\left[\frac{a}{2}(\Delta\tau-\mathrm{i}\epsilon)\right]}\right] \\
&= g^2|\langle E_1|m(0)|E_2\rangle|^2 \frac{\Delta E}{2\pi}\frac{1}{\mathrm{e}^{2\pi\Delta E/a}-1}.
\end{aligned}
\tag{5.2}
$$

In the above integrals, the $\mathrm{i}\epsilon$ term plays a fundamental role in regularizing the denominator of the integrand, which otherwise would lead to a divergence as $\Delta\tau\to0$. This could be seen as an indication that ultrashort (sub-Planckian) distances $(\Delta\tau)^2 < \ell_{\mathrm{P}}^2$ (ℓ_{P} is the Planck length) play a relevant role in the outcome of those integrals. A Planckian cutoff for $\Delta\tau$ in the above integrals substantially modifies the thermal result. However, the integral representation of the transition probability rates provided by the $\mathrm{i}\epsilon$ prescription cannot be properly used to evaluate the effect of such a cutoff. The $\mathrm{i}\epsilon$ prescription is incompatible with cutting out part of the integration path [18]. The distributional character of the integrand, in contrast with the smooth integrand of (4.12), prevents us from properly evaluating the relative contribution of trans-Planckian physics in terms of the above integral expressions.

In contrast, when the different contributions to the transition processes are worked out separately (see expressions (4.11) and (4.12) in section 4), the integrals are well-defined smooth functions. This implies that expression (4.11), as pointed out in [19, 20], can be used to properly

estimate the interval of $\Delta\tau$ that significantly contributes to the overall integral for the induced emission rate,

$$\int_{-\infty}^{+\infty} d\Delta\tau \, e^{-i\Delta E \Delta\tau} \left[\frac{-(a/2)^2}{4\pi^2 \sinh^2 [(a/2)\Delta\tau]} + \frac{1}{4\pi^2 (\Delta\tau)^2} \right] = \frac{\Delta E}{2\pi} \frac{1}{e^{2\pi \Delta E/a} - 1}. \tag{5.3}$$

One finds that values of $\Delta\tau$ giving the dominant contribution to the above integral are of the same order as the acceleration a itself (for non-extreme values of ΔE). That is, neither very large nor very small values of $\Delta\tau$, in comparison with a, are important for obtaining the result. To be more precise, one can compute the integral (5.3), excluding the contribution of ultrashort proper time lapses $|\Delta\tau| < \ell_P$, and the result for the induced emission probability rate is

$$\dot{P}_{2\to 1} \, (\text{induced}) = g^2 |\langle E_1 | m(0) | E_1 \rangle|^2 \frac{\Delta E}{2\pi} \left(\frac{1}{e^{2\pi \Delta E/a} - 1} - \frac{a \, \ell_P}{48\pi^3 \Delta E/a} + O(a\ell_P)^3 \right), \tag{5.4}$$

where we can see that the correction term is completely negligible relative to the thermal term if $a \ll \ell_P^{-1}$ and if the energy gap ΔE is not much larger than the temperature, $a/(2\pi)$, of the thermal spectrum. Exactly the same result is obtained for the excitation probability rate. Obviously, the spontaneous emission is robust against trans-Planckian physics.[11] Therefore, we conclude that for an accelerated detector, the behavior of the two-point function relative to Planckian lapses of proper time does not affect the bulk of the thermal radiation.

The above analysis indicates that the spectrum of thermal radiation felt by a uniformly accelerated observer in Minkowski spacetime is rooted on energy scales of the same order as the energy scale defined by the temperature of the spectrum. We want to point out that the derivation of the transition probability rates in terms of two-point functions in the previous section has allowed us to introduce a cutoff in such a way that Lorentz invariance is manifestly respected. This contrasts with the introduction of a non-Lorentz-invariant cutoff as is sometimes felt to be necessary to avoid appealing to trans-Planckian frequencies in computing the Bogolubov coefficients. Such a cutoff that distinguishes a particular inertial frame also produces a substantial modification of the thermal spectrum at late times. This late-time modification is an explicit consequence of the breakdown of Lorentz invariance, because different instants along the trajectory are related by a Lorentz boost. However, when defined in a Lorentz-invariant way, as in our analysis, trans-Planckian physics does not play a fundamental role in obtaining the thermal spectrum of the acceleration radiation.

In the remaining part of this section, we want to extend the previous analysis of the trans-Planckian contribution to the thermal spectrum, as measured by a particle detector, to the method of calculating the mean number per mode of 'Rindler' particles n_w present in the Minkowski vacuum state. The derivation of this mean number using Bogolubov coefficients has already been summarized in section 2. We will also make some comments about the extension of our analysis to the computation of Hawking radiation by black holes.

In the first paragraph of this section, we showed that introducing a high-frequency cutoff in the calculation of Bogolubov coefficients strongly affects the thermal spectrum. This contrasts

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[11] The result $\dot{P}_{2\to 1} \, (\text{spontaneous}) = g^2 |\langle E_1 | m(0) | E_2 \rangle|^2 ((E_2 - E_1)/2\pi)$ stems essentially from evaluating the amplitude of the modes defining the vacuum at the (low) frequency $w = \Delta E \equiv E_2 - E_1$. Moreover, the derivation in terms of the accelerated detector indicates that the integral in (3.11) does not require very large $|\vec{k}_\perp|$ due to the exponential decay of the Bessel function. Additionally, the spontaneous emission rates of all microscopic systems calculated in the conventional low-energy frameworks agree with observations.

with our previous conclusion. This apparent conflict boils down to whether or not one insists on respecting Lorentz invariance. We prefer to preserve this symmetry. To achieve this, we reformulate the analysis of the mean number distribution of quanta obtained in section 2 in such a way that a study of the trans-Planckian contribution to the thermal spectrum can be done using invariant quantities, thus paralleling our analysis in terms of particle detectors.

As originally shown by Fulling [2], the spectrum of acceleration radiation can be derived by computing the content of 'Rindler' particles in the Minkowski vacuum state. This mean number of Rindler particles per mode n_w can be expressed as $n_w \equiv \langle 0_{\rm M}|N_w^{\rm R}|0_{\rm M}\rangle$, where $N_w^{\rm R}$ is the Rindler particle number operator. In terms of Bogolubov coefficients, that quantity is evaluated as

$$n_w = \langle 0_{\rm M}|N_w^{\rm R}|0_{\rm M}\rangle = \sum_{w'} |\beta_{ww'}|^2. \tag{5.5}$$

On the other hand, as explained in [19, 20, 24], one can rewrite the previous expression in terms of two-point functions as

$$\langle 0_{\rm M}|N_w^{\rm R}|0_{\rm M}\rangle = \int_\Sigma {\rm d}\Sigma_1^\mu {\rm d}\Sigma_2^\nu [u_{w,\vec{k}_\perp}^{\rm R}(x_1)\overset{\leftrightarrow}{\partial}_\mu][u_{w,\vec{k}_\perp}^{\rm R*}(x_2)\overset{\leftrightarrow}{\partial}_\nu](G_{\rm M}(x_1,x_2) - G_{\rm R}(x_1,x_2)), \tag{5.6}$$

where Σ is a Cauchy hypersurface, $u_{w,\vec{k}_\perp}^{\rm R}(x)$ are the Rindler modes defined in (2.4) and $G_{\rm M}$, $G_{\rm R}$ are the two-point functions for the Minkowski and Rindler vacuum states, respectively. Choosing the null plane H^-, defined by $V \equiv t + x = 0$, as the initial data hypersurface, we obtain[12]

$$n_w = \langle 0_{\rm M}|N_w^{\rm R}|0_{\rm M}\rangle = \frac{2\pi}{w} \int_{-\infty}^{+\infty} {\rm d}\Delta u \, {\rm e}^{-{\rm i}w\Delta u} \left[\frac{-(a/2)^2}{4\pi^2 \sinh^2[(a/2)\Delta u]} + \frac{1}{4\pi^2(\Delta u)^2}\right] = \frac{1}{{\rm e}^{2\pi w/a} - 1}, \tag{5.7}$$

where u is the null coordinate $u \equiv \tau - \xi$ and $\Delta u \equiv u_1 - u_2$. Note, in passing, that if we project the acceleration trajectory ($\xi = 0$) onto the horizon H^-, then the point on H^- characterized by the coordinate u corresponds to the point on the uniformly accelerated trajectory characterized by coordinate τ.

We want to point out now that the previous derivation of the thermal spectrum using equations (5.6) and (5.7) is closely related to the derivation presented in section 4 using particle detectors and two-point functions. To be more precise, if we compare the generic relation $\dot{P}_{1\to 2} = B u_w$ (recalling that u_w represents the energy density per mode w of the radiation) with the result for the excitation emission rate (4.13), we see that the mean number of particles per mode w for the thermal distribution corresponds to the integral

$$n_w \equiv \frac{1}{{\rm e}^{2\pi w/a} - 1} = \frac{2\pi}{w} \int_{-\infty}^{+\infty} {\rm d}\Delta\tau \, {\rm e}^{-{\rm i}w\Delta\tau} \left[\frac{-(a/2)^2}{4\pi^2 \sinh^2[(a/2)\Delta\tau]} + \frac{1}{4\pi^2(\Delta\tau)^2}\right]. \tag{5.8}$$

This integral coincides with (5.7), with $\Delta\tau$ replaced by Δu. Note that, along H^-, the quantity Δu is invariant under Lorentz transformation. Thus, we see that there is a clear relation between the derivation of the acceleration radiation using accelerated particle detectors and the derivation based on the Rindler particle number. The former derivation showed that invariantly defined trans-Planckian physics does not significantly affect the observed radiation. This implies that

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[12] We neglect an infinite factor $\delta(0)$ arising in the integral as a consequence of using plane-wave modes. By using the standard normalizable wave packets, that factor disappears.

we use the condition $|\Delta u| < \ell_P$ to characterize in a Lorentz-invariant way the trans-Planckian physics in equations (5.6) and (5.7). A further discussion of the Lorentz-invariant cutoff introduced here and the comparison with the cutoff $|\Delta U| < \ell_P$, with $U \equiv t - x$, can be found in [21, 22].

The above discussion offers some hints for the study of the trans-Planckian question in Hawking radiation by black holes. For a spherically symmetric black hole, the average number of particles observed at late times in the state in which no particles are present at early times is given by an expression analogous to (5.5), but in the black hole geometry [4]. A steady rate of radiation is obtained from an explicit computation of the corresponding Bogolubov coefficients and it turns out to be thermal. However, to get this, one needs to perform an unbounded integration in the frequencies w', as discussed, for example, in [9]–[11], [23], in parallel to the unbounded integration in k'_X for the acceleration radiation effect in (2.8). A cutoff in the frequencies w' will change the Hawking effect completely. It will introduce a damping time-dependent factor modulating the thermal radiation. The Hawking radiation is then converted into a transient phenomenon (see, for instance, [24, 25]).

In analogy with the acceleration radiation effect, it is possible to derive the Hawking effect in terms of smooth integrals involving the difference between the two-point functions of the two vacuum states involved. In fact, the general expression (5.6) can also be applied to the black hole, with the Minkowski observer replaced by the so-called *in* observer, the Rindler observer by the *out* observer, and the acceleration a by the surface gravity κ of the black hole (for details, see [21, 24]). Exactly the same expression (5.7) is then obtained,[13] where now u stands for the retarded null coordinate $u \equiv t - r^*$, with t the Schwarzschild time and r^* the tortoise coordinate. The analysis performed for the acceleration radiation then suggests that (as was done in [20, 24]) the condition $|\Delta u| < \ell_P$ characterizes the regime of trans-Planckian physics entering into the derivation of the thermal spectrum and that altering physics in this trans-Planckian regime will not modify the fundamental properties of the Hawking radiation.

6. Conclusions

In this work, we have analyzed the trans-Planckian question for a uniformly accelerated detector. We have split the transition probability rates into spontaneous and induced contributions. The latter can be expressed as a Fourier integral with a smooth integrand involving the difference between two-point functions. This permits us to estimate in a new way the contribution of trans-Planckian physics to the induced probability rates and allows us to show that the main contribution to the induced rates comes from the low energy scale defined by the acceleration a. Trans-Planckian (and ultralow energy) contributions do not seem to play a central role. Nevertheless, one cannot discard the fact that new effects could arise at the Planck scale if one admits that at such high energies non-linear couplings of the field and detector emerge or, even more, if the very notion of the spacetime and Lorentz invariance dissolve into more elementary structures. In other words, we have assumed the validity of the field-detector model up to energies well above the natural scale of the system. On the other hand, the close analogy between the acceleration radiation and the Hawking effect suggests that the above arguments

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[13] The expression analogous to (5.7) for black holes is also relevant [20, 24] to preserve the near-horizon 2D conformal symmetry of black holes, which seems to play a crucial role in understanding the Bekenstein–Hawking entropy (see, for instance, [26]–[29]).

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also support the view of the Hawking effect as a low-energy phenomenon, in agreement with recent results coming from a different perspective [30].

Acknowledgments

Appendix. Spontaneous emission of a detector in de Sitter space

In this appendix we evaluate the spontaneous emission rate of the detector in a de Sitter space described by the static metric $ds^2 = -(1 - \tilde{r}^2 H^2)d\tilde{t}^2 + (1 - \tilde{r}^2 H^2)^{-1}d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$. To properly compare this emission rate with that of the massless (conformal) field in Minkowski space analyzed in section 2, we have to consider here a massless field with a conformal coupling $\xi = 1/6$ to the curvature. In this situation, the form of the modes $u_{wlm}(\tilde{t}, \tilde{r}, \theta, \phi)$ on the detector's trajectory $\tilde{t} = \tau, \tilde{r} = 0$ (detector at rest and at the origin of static coordinates) is[14]

$$u_{wlm}(\tau) = \sqrt{\frac{w}{\pi}} e^{-iw\tau} \frac{1}{\sqrt{4\pi}} \delta_{l0}. \tag{A.1}$$

The transition probability for all possible one-particle final states is given by

$$\dot{P}_{2\to 1} \text{(spontaneous)} = g^2 |\langle E_f | m(0) | E_i \rangle|^2 \sum_{lm} \int_0^{+\infty} dw \frac{w}{2\pi} \delta(E_1 - E_2 + w) \delta_{l0}$$

$$= g^2 |\langle E_1 | m(0) | E_2 \rangle|^2 \frac{(E_2 - E_1)}{2\pi}, \tag{A.2}$$

which, as expected, coincides with (3.8).

References

[1] Parker L 1968 *Phys. Rev. Lett.* **21** 562
    Parker L 1969 *Phys. Rev.* **183** 1057
[2] Fulling S A 1973 *Phys. Rev.* D **7** 2850
[3] Davies P C W 1975 *J. Phys. A: Math. Gen.* **8** 609
[4] Hawking S W 1975 *Commun. Math. Phys.* **43** 199
[5] Parker L L and Toms D J 2009 *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge: Cambridge University Press)
[6] Unruh W G 1976 *Phys. Rev.* D **14** 870
[7] Crispino L C B, Higuchi A and Matsas G E A 2008 *Rev. Mod. Phys.* **80** 787
[8] Fredenhagen K and Haag R 1990 *Commun. Math. Phys.* **127** 273
[9] Parker L 1977 *Asymptotic Structure of Space-Time* ed F P Esposito and L Witten (New York: Plenum)
[10] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)

[14] At an arbitrary point the s-wave mode is $u_{wl=0}(\tilde{t}, \tilde{r}) = \tilde{r}^{-1}(\pi w)^{-1/2}e^{-iw\tilde{t}} \sin w(H^{-1}\tanh^{-1}(H\tilde{r}))Y_{00}$.

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[11] Jacobson T 1991 *Phys. Rev.* D **44** 1731
      Jacobson T 1993 *Phys. Rev.* D **48** 728
[12] Unruh W G 1995 *Phys. Rev.* D **51** 2827
      Brout R, Massar S, Parentani R and Spindel P 1995 *Phys. Rev.* D **52** 4559
      Corley S and Jacobson T 1996 *Phys. Rev.* D **54** 1568
      Corley S and Jacobson T 1999 *Phys.Rev.* D **59** 124011
      Corley S 1998 *Phys. Rev.* D **57** 6280
      Balbinot R, Fabbri A, Fagnocchi S and Parentani R 2005 *Riv. Nuovo Cimento* **28** 1
      Schützhold R and Unruh W G 2008 *Phys. Rev.* D **78** 041504
      Rinaldi M 2008 *Phys. Rev.* D **77** 124029
[13] Wald R M 2001 *Living Rev. Rel.* **4** 6
[14] Unruh W G and Wald R M 1984 *Phys. Rev.* D **29** 1047
[15] DeWitt B S 1979 *General Relativity* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[16] Birrel N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[17] Campo D 2010 arXiv:1004.5324
[18] Kay B S and Wald R M 1991 *Phys. Rep.* **207** 49
[19] Agullo I, Navarro-Salas J, Olmo G J and Parker L 2008 *Phys. Rev.* D **77** 104034
[20] Agullo I, Navarro-Salas J, Olmo G J and Parker L 2008 *Phys. Rev.* D **77** 124032
[21] Agullo I, Navarro-Salas J, Olmo G J and Parker L 2009 *Phys. Rev.* D **80** 047503
      Agullo I, Navarro-Salas J, Olmo G J and Parker L 2010 *Phys. Rev.* D **81** 108502
[22] Helfer A 2010 *Phys. Rev.* D **81** 108501
[23] Fabbri A and Navarro-Salas J 2005 *Modeling Black Hole Evaporation* (London: ICP-World Scientific)
      Agullo I, Navarro-Salas J, Olmo G J and Parker L 2007 *Phys. Rev.* D **76** 044018
[24] Agullo I, Navarro-Salas J and Olmo G J 2006 *Phys. Rev. Lett.* **97** 041302
[25] Barcelo C, Garay L J and Jannes G 2009 *Phys. Rev.* D **79** 024016
[26] Carlip S 2007 *Gen. Relativ. Gravit.* **39** 1519
      Carlip S 1999 *Phys. Rev. Lett.* **82** 2828
[27] Skenderis K 2000 *Lect. Notes Phys.* **541** 325
[28] Navarro-Salas J and Navarro P 2000 *Nucl. Phys.* B **579** 250
      Navarro D J, Navarro-Salas J and Navarro P 2000 *Nucl. Phys.* B **580** 311
[29] Agullo I, Borja E F and Diaz-Polo J 2009 *J. Cosmol. Astropart. Phys.* **0907** 016
[30] Balbinot R, Fabbri A, Fagnocchi S, Recati A and Carusotto I 2008 *Phys. Rev.* A **78** 021603
      Carusotto I, Fagnocchi S, Recati A, Balbinot R and Fabbri A 2008 *New J. Phys.* **10** 103001