Logical Metatheorems for Abstract Spaces axiomatized in Positive Bounded Logic

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Abstract

In this paper we show that normed structures which can be axiomatized in positive bounded logic (in the sense of Henson and Iovino) admit proof-theoretic metatheorems (as developed by the 2nd author since 2005) on the extractability of explicit uniform bounds from proofs in the respective theories. We apply this to design such metatheorems for abstract Banach lattices, L^p- and $C(K)$-spaces as well as bands in $L^p(L^q)$-Bochner spaces. We also show that a proof-theoretic uniform boundedness principle can serve in many ways as a substitute for the model-theoretic use of ultrapowers of Banach spaces.

Keywords: proof mining, positive bounded logic, ultrapower, uniform boundedness principle

1. Introduction

During the last decade, proof-theoretic results (so-called logical metatheorems due to the 2nd author) have been developed which allow one to extract finitary computational content in the form of explicit uniform bounds from prima facie noneffective proofs in abstract nonlinear analysis (see [29] and the subsequent extensions in [16] and [31] as well as [34, 33, 25, 32, 35] for some recent applications). 'Abstract' here refers to the fact that the proofs analyzed concern general classes of metric structures X (in addition to concrete structures such as $\mathbb{R}$ or $C[0,1]$ whose proof-theoretic treatment is covered already by e.g. [28]). As the proof-theoretic methods used in this context are based on extensions and variants of Gödel's functional ('Dialectica') interpretation, the basic condition on the classes of structures to be admissible is that they can be axiomatized by axioms having a (simple) computable solution of their (monotone) functional interpretation (given enrichments by suitable moduli e.g. of uniform convexity, uniform smoothness etc.). Structures treated so far include metric and normed spaces and their completions, W-hyperbolic spaces and CAT(0)-spaces, uniformly convex normed and hyperbolic spaces, uniformly smooth spaces, compact metric spaces. Notably absent in this list are the classes of smooth (but in general not uniformly smooth) or strictly convex (but in general not uniformly convex), separable (but in general not boundedly compact and hence not finite dimensional) normed spaces, incomplete metric spaces etc. These are classes of structures which are not closed under taking ultrapowers (w.r.t. a nonprincipal ultrafilter) of a normed (or metric) structure, since e.g. an ultrapower of a Banach space X is strictly convex iff X is uniformly convex. This already indicates a first point of connection between the proof-theoretic approach to metric and normed structures and the model theory of such structures as developed in the framework of continuous logic (due to [10], adapted by [6]) or positive bounded

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logic ([19]).
The proof-theoretic metatheorems referred to above and adapted to new classes of spaces in this paper, have the form to guarantee the extractability of explicit uniform effective bounds from proofs of large classes of statements provided that the proof can be formalized in a suitable formal framework (the permitted frameworks are so strong that this is no restriction in practice). The complexity and - in particular - the growth of the extracted bound reflect the computational content of the given proof. The metatheorems are applied via specialized formats adapted to the concrete situation at hand (see Corollaries 17.54, 17.55, 17.59, 17.70, 17.71 in [31]). E.g. we may have some iterative procedure (x_n) (e.g. the Krasnoselski, Mann, Ishikawa, Halpern or Bruck iteration) based on a map $T : X \to X$ starting from some $x_0 \in X$ is considered for which either asymptotic regularity results of the form $\|x_n - T(x_n)\| \to 0$ or strong convergence results for (x_n) can be proven. Then the metatheorems can be applied to extract rates of asymptotic regularity resp. of metastability in the sense of T. Tao which, as far as x_0, X, T are concerned, only depend on a bound $\mathbb{N} \ni b \geq \|x_0\|, \|x_0 - T(x_0)\|$ and some so-called majorant T^* for T. In the important case (both in fixed point theory as well as ergodic theory) where T is nonexpansive, T^* can be defined as $T^*(n) := n + 3b$ (see [31], p. 419). This approach has been applied to fixed point theory, ergodic theory, topological dynamics, geodesic geometry, convex optimization, image recovery problems and abstract Cauchy problems in more than 50 papers during the last decade. A survey of applications up to 2008 can be found in [31]. For some more recent applications of logical metatheorems (and related techniques) see e.g. [2, 4, 34, 33, 32, 35].

Let us now come to a second point of connection between the proof-theoretic approach to metric and normed structures and positive bounded logic, namely the treatment of extensionality: in the proof-theoretic framework metric structures X are treated as pseudo-metric spaces with a defined equality relation $x =_X y := d_X(x, y) =_{\mathbb{R}} 0$. To state then that e.g. $f : X \to X$ is a selfmap of a metric space X means to state the extensionality of f w.r.t. $=_X$

$$x =_X y \to f(x) =_X f(y)$$

which must not be included as a general axiom for all f of 'type' $X \to X$ to hold (see the discussion in [29] on the collapse of the proof-theoretic metatheorems in the presence of such an axiom). The issue is that the proof-theoretic method, which extracts uniform quantitative bounds from proofs, would automatically translate such an axiom into the uniform quantitative form of extensionality, *i.e.* uniform continuity on bounded subsets. One possible solution to this is to assume (in the case of bounded metric structures) as an axiom that all the functions considered are uniformly continuous (or even Lipschitzian) with given moduli of uniform continuity which is what is done in the model-theoretic framework (see [19]). This scenario is also the most comfortable one in the proof-theoretic context where the latter, however, also allows for a less radical solution by weakening the extensionality axiom to a (permitted) rule of extensionality:

$$\text{from a proof of } s =_X t \text{ infer that } f(s) =_X f(t),$$

which does not seem to have a natural model-theoretic counterpart.
Related to this extensionality issue is the treatment of relations R (say for simplicity binary ones): if one adds a new constant χ_R for its characteristic function to the formal system, then - again - we are only allowed to use the rule of extensionality. This problem is circumvented when χ_R can be extended to a uniformly continuous real-valued function. A prime example for this in the model-theoretic approach is the relation $x \leq y$ in a Banach lattice which can be expressed in a uniformly continuous way as $x \sqcup y =_X y$ using the supremum operation $\sqcup$ (see below). Here again, continuous logic (or positive bounded logic) solves the extensionality problem by taking the radical approach of demanding things to be expressed in continuous real-valued terms whereas in proof theory we can also follow this line but are not obliged to (using instead an extensionality rule).

The proof-theoretic approach is particularly simple if one only deals with bounded metric structures X as is done in [29], where, however, also normed spaces are included (but mainly via norm-bounded balls). This is due to the fact that one can use a trivial notion of majorizability - which is the key concept for keeping track of uniform boundedness relations throughout a given proof - for objects in X, maps $X \to X$ etc. Nevertheless, things can be adapted to unbounded metric structure as is done in [16], where one then uses a ternary majorizability relation relative to a reference point $a \in X$ (which in the case of normed spaces is always taken to be a zero vector). Much of the model theory for metric and normed structures relies on boundedness restrictions right from the beginning and continuous logic has been adapted to unbounded metric structures only in [5].

While the proof-theoretic framework, which is not restricted to uniformly continuous functions, can deal with classes of functions and metric structures which are not included in the present set-up of positive bounded and continuous logic, we show in this paper that, conversely, all structures which have an axiomatization in positive bounded logic admit proof-theoretic metatheorems tailored at the respective structures. We exemplify this first by treating abstract (real) L^p-spaces and abstract spaces $C(K)$ (of continuous real-valued functions on an abstract compact space K) which are model-theoretically particularly well-studied but have not yet been considered from the proof-theoretic side. Seminal characterizations due to Bohnenblust [8], Kakutani [23, 24], Nakano [44], Gordon [17] and subsequent work of Krivine [9] (see also [6]) are the starting point of our axiomatization of the aforementioned abstract spaces. Using real Banach lattices and some additional axioms in the language of Banach lattices, it is possible to characterize L^p-spaces $1 \leq p < \infty$) in a way that we can design logical metatheorems in the spirit of [31],[16] and [29] (Section 5). To this end, we give a set of universal axioms for Banach lattices (Section 2), which is proven to be equivalent to the standard approach ([45, Schaefer]). By adding the inequality

$$\|x \sqcup y\|^p \leq \|x\|^p + \|y\|^p \leq \|x + y\|^p, \qquad \text{for all positive } x, y \in X,$$

where $x \sqcup y$ denotes the supremum of x and y (elements of a Banach lattice X), it is known from [24] that any model of the theory is isometrically order-isomorphic to an L^p-space (Section 3). Similarly for the spaces $C(K)$ (Section 4).

We then prove that **generally** axioms in positive bounded logic (which has the same expressive power as Chang and Keisler's continuous logic, see [10]) can be translated into (adding appropriate 'Hilbert ε-operators' to the language) axioms Δ of a logical form which guarantees a trivial (monotone) functional interpretation (Proposition 6.17). Moreover, the latter axioms are more expressive as they allow for quantification over $\mathbb{N}$ and X (rather than only over $B_{\overline{n}}(0)$ for each fixed numeral $\overline{n}$). This is crucial for the domain of applicability of the metatheorems as it makes many $\forall n \in \mathbb{N} \forall x \in B_n(0) \exists m \in \mathbb{N}\, A_\exists$-theorems ($A_\exists$ purely existential) provable to which the extractability of explicit uniform bounds $\Phi(n) \geq m$ then applies.

Using this, we adapt the logical metatheorems developed by [29] and [16] not only to (real) Banach lattices, abstract L^p-spaces and abstract $C(K)$-spaces but to any structure axiomatized in positive bounded logic in the sense of [19] (Theorem 6.18). In particular, we give a proof-theoretic account of the technically very involved model-theoretic treatment (due to [20]) of the theory of $L^p(L^q)$-Banach lattices and we establish a proof-theoretic bound extraction theorem for the theory of bands of $L^p(L^q)$-Bochner spaces. Henson and Raynaud presented in [20] an infinite list of axioms, also using Banach lattices, which axiomatizes bands of $L^p(L^q)$-Bochner spaces. In our formal framework we can express their list of axioms by one sentence.

When we talked so far about structures axiomatized by sentences in positive bounded logic we referred to the usual notion of satisfaction. In the model-theoretic literature ([19]), however, a different notion of approximative satisfaction is used which means the satisfaction of the family of all 2^{-k}-approximations φ_k to a sentence φ in positive bounded logic rather than that of φ itself. For the axiomatizations discussed so far, this makes no difference as the axioms are already in approximate form. In general, however, a structure may satisfy all φ_k without satisfying φ. Henson and Iovino [19] showed that the validity of all approximations φ_k for each fixed $k \in \mathbb{N}$ in a normed space structure $\mathcal{M}$ is equivalent to the validity of φ in any ultrapower $\mathcal{M}_{\mathcal{U}}$ of $\mathcal{M}$ w.r.t. a nonprincipal ultrafilter $\mathcal{U}$ (see [19], Proposition 9.26). So the class of structures axiomatized by the approximate version of the axioms is also closed under taking ultraroots while the class axiomatized (in the usual sense) by positive bounded axioms is only closed under ultraproducts. We show that in the proof-theoretic framework, $\varphi(k)$ can be written as a single formula with parameter k and establish that - over our deductive framework - a certain nonstandard uniform boundedness principle, more precisely Σ^0_1-UB$^X_-$ (going back - for the case of bounded metric structures - to the 2nd author [30]) establishes the equivalence between φ and $\forall k \in \mathbb{N}\, \varphi(k)$ (Theorem 6.33). This suggests that Σ^0_1-UB$^X_-$, which can be safely added as an axiom to the formal systems in our metatheorems without any contribution to the complexity of the extracted bounds, can be viewed as a proof-theoretic analogue to the model-theoretic use of ultrapowers (for proof-theoretic investigations on the strength of the existence of a nonprincipal ultrafilter see [37, 49]). In fact, we show that we may safely use the full strength of axioms φ in positive bounded logic in proofs from which we extract uniform bounds while the resulting bound then will be valid also in the (in general larger) class of all structures which only satisfy the weaker axioms $\forall k \in \mathbb{N}\, \varphi(k)$. We also show that a number of other uses of ultrapowers can be replaced by the use of Σ^0_1-UB$^X_-$: e.g. Σ^0_1-UB$^X_-$ implies that a Banach space X is uniformly convex (uniformly smooth resp.) if and only if it is strictly convex (resp. smooth) which corresponds to the respective equivalences in ultraproducts of Banach spaces (see Section 6.4).

To summarize things, the present paper shows that, to a certain extent, the proof-theoretic approach, in the case of uniformly continuous functions and structures axiomatizable in positive bounded logic, can be viewed as a constructive explicit finitary counterpart to the model-theoretic and ultrapower-based techniques which, conversely, can be used in this case, as has recently been pointed out in [3], to establish qualitative uniformity results corresponding to the quantitative uniformity results extracted proof-theoretically. Let us emphasize though, that the proof-theoretic framework, which is based on the language of functionals in all finite types, also allows for higher order axiomatizations of structures and functions, whereas the model-theoretic context is essentially first-order. Also, as mentioned already above, the proof-theoretic analysis also works in a weakly extensional framework and only requires uniform quantitative versions of those instances of extensionality actually used in the proof which in general is much weaker than to assume the uniform continuity of all the constants involved (see [35] for a recent use of this feature).

For simplicity, we only consider one abstract space X (in addition to the concrete space $\mathbb{R}$) and selfmaps $f : X \to X$ in this paper. However, following the approach in [18], everything can be extended to several (possibly different) normed spaces X_i and functions $f : X_{i_1} \times \ldots \times X_{i_k} \to X_{i_j}$ (where some of these spaces could also be $\mathbb{R}$).

2. Banach lattices

We follow Schaefer [45] to define real Banach lattices. We do not consider complex Banach lattices since the additional structure is irrelevant in our context and a complex Banach lattice can be viewed as a real Banach lattice.

Definition 2.1 ([45, II, Section 1])**.** The set X with a binary relation $\leq$ is called a *lattice* if there are binary operations $\sqcup, \sqcap$ on X such that the following axioms hold:

(B1) $\forall x, y, z \in X\, (x \leq y \land y \leq z \to x \leq z)$,

(B2) $\forall x \in X\, (x \leq x)$,

(B3) $\forall x, y \in X\, (x \leq y \wedge y \leq x \rightarrow x = y)$,

(B4) $\forall x, y \in X\, (x \leq x \sqcup y \wedge y \leq x \sqcup y)$,

(B5) $\forall x, y \in X\, (x \sqcap y \leq x \wedge x \sqcap y \leq y)$,

(B6) $\forall m, x, y \in X\, (x \leq m \wedge y \leq m \rightarrow x \sqcup y \leq m)$,

(B7) $\forall m, x, y \in X\, (m \leq x \wedge m \leq y \rightarrow m \leq x \sqcap y)$.

Note that $\sqcup, \sqcap$ are uniquely determined.

Definition 2.2 ([45, II, Definition 1.2])**.** Let X be a vector space X over $\mathbb{R}$ together with an order relation $\leq$. X is called an *ordered vector space* if the following hold:

(B8) (LO_1) $\forall x, y, z \in X\, (x \leq y \rightarrow x + z \leq y + z)$,

(B9) (LO_2) $\forall x, y \in X \forall \lambda \in \mathbb{R}_+\, (x \leq y \rightarrow \lambda x \leq \lambda y)$.

If in addition X is a lattice in the sense of Definition 2.1, we call X a *vector lattice* or *Riesz space*.

Remark 2.3 ([45, p. 50]). The following is true in all vector lattices X (implied by axiom LO_1): For all $x \in X$ and for any nonempty subset $A \subseteq X$ it holds that $x + \sup(A) = \sup(x + A)$, $x + \inf(A) = \inf(x + A)$ and $\sup(A) = -\inf(-A)$ provided that $\sup(A)$ and $\inf(A)$ resp. exist.

Notation 2.4. The following abbreviations are introduced.

1. $x^+ := x \sqcup 0,\ \ x^- := (-x) \sqcup 0$ and $|x| := x \sqcup (-x)$,
2. $a \sqcup b \pm c \sqcup d := (a \sqcup b) \pm (c \sqcup d)$ and $a \sqcap b \pm c \sqcap d := (a \sqcap b) \pm (c \sqcap d)$.

Definition 2.5 ([45, p. 81])**.** Let X be a vector lattice. A norm $\|\cdot\|$ on X is called a lattice norm if

(B10) $\forall x, y \in X\, (\|x\| = \||x|\| \wedge (0 \leq x \leq y \rightarrow \|x\| \leq \|y\|))$.

If $\|\cdot\|$ is a lattice norm, then the pair $(X, \|\cdot\|)$ is called a *normed (vector) lattice*; if, in addition, $(X, \|\cdot\|)$ is complete w.r.t. the norm it is called a *Banach lattice*.

2.1. Formal representation of Banach lattices

We introduce an extension of the theory $\mathcal{A}^\omega[X, \|\cdot\|, \mathfrak{C}]$ ([31, pp. 410-412 and pp. 432-434] or [29]), consisting of an axiomatization of normed spaces together with an operator C assigning a limit point to each Cauchy sequence with Cauchy rate 2^{-n} (thereby axiomatizing the completeness of X).

Definition 2.6. Define the set of *finite types* $\mathbf{T}^X$ of $\mathcal{A}^\omega[X, \|\cdot\|, \mathfrak{C}]$ by

1. defining ground types: $\mathbb{N}$, X, *i.e.* $\mathbb{N}, X \in \mathbf{T}^X$, and
2. building up higher types inductively: $\rho, \tau \in \mathbf{T}^X \Rightarrow \tau(\rho) \in \mathbf{T}^X$.

The type $\tau(\rho)$ can be written as $\rho \rightarrow \tau$ and objects of type $\tau(\rho)$ can be understood as functions mapping arguments of type ρ to an object of type τ.

Notation 2.7. We define the following abbreviations:

1. Type 1 is an abbreviation for the type $\mathbb{N}(\mathbb{N})$. Using encoding techniques we always allow finitely many arguments of the same type.

2. We write "$+,-,\dots$" instead of "$+_{\mathbb{R}},-_{\mathbb{R}},\dots$", whenever the interpretation is obvious and we use "$\|\cdot\|,\sqcup$" instead of "$\|\cdot\|_X,\sqcup_X$".
3. For the base type X define $x =_X y :\equiv \|x-y\| =_{\mathbb{R}} 0_{\mathbb{R}}$.
4. Define higher-type equalities inductively for types $\rho = \mathbb{N}\tau_k\dots\tau_1$, respectively $\rho = X\tau_k\dots\tau_1$, we set $x =_\rho y$ as
$$\forall z_1^{\tau_1},\dots,z_k^{\tau_k}(x(z_1,\dots,z_k) =_{\mathbb{N}} y(z_1,\dots,z_k)),$$
respectively $\forall z_1^{\tau_1},\dots,z_k^{\tau_k}(x(z_1,\dots,z_k) =_X y(z_1,\dots,z_k))$.
5. Finite tuples of variables are denoted by $\underline{x}^{\underline{\sigma}}$, where $x = x_1^{\sigma_1}\dots x_n^{\sigma_n}$ and $\underline{\sigma} = \sigma_1\dots\sigma_n$ (where the types σ_i are identical if not specified otherwise).

To represent Banach lattices one could add the constants and axioms (B1)-(B10) to our theory. However, the binary relation "$\leq$", or more explicitly its characteristic function, is not computable (since it is not continuous). Since the main goal is to produce computable functionals bounding existential quantified variables, this is an obstacle. Thus, we introduce a constant for the supremum operation, then define the infimum and the binary order relation in terms of the supremum. To this end, we have to add different axioms, for which we will show that they are true in all Banach lattices in the sense of [45] and that the usual axioms for Banach lattices are provable in our theory.

Definition 2.8. We extend the theory $\mathcal{A}^\omega[X,\|\cdot\|,\mathfrak{C}]$ to $\mathcal{A}^\omega[X,\|\cdot\|,\sqcup]$ to represent Banach lattices. The language of $\mathcal{A}^\omega[X,\|\cdot\|,\sqcup]$ has the following constants: All constants inherited from $\mathcal{A}^\omega[X,\|\cdot\|,\mathfrak{C}]$ and the supremum operation "$\sqcup$" of type $X(X)(X)$.

Definition 2.9. We introduce the following symbols as abbreviations:

1. Set "$\sqsubseteq$" as a binary relation as follows: $x \sqsubseteq y :\equiv x \sqcup y =_X y$.
2. Set "$\sqcap$" as operation of type $X(X)(X)$: $x \sqcap y :\equiv -_X((-_X x)\sqcup(-_X y))$.
3. $(x^X)^+ :\equiv x \sqcup 0_X$ and $(x^X)^- :\equiv (-_X x)\sqcup 0_X$,
4. $|x^X|_X :\equiv x \sqcup (-_X x)$.

Definition 2.10. We add the following axioms to the theory $\mathcal{A}^\omega[X,\|\cdot\|,\sqcup]$:

(A1) $\forall x^X(x \sqcup x =_X x)$,

(A2) $\forall x^X, y^X\,(x\sqcup y =_X y \sqcup x)$,

(A3) $\forall x^X, y^X, z^X\,(x \sqcup (y\sqcup z) =_X (x\sqcup y)\sqcup z)$,

(A4) $\forall x^X, y^X\,(x\sqcup(x\sqcap y) =_X x)$ and $\forall x^X, y^X\,(x\sqcap(x\sqcup y) =_X x)$,

(A5) $\forall x^X, y^X, z^X\,(x +_X (y\sqcup z) =_X (x +_X y)\sqcup(x +_X z))$,

(A6) $\forall \lambda^1, x^X, y^X(|\lambda|_{\mathbb{R}}\, x \sqcup |\lambda|_{\mathbb{R}}\,(x\sqcup y) =_X |\lambda|_{\mathbb{R}}\,(x\sqcup y))$.

(A7) $\forall x^X\,(\||x|_X\| =_{\mathbb{R}} \|x\|)$,

(A8) $\forall x^X, y^X\,(\|0_X \sqcup x\| \leq_{\mathbb{R}} \|(0_X\sqcup x)\sqcup y\|)$,

(A9) $\forall x_1^X, x_2^X, y_1^X, y_2^X(\|x_1\sqcup y_1 -_X x_2\sqcup y_2\| \leq_{\mathbb{R}} \|x_1 -_X x_2\| + \|y_1 -_X y_2\|)$.

Proposition 2.11. *The operations "$\sqcup$","$\sqcap$" and "$\sqsubseteq$" are (provably) extensional.*

Proof. Follows directly from axiom (A9) in Definition 2.10 (which we included for this very reason as the usual proof of (A9) from the other axioms uses already extensionality). ☐

Corollary 2.12 (Majorization of "$\sqcup$")**.**

$$\forall x^X, y^X \forall n, m \in \mathbb{N}\, (\|x\| \leq_{\mathbb{R}} n \land \|y\| \leq_{\mathbb{R}} m \to \|x \sqcup y\| \leq_{\mathbb{R}} n + m)\,.$$

Proof. Follows from axioms (A1), (A9) and Proposition 2.11. ☐

For the general definition of majorizability we refer the reader to Definition 5.6. Since "$\sqcap$" is defined via "$\sqcup$" (and "$-_X$") it is majorizable (see [31, Lemma 17.84]). In fact even the same majorant can be used.

Proposition 2.13. *The axioms (B1)-(B10) are provable in* $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$ *and the axioms from Definition 2.10 are true in any Banach lattice (and the order in the lattice coincides with the one defined in terms of* $\sqcup$*).*

Proof. See Appendix Propositions A.1 and A.2. ☐

Definition 2.14 (cp. [29, Definition 3.1])**.** The *full set-theoretic type structure* $\mathcal{S}^{\omega,X} := \langle S_\rho \rangle_{\rho \in \mathbf{T}^X}$ over $\mathbb{N}$ and the space X is defined by:

$$S_{\mathbb{N}} := \mathbb{N}, \quad S_X := X, \quad S_{\tau(\rho)} := S_\tau^{S_\rho},$$

where we denote all set-theoretic functions $S_\rho \to S_\tau$ by $S_\tau^{S_\rho}$.

Proposition 2.15 (cp. [29, Definition 3.21])**.** *Let* $(X, \|\cdot\|, \sqcup)$ *be a nontrivial Banach lattice. Then* $\mathcal{S}^{\omega,X}$ *becomes a model of* $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$ *by letting the variables of type* ρ *range over* S_ρ *if all interpretations for the constants used for normed spaces are obtained from [29, Definition 3.21], and if* $x \sqcup y$ *with* $x, y \in X$ *is interpreted by* $\sup\{x, y\}$.

Proof. Follows from Proposition 2.13. ☐

Definition 2.16 ([29, cp. Definition 3.21])**.** A sentence of the language of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$ *holds* in a nontrivial Banach lattice $(X, \|\cdot\|, \sqcup)$ if it is true in the models of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$ obtained from $\mathcal{S}^{\omega,X}$ as specified in Proposition 2.15.

Remark 2.17. For all subsequent theories and their interpretations we assume an analogue of the previous definition of "*holds*".

3. L^p spaces as Banach lattices

Following Ben-Yaacov et al. [6, Section 17] let $1 \leq p < \infty$, Ω be a set, U a σ-algebra on Ω and μ a σ-additive measure on U. Denote by $L^p(\Omega, U, \mu)$ the space of (equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$ with $\|f\| := \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}$.

Definition 3.1 ([6, pp. 414-415])**.** We write BL^p (for $p \geq 1$) for the theory consisting of the axioms (B1)-(B10) for Banach lattices and

(B11) $\forall x, y \in X\ (x, y \geq 0 \to \|x \sqcup y\|^p \leq \|x\|^p + \|y\|^p \leq \|x + y\|^p)$.

To exclude measures with atoms, *i.e.* the existence of so-called atoms, which are sets $A \subseteq \Omega$ with $\mu(A) > 0$ such that no subset $B \subseteq A$ exists with $0 < \mu(B) < \mu(A)$, one can add another axiom to the theory expressing that (Ω, U, μ) is atomless. An important example for atomless measures is the Lebesgue measure on the real line.

Definition 3.2 ([6, p. 415])**.** The theory BL^p together with the following axiom is denoted by ABL^p.

(B12) $\sup_{x\in X}\inf_{y\in X}\left(\max\{\left|\|y\| - \|x^+ - y\|\right|, \|y \sqcap (x^+ - y)\|\}\right) =_{\mathbb{R}} 0$.

The next theorem goes back to [8],[44],[17] (for $1 \le p < \infty$) and (for the special case $p = 1$) to [23] (although we use a variant axiomatization due to [9], see [38] for more information on the historical background):

Theorem 3.3 (cp. [9, Theorem 3] and [6, Propositions 17.3 and 17.4])**.** *Let $\mathfrak{M}$ be a Banach lattice. Then $\mathfrak{M}$ is a model of the theory $(A)BL^p$ if and only if there is a (atomless) measure space (Ω, U, μ) such that $\mathfrak{M}$ is isometric and lattice isomorphic to $L^p(\Omega, U, \mu)$ where $1 \le p < \infty$ (here $\sqcup$ in $L^p(\Omega, U, \mu)$ is defined up to measure zero sets as pointwise maximum).*

Proof. We refer to the proof of [9, Theorem 3] for BL^p and to the proof of [6, Proposition 17.4] for ABL^p. ☐

3.1. Formal theory for L^p spaces

Definition 3.4. We define the extension $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]$ of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$ by adding a constant c_p of type 1 with the axioms (cp. axiom (B11)):

(A10) $c_p \ge_{\mathbb{R}} 1_{\mathbb{R}}$,

(A11) $\forall x^X, y^X \left(\||x| \sqcup |y|\|^{c_p} \le_{\mathbb{R}} \||x|\|^{c_p} +_{\mathbb{R}} \||y|\|^{c_p} \le_{\mathbb{R}} \||x| +_X |y|\|^{c_p}\right)$.

Note that in Definition 3.1 axiom (B11) is stated without the absolute value but with the restriction to positive $x, y \in X$ which is obviously equivalent. Our version is purely universal, thus it is its own functional interpretation.

Proposition 3.5 (cp. [29, Definition 3.21])**.** *Let Ω be a nonempty set, U a σ-algebra on Ω and μ a nontrivial measure on Ω. Let $1 \le p < \infty$ and let X be the space $L^p(\Omega, U, \mu)$. Then $\mathcal{S}^{\omega,X}$ becomes a model of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]$ by letting the variables of type ρ range over S_ρ as specified in Proposition 2.15, with the exception that $f \sqcup g$ with $f, g \in X$ is interpreted by $\max\{f, g\}$, μ-almost everywhere. The constant c_p is interpreted by $(p)_\circ$, where $(r)_\circ$ for $r \in \mathbb{R}_+$ is the function mapping every real number to a representing element of $\mathbb{N}^{\mathbb{N}}$ (see [29, Definition 2.9]).*

Proof. It is easy to see that the interpretation defined above fulfills all axioms from Definitions 2.10 and 3.4. ☐

Definition 3.6. We define the extension $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_a$ of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]$ by adding the following axiom to ensure that the measure μ is atomless:

(A12) $\forall x^X \forall k^{\mathbb{N}} \exists y \preceq_X 1_X(\|x\| + 1) \left(\left|\|y\| - \|x^+ - y\|\right|, \|y \sqcap (x^+ - y)\| \le_{\mathbb{R}} 2^{-k}\right)$

Proposition 3.7. *The axioms (A12) and (B12) are (after expressing the use of* sup, inf *equivalently using quantifiers) provably equivalent in $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$.*

Proof. By unwinding sup and inf we see that (A12) implies (B12). For the converse we have to prove the bound for y. Observe that $\|(x^+ - y) - (-y)\| =_{\mathbb{R}} \|x^+\| =_{\mathbb{R}} \|x \sqcup 0\| \le_{\mathbb{R}} \||x|\| =_{\mathbb{R}} \|x\|$. By the nonexpansiveness of "$\sqcap$" (axiom (A9)) and (B10) this implies: $\|y \sqcap (x^+ - y) - y \sqcap (-y)\| \le_{\mathbb{R}} \|x\|$ which yields by the reverse triangle inequality $\|y \sqcap (-y)\| - \|x\| \le_{\mathbb{R}} \|y \sqcap (x^+ - y)\|$.
Since $\|y \sqcap (-y)\| =_{\mathbb{R}} \|y \sqcup (-y)\| =_{\mathbb{R}} \||y|\| =_{\mathbb{R}} \|y\|$, this implies $\|y\| \le_{\mathbb{R}} \|x\| + \|y \sqcap (x^+ - y)\|$.
Hence, the axioms (B12) and (A12) are equivalent. ☐

Theorem 3.8 (cp. [9, Theorem 3] and [6, Propositions 17.3 and 17.4])**.** *The structure $\mathcal{S}^{\omega,X}$ is a model of the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_{(a)}$ as defined in Proposition 2.15 if and only if there is a (atomless) measure space (Ω, U, μ) such that $(X, \|\cdot\|, \sqcup)$ is isometric lattice isomorphic to $L^p(\Omega, U, \mu)$.*

Proof. Since we have shown that all axioms of the theory $(A)BL^p$ from Definition 3.1 can be proven in the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_{(a)}$, and also that axioms from $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_{(a)}$ hold in a Banach lattice in the sense of $(A)BL^p$ together with an equivalent formulation of the atomless axiom, the result follows from Theorem 3.3. $\square$

4. C(K) spaces

Similarly to L^p spaces one can also represent $C(K)$ spaces of continuous real-valued functions, where K is an abstract compact space, by Banach lattices.

Definition 4.1 ([45, Definition II.7.1])**.** A lattice norm $x \mapsto \|x\|$ on a vector lattice E is called an M-norm if it satisfies the axiom

$$\text{(M)} \quad \|x \sqcup y\| = \max\{\|x\|, \|y\|\} \qquad (x, y \in E_+).$$

A Banach lattice whose norm fulfills (M) is called an abstract M-space (AM-space). If the unit ball contains a largest element and that element has norm 1, it is called the unit of E.

Theorem 4.2 ([24, Theorem 2])**.** *For any AM-space with unit there exists a compact Hausdorff space K such that (AM) is isometric and lattice isomorphic to the space $C(K)$ of all bounded continuous real-valued functions defined on K with $\|\cdot\|_\infty$ and pointwise supremum $\sqcup$.*

Definition 4.3. We extend the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$ to $\mathcal{A}^\omega[X, \|\cdot\|, C(K)]$ by adding the following axioms (note that $\|1_X\| =_{\mathbb{R}} 1$ is already an axiom of $\mathcal{A}^\omega[X, \|\cdot\|]$)

(A13) $0_X \sqsubseteq 1_X$ and $\forall x^X (\tilde{x} \sqsubseteq 1_X)$, where $\tilde{x} := \dfrac{x}{\max_{\mathbb{R}}\{\|x\|, 1\}}$.

(A14) $\forall x^X, y^X (\| |x| \sqcup |y| \| =_{\mathbb{R}} \max_{\mathbb{R}}\{\|x\|, \|y\|\})$.

Proposition 4.4. *The axioms (A13) and (A14) are true in any AM-space with unit in the sense of Definition 4.1 and the theory $\mathcal{A}^\omega[X, \|\cdot\|, C(K)]$ proves axiom* (M) *and the existence of a unit, namely 1_X.*

Proof. The axioms (A13) and (A14) are direct formalization of axiom (M) and the fact that the unit element is the largest element in the unit ball. $\square$

Proposition 4.5. *Let $(X, \|\cdot\|, \sqcup, e)$ be an AM-space. Then $\mathcal{S}^{\omega,X}$ becomes a model of the theory $\mathcal{A}^\omega[X, \|\cdot\|, C(K)]$ by letting the variables of type ρ range over S_ρ if all conditions of Proposition 2.15 hold with the exception of the interpretation of the constant 1_X[1] which is interpreted by the element $e \in X$ with $\|e\| = 1$ and the property $\forall x \in X\, (\|x\| \leq 1 \to x \sqsubseteq e)$.*

Theorem 4.6. *Let $(X, \|\cdot\|, \sqcup, e)$ be a Banach lattice with a unit e. The structure $\mathcal{S}^{\omega,X}$ is a model of the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, C(K)]$ as defined in Proposition 4.5 if and only if there exists a compact Hausdorff space K such that $(X, \|\cdot\|, \sqcup, e)$ is isometric and lattice isomorphic to the space $C(K)$ of all bounded continuous real-valued functions defined on K, where $e \in X$ is the interpretation of the constant e according to Proposition 4.5.*

---

[1] Note that 1_X is a constant in the language of $\mathcal{A}^\omega[X, \|\cdot\|]$ having norm 1, see [29].

Proof. Follows from Theorem 4.2 and Proposition 4.4. □

Remark 4.7. It is also possible to use the following axiom without involving a constant for the unit element and so staying in the signature of Banach lattices (note that $0_X \sqsubseteq e$ follows already from $x := 0_X$ and $\|e\|_X = 1$ follows from $x := 1_X$ and $\|1_X\| = 1$):

$$\exists e \preceq_X 1_X \forall x^X \left(\tilde{x} \sqsubseteq e\right), \text{ where } \tilde{x} := \frac{x}{\max_{\mathbb{R}}\{\|x\|, 1\}}. \tag{1}$$

5. Logical Metatheorem for L^p, $C(K)$ and Banach lattices

As previewed in Corollary 2.12 we define majorization, which is crucial for proving the forthcoming metatheorem.

Definition 5.1 ([31, Definition 17.32])**.** We define inductively for each type $\rho \in \mathbf{T}^X$ the corresponding majorization type $\widehat{\rho} \in \mathbf{T}$:

$$\widehat{\mathbb{N}} := \mathbb{N}, \qquad \widehat{X} := \mathbb{N}, \qquad \widehat{\tau(\rho)} := \widehat{\tau}(\widehat{\rho}).$$

Definition 5.2. We define two important classes of finite types $\rho \in \mathbf{T}^X$:

1. Define the class of *small* types consisting of the following finite types: $\mathbb{N}$, $\mathbb{N}(\mathbb{N})\ldots(\mathbb{N})$, X and $X(\mathbb{N})\ldots(\mathbb{N})$.
2. Define the class of *admissible* types consisting of the following finite types: $\mathbb{N}(\rho_k)\ldots(\rho_1)$ and $X(\rho_k)\ldots(\rho_1)$ where $\rho_1,\ldots,\rho_k$ are *small* types. Also the type $\mathbb{N}, X$ are admissible (in particular, therefore, every small type is admissible).

Definition 5.3 ([29, Definition 3.22])**.** For functionals x^ρ,y^ρ of type $\rho \in \mathbf{T}^X$ define $x \preceq_\rho y$ by

$$\begin{aligned}
\rho = \mathbb{N}: &\quad x \preceq_{\mathbb{N}} y :\equiv x \leq y,\\
\rho = X: &\quad x \preceq_X y :\equiv \|x\| \leq_{\mathbb{R}} \|y\|,\\
\rho = \tau(\sigma): &\quad x \preceq_{\tau(\sigma)} y :\equiv \forall z^\sigma \left(x(z) \preceq_\tau y(z)\right).
\end{aligned}$$

Definition 5.4 ([31, cp. p. 142 and Theorem 10.26])**.**

1. We define Δ to be the set of all sentences of the form

   $$\forall \underline{a}^{\underline{\delta}} \exists \underline{b} \preceq_{\underline{\sigma}} \underline{ra} \forall \underline{c}^{\underline{\gamma}} B_0(\underline{a}, \underline{b}, \underline{c}),$$

   where B_0 is quantifier-free and does not contain any further free variables, $\underline{r}$ is a closed term (of suitable types) of $\mathcal{A}^\omega[X, \|\cdot\|, \ldots]$. The types $\underline{\delta}, \underline{\sigma}, \underline{\gamma}$ can be at most admissible.

2. We denote the Skolem normal forms of the sentences in Δ by

   $$\tilde{\Delta} :\equiv \left\{\exists \underline{B} \preceq_{\underline{\sigma}(\underline{\delta})} \underline{r} \forall \underline{a}^{\underline{\delta}}, \underline{c}^{\underline{\gamma}} B_0(\underline{a}, \underline{Ba}, \underline{c}) : \forall \underline{a}^{\underline{\delta}} \exists \underline{b} \preceq_{\underline{\sigma}} \underline{ra} \forall \underline{c}^{\underline{\gamma}} B_0(\underline{a}, \underline{b}, \underline{c}) \in \Delta\right\}$$

Remark 5.5. The atomless axiom (A12) is syntactically in the class Δ, in contrast to axiom (B12).

Definition 5.6 ([16, Definition 9.1]). The *type structure* $\mathcal{M}^{\omega,X}$ of all (strongly) majorizable set-theoretic functions of finite type $\rho \in \mathbf{T}^X$ over a normed space $(X, \|\cdot\|)$ is defined as:

$$\begin{cases} M_{\mathbb{N}} := \mathbb{N}, \quad n \gtrsim_{\mathbb{N}} m :\equiv n \geq m \wedge n, m \in \mathbb{N}, \\ M_X := X, \quad n \gtrsim_X x :\equiv n \geq \|x\| \wedge n \in M_{\mathbb{N}}, x \in M_X, \\ x^* \gtrsim_{\tau(\rho)} x :\equiv x^* \in M_{\widehat{\tau}}^{M_{\widehat{\rho}}} \wedge x \in M_\tau^{M_\rho} \\ \qquad \wedge \forall y^* \in M_{\widehat{\rho}}, y \in M_\rho \, (y^* \gtrsim_\rho y \to x^* y^* \gtrsim_\tau xy) \\ \qquad \wedge \forall y^*, y \in M_{\widehat{\rho}} \Big(y^* \gtrsim_{\widehat{\rho}} y \to x^* y^* \gtrsim_{\widehat{\tau}} x^* y \Big), \\ M_{\tau(\rho)} := \Big\{ x \in M_\tau^{M_\rho} : \exists x^* \in M_{\widehat{\tau}}^{M_{\widehat{\rho}}} \left(x^* \gtrsim_{\tau(\rho)} x \right) \Big\} \quad \Big(\tau, \rho \in \mathbf{T}^X \Big). \end{cases}$$

Note that without adding the base type X, the type structure of (strongly) majorizable functions of finite type is denoted by $\mathcal{M}^\omega$ defined first by Bezem [7]. We read $x^* \gtrsim_X x$ as "x^* (strongly) majorizes x".

Lemma 5.7.

1. *Let ρ be a small type. Then $M_\rho = S_\rho$.*
2. *Let ρ be an admissible type. Then $M_\rho \subseteq S_\rho$.*

Proof. This is proven in [31, Proposition 3.70] for types $\mathbf{T}$ and for types $\mathbf{T}^X$ in [16, Proof of Theorem 4.10]. □

Lemma 5.8 (cp. [16, Lemma 9.11]). *All closed terms t in the language of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_a$ are majorizable by closed terms in $\mathcal{A}^\omega$ when interpreted in $\mathcal{M}^{\omega,X}$ (depending on p only via an upper bound $\mathbb{N} \ni b \geq p$).*

Proof. We can refer to the proof of [16, Lemma 9.11] which is done by induction on the complexity of the closed terms for $\mathcal{A}^\omega[X, \|\cdot\|]$ (and for $\mathcal{A}^\omega[X, \|\cdot\|, \mathfrak{C}]$ see [31, p. 434]). Thus, it remains to show that newly introduced constants are majorizable. For the supremum operation this is shown in Corollary 2.12. The constant c_p is majorized (see [31, Lemma 17.8]) by $M(b) \gtrsim_1 [c_p]_{\mathcal{M}^{\omega,X}} = [c_p]_{\mathcal{S}^{\omega,X}} = (p)_\circ$, with $b \in \mathbb{N}$ such that $b \geq p$ and $M(b) := \lambda n. j(b 2^{n+2}, 2^{n+1} - 1)$, where $j(\cdot, \cdot)$ denotes the Cantor pairing function. As b we can always take e.g. $b := \lceil (c_p(0))_{\mathbb{Q}} \rceil + 1$. □

Definition 5.9 (cp. [27, Definition 3.10]). We define the bounded axiom of choice:

$$\begin{aligned} \text{b-AC}_X &:\equiv \bigcup_{\delta, \rho \in \mathbf{T}^X} \left\{ \text{b-AC}^{\delta,\rho} \right\}, \quad \text{where} \\ \text{b-AC}^{\delta,\rho} &:\equiv \forall Z^{\rho(\delta)} \left(\forall x^\delta \exists y \preceq_\rho Zx A(x, y, Z) \to \exists Y \preceq_{\rho(\delta)} Z \forall x A(x, Yx, Z) \right). \end{aligned}$$

Lemma 5.10 (cp. [27, Application 3.12]). $\mathcal{M}^{\omega,X} \models \text{b-AC}_X$.

Proof. Analogous to the proof of [27, Application 3.12]). □

Lemma 5.11. *For the sentences Δ as defined in Definition 5.4 the following holds: $\mathcal{S}^{\omega,X} \models \Delta \;\Rightarrow\; \mathcal{M}^{\omega,X} \models \tilde{\Delta}$.*

Proof. We first want to prove $\mathcal{S}^{\omega,X} \models \Delta \;\Rightarrow\; \mathcal{M}^{\omega,X} \models \Delta$. Recall that all sentences in Δ (here we only implictly refer to the tuple notation) have the format $A := \forall a^\delta \exists b \preceq_\sigma ra \forall c^\gamma B_0(a, b, c)$. From Lemma 5.7 we know that for small types ρ we have $M_\rho = S_\rho$ and for admissible types σ we have $M_\sigma \subseteq S_\sigma$. So if all types are small, the assertion holds trivially (see also Lemma 17.84 in [31]).

For the universal variables a^δ and c^γ the sentence A is weakened since the scope of the universal quantifier is reduced from S_δ to M_δ (resp. for γ). Note that this inclusion does not hold for higher types (see Howard [21]). Then we check the definition of the statement $\exists b \preceq_\sigma ra$ which is defined (for $\sigma = \tau\rho_n \dots \rho_1$, where $\tau \in \{\mathbb{N}, X\}$) as $\exists b^\sigma \forall \underline{z}^{\underline{\rho}}(b\underline{z} \preceq_\tau ra\underline{z})$.
Here we see that it is important to have only small types ρ_i since otherwise the scope of the universal quantified variables $\underline{z}$ would be not identical. Since type of b is admissible we have a smaller domain for finding a witness, thus we show that any b making A true is majorizable and therefore an element of $\mathcal{M}^{\omega,X}$. Because the term r and the variables a, z can only take values in $\mathcal{M}^{\omega,X}$, they are majorizable by definition. From [31, Lemma 17.65] we get that b is majorizable.
Now we show $\mathcal{M}^{\omega,X} \models \Delta \;\Rightarrow\; \mathcal{M}^{\omega,X} \models \tilde{\Delta}$. Recall that all sentences in $\tilde{\Delta}$ have the form $\exists B \preceq_{\sigma(\delta)} r \forall a^\delta, c^\gamma B_0(a, Ba, c)$. Then by using the bounded axiom of choice (Lemma 5.10) we see that $\Delta + \text{b-AC}_X \vdash \tilde{\Delta}$ and thus $\mathcal{M}^{\omega,X} \models \tilde{\Delta}$. □

Definition 5.12 (cp. [29, Definition 3.6]). A formula F is called a *$\forall$-formula* (resp. *$\exists$-formula*) if it has the form $F \equiv \forall \underline{a}^{\underline{\sigma}} F_{\text{qf}}(\underline{a}, \underline{b})$ (resp. $F \equiv \exists \underline{a}^{\underline{\sigma}} F_{\text{qf}}(\underline{a}, \underline{b})$) where F_{qf} does not contain any quantifiers and the types in $\underline{\sigma}$ are admissible and $\underline{b}$ are parameters of arbitrary finite type.

Now we prove our first logical metatheorem, extending the scope of the logical metatheorems due to [16, Theorem 6.3] and [29, Theorem 3.7].

Theorem 5.13 (Logical Metatheorem for L^p, $C(K)$ and Banach lattices). *Let $\rho \in \boldsymbol{T}^X$ be an admissible finite type. Let $B_\forall(x, u)$, resp. $C_\exists(x, v)$, be $\forall$- resp. $\exists$-formulas that contain only the variables x, u resp. x, v free. Assume*

$$\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_{(a)} \vdash \forall x^\rho \left(\forall u^{\mathbb{N}} B_\forall(x, u) \to \exists v^{\mathbb{N}} C_\exists(x, v)\right) \tag{2}$$

then one can extract a partial functional $\Phi : S_{\widehat{\rho}} \rightharpoonup \mathbb{N}$ whose restriction to the strongly majorizable elements of $S_{\widehat{\rho}}$ is a (bar recursive) computable functional of $\mathcal{M}^\omega$ and the following holds in all nontrivial (atomless) $L^p(\Omega, U, \mu)$ spaces: for all $x \in S_\rho$, $x^* \in S_{\widehat{\rho}}$ if $x^* \gtrsim_\rho x$ then

$$\forall u \leq \Phi(x^*) B_\forall(x, u) \to \exists v \leq \Phi(x^*) C_\exists(x, v).$$

Φ depends on p only via an upper bound $\mathbb{N} \ni b \geq p$.
Moreover,

1. *if $\widehat{\rho}$ is type 1, then $\Phi : S_{\widehat{\rho}} \to \mathbb{N}$ is a total computable functional (in the ordinary sense of type-2 recursion theory) defined by bar recursion.*
2. *All variables may occur as finite tuples satisfying the same type restrictions.*
3. *If* (2) *holds for the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$, resp. $\mathcal{A}^\omega[X, \|\cdot\|, C(K)]$, instead of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_{(a)}$ the conclusion holds in all nontrivial Banach lattices $(X, \|\cdot\|, \sqcup)$, resp. all spaces $C(K)$ of continuous real-valued functions on an abstract compact space K.*
4. *If the statement in* (2) *can be proven without the axiom of dependent choice, one does not need bar recursion. Thus, we could then allow the type ρ to be an arbitrary finite type. Moreover, all restrictions to $\mathcal{M}^{\omega,X}$ can be omitted, and so everything follows in the full $\mathcal{S}^{\omega,X}$. Then the functional $\Phi : S_{\widehat{\rho}} \to \mathbb{N}$ is primitive recursive (in the sense of Gödel).*

Proof. We extend the proof of [16, Theorem 6.3]. We need the model $\mathcal{M}^{\omega,X}$ for bar recursion to be true (which does not hold in $\mathcal{S}^{\omega,X}$, see [31, p. 214]), which in turn is necessary to solve the functional interpretation of dependent choice (see [31, Chapter 11]). As stated in 4., without dependent choice we can omit all restrictions to the types and use the model $\mathcal{S}^{\omega,X}$ instead. Theorem 3.8 shows that

the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_{(a)}$ is the correct axiomatization for nontrivial (atomless) $L^p(\Omega, U, \mu)$ spaces, similarly in Theorem 4.6 we have shown that the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, C(K)]$ axiomatizes abstract spaces of continuous functions on a compact set K and for Banach lattices the same is proven in Proposition 2.13. Since all terms of the theories are majorizable (see Lemma 5.8), we can refer to the proof of [16, Theorem 6.3] with the exception of the sentences Δ, which are necessary for the atomless axiom. All new axioms of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]$ are universal and are, therefore unchanged by the functional interpretation, which is one of the key ingredients of the proofs of [16, Theorem 6.3] and [29, Theorem 3.7]. By [31, Theorem 10.21] (since this theorem applies negative translation - which only weakens Δ - and the monotone functional interpretation) all sentences Δ are upgraded to $\tilde{\Delta}$. This is not a major concern, see Lemma 5.11. The newly added Skolem functionals B for each sentence in $\tilde{\Delta}$ have to be added as new constants to the language to witness the existential quantifier. Of course, none of these new constants is expected to be provably extensional. However, since in the proof (2) those constants are not in the language, they cannot be used in the proof anyway. They are majorizable, since they are smaller than closed terms r which are in turn majorizable by primitive recursive terms, which follows from Lemma 5.8. This implies that the newly added Skolem constants are majorizable and so can be interpreted in $\mathcal{M}^{\omega,X}$. Since with the added constants all axioms $\tilde{\Delta}$ are universal sentences, they are unchanged by the functional interpretation.
The axiom not involving the existence of a unit constant (the 1_X is an arbitrary element of norm 1) from Remark 4.7 is in the class Δ, thus this axiomatization of $C(K)$ is also admissible. □

As a corollary to the proof of Theorem 5.13 we see that we may explicitly allow arbitrary axioms of the form Δ which can be added to the theory.

Corollary 5.14. *Assume the same setting as in Theorem 5.13. If*

$$\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_{(a)} + \Delta \vdash \forall x^\rho \left(\forall u^{\mathbb{N}} B_\forall(x, u) \to \exists v^{\mathbb{N}} C_\exists(x, v)\right)$$

then one can extract a partial functional $\Phi : S_{\widehat{\rho}} \rightharpoonup \mathbb{N}$ *whose restriction to the strongly majorizable elements of* $S_{\widehat{\rho}}$ *is a (bar recursive) computable functional of* $\mathcal{M}^\omega$ *and the following holds in all nontrivial (atomless)* $L^p(\Omega, U, \mu)$ *spaces* X *s.t.* $\mathcal{S}^{\omega,X} \models \Delta$: *for all* $x \in S_\rho$, $x^* \in S_{\widehat{\rho}}$ *if* $x^* \gtrsim_\rho x$ *then*

$$\forall u \leq \Phi(x^*) B_\forall(x, u) \to \exists v \leq \Phi(x^*) C_\exists(x, v).$$

The supplements (1)-(4) *remain valid in this setting. The theory* $\mathcal{A}^\omega[X, \|\cdot\|]$ *and all extensions defined in this work are also admitted.*

Proof. Follows directly from the proof of Theorem 5.13. □

As discussed already in the introduction, logical metatheorems of the type of Theorem 5.13 are applied to nonlinear analysis not in the abstract form stated but via specialized corollaries that refer to concrete formats such as the convergence of some iterative procedure (x_n) involving some nonlinear operator $T : X \to X$ as is common in fixed point and ergodic theory and continuous optimization (see the introduction).

6. Positive bounded logic

After having formalized directly some theories of abstract spaces studied in model theory we will next analyze positive bounded logic, which is a restriction of first-order logic, more systematically from the proof-theoretic point of view. In this framework there are only bounded quantifiers, no negations and all functions are uniformly continuous. After discussing the definitions due to [19] we show how we can mimic positive bounded logic in the formal theory $\mathcal{A}^\omega[X,\|\cdot\|]$ (axiomatizing normed spaces without completeness) and show that the logical metatheorem for normed spaces with additional axioms Δ covers the expressive power of positive bounded logic (or continuous logic, adapted by [6], having the same expressive power).

Remark 6.1. From now on, whenever we refer to the theory $\mathcal{A}^\omega[X,\|\cdot\|]$ it is permitted to use all extensions (with the truth in the respective models) defined in this work, instead.

6.1. Model-theoretic view

First we introduce the models of positive bounded logic, *i.e.* families of normed spaces together with some functions.

Definition 6.2 ([19, Definition 2.1])**.** A *normed space structure* $\mathfrak{M}$ consists of a set $\left(M^{(s)} \mid s \in S\right)$ of normed spaces $M^{(s)}$ (one of which is always $\mathbb{R}$), which are also called *sorts* with sort index set S and a set of uniformly continuous (on bounded domains) functions $F : M^{(s_1)}\times\cdots\times M^{(s_m)} \to M^{(s_0)}$.

For simplicity reasons we will focus on a single normed space which corresponds to the abstract type X (in addition to $\mathbb{R}$). As indicated in [16, Section 7] and executed in [18] one can have multiple abstract types X_i to treat several normed spaces simultaneously.

Definition 6.3 ([19, Definition 5.2])**.** Let L be a signature for normed space structures. We define *positive bounded (L-)formulas* via induction on the complexity.

1. The prime formulas are $r \leq t$ and $t \leq r$, where t is a real-valued term and $r \in \mathbb{Q}$.
2. If φ_1 and φ_2 are positive bounded formulas, x is a variable, and $r \in \mathbb{Q}$ with $r > 0$ then the following are positive bounded formulas: $(\varphi_1 \wedge \varphi_2)$ and $(\varphi_1 \vee \varphi_2)$, $\exists x\,(\|x\| \leq r \wedge \varphi_1)$, $\forall x\,(\|x\| \leq r \to \varphi_1)$.

Notation 6.4. We introduce the following abbreviations:

$$\begin{aligned} &\exists_r x\varphi :\equiv \exists x\,(\|x\| \leq r \wedge \varphi)\,, \qquad \forall_r x\varphi :\equiv \forall x\,(\|x\| \leq r \to \varphi)\,, \\ &t = r :\equiv t \leq r \wedge r \leq t. \end{aligned}$$

Definition 6.5 ([19, Section 5])**.** If φ is a positive bounded formula, we define the positive bounded formula φ' to be an *approximation* of φ, which is denoted by $\varphi \sqsubset \varphi'$, as follows:

- For $\varphi \equiv r \leq t$, approximations of φ are $r' \leq t$, where $r' < r$.
- For $\varphi \equiv t \leq r$, approximations of φ are $t \leq r'$, where $r < r'$.
- For $\varphi \equiv \psi_1 \Box \psi_2$, approximations of φ are $\psi_1' \Box \psi_2'$, where $\psi_i \sqsubset \psi_i'$, for $i = 1, 2$ and $\Box \in \{\wedge, \vee\}$.
- For $\varphi \equiv \exists_r x\psi$, approximations of φ are $\exists_{r'} x\psi'$, where $r < r'$ and $\psi \sqsubset \psi'$.
- For $\varphi \equiv \forall_r x\psi$, approximations of φ are $\forall_{r'} x\psi'$, where $r' < r$ and $\psi \sqsubset \psi'$.

Definition 6.6 ([19, Definition 5.9])**.** Let $\mathfrak{M}$ be a normed space structure and let $\varphi(x_1,\ldots,x_n)$ be a positive bounded formula. If $\varphi'[a_1,\ldots,a_n]$ is true in $\mathfrak{M}$ for every approximation φ' of φ we say that $\mathfrak{M}$ *approximately satisfies* $\varphi(x_1,\ldots,x_n)$ *at* $a_1,\ldots,a_n$ (where $a_i \in M^{(s_i)}$), which is denoted by $\mathfrak{M} \models_{\mathcal{A}} \varphi[a_1,\ldots,a_n]$.

Definition 6.7 ([19, Definition 13.5])**.** For two classes of normed spaces structures $\mathcal{C}, \mathcal{D}$ with $\mathcal{C} \subseteq \mathcal{D}$ we say that $\mathcal{C}$ is *axiomatizable in* $\mathcal{D}$ *by positive bounded sentences*, if there exists a set of such sentences Γ such that for all structures $\mathfrak{C} \in \mathcal{D}$ it holds that $\mathfrak{C} \in \mathcal{C}$ iff $\mathfrak{C} \models_{\mathcal{A}} \Gamma$.

Remark 6.8. All approximate models of the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p]_a$ are also models since for universal formulas this is equivalent [19, Section 5]. The atomless axiom (A12) is - via its formulation from Proposition 3.7 - equivalent to its own approximate version (in the sense of Lemma 6.13).

Lemma 6.9 ([19, Section 13])**.** *Let $\varphi(x_1, \ldots, x_n)$ be a positive bounded formula. There exists an equivalent prenex normal formula $\psi(x_1, \ldots, x_n)$ of the form*

$$Q_1 y_1 Q_2 y_2 \ldots Q_k y_k \theta(x_1, \ldots, x_n, y_1, \ldots, y_k),$$

where $Q_i \in \{\exists_{r_i}, \forall_{r_i}\}$ for $i \in \{1, \ldots, k\}$, and $\theta(x_1, \ldots, x_n, y_1, \ldots, y_k)$ is quantifier-free in the sense of positive bounded logic.

6.2. Positive bounded logic in proof theory

Since in positive bounded logic real numbers and abstract normed spaces are treated identically (from a syntactic point of view) we want to stay close to this approach. Therefore, we introduce the type $\tilde{X}$ which stands for the two types X and 1, where the latter is used to represent real numbers. If x is of type 1 but is interpreted as a representative of a real number, equality has to be understood in the following sense: $x =_{\mathbb{R}} y$ (instead of $x =_1 y$), as well as $\|x\| = |x|_{\mathbb{R}}$ and $x \preceq_{\tilde{X}} y := |x| \leq_{\mathbb{R}} |y|$ (instead of $x \preceq_1 y$). For all further details we refer to [31]. This double role of the type 1, namely as the type of number-theoretic functions, which it officially represents, and its use to represent real numbers causes certain technical complications (see Lemma 6.16) as we will have to translate bounds in the sense of $\leq_{\mathbb{R}}$ into bounds in the sense of $\preceq_1$ which is needed in majorizability arguments.

Notation 6.10. Since rational numbers are encoded by natural numbers using the Cantor pairing function (see [28] for the details) we introduce the following abbreviation: $\forall a \in \mathbb{Q}^*_+ \varphi(a) := \forall a^{\mathbb{N}} \Big(|a|_{\mathbb{Q}} >_{\mathbb{Q}} 0 \to \varphi(|a|_{\mathbb{Q}}) \Big)$.

Definition 6.11. We define a class of formulas in the language of $\mathcal{A}^\omega[X, \|\cdot\|]$ denoted by $\mathcal{PBL}$:

$$\Theta_m(T, \underline{r}, \underline{s}) :\equiv \forall l^{\mathbb{N}} \forall_{r_1} x_1^{\tilde{X}} \exists_{s_1} y_1^{\tilde{X}} \ldots \forall_{r_m} x_m^{\tilde{X}} \exists_{s_m} y_m^{\tilde{X}} \left(T(\underline{x}, \underline{y}, l) =_{\mathbb{R}} 0 \right),$$

where $r_i(l), s_i(l)$ are terms containing only l free denoting functions $\mathbb{N} \to \mathbb{Q}^*_+$ (see Notation 6.10) and T is a function of type $1(\mathbb{N})(\tilde{X}) \ldots (\tilde{X})$. In the following, we will for better readability surpress the dependence of $r_i(l), s_i(l)$ on l and simply write r_i, s_i.

Whenever dealing with a formula of the class $\mathcal{PBL}$ we assume to have a modulus of uniform continuity $\omega_T : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t.

$$\begin{aligned} U_m(T, \omega_T) := {} & \forall n^{\mathbb{N}}, b^{\mathbb{N}}, l^{\mathbb{N}} \forall_b x_1^{\tilde{X}}, \bar{x}_1^{\tilde{X}}, y_1^{\tilde{X}}, \bar{y}_1^{\tilde{X}} \ldots, y_m^{\tilde{X}}, \bar{y}_m^{\tilde{X}} \\ & \Big(\bigwedge_{i=1}^{m} \|x_i - \bar{x}_i\|, \|y_i - \bar{y}_i\| <_{\mathbb{R}} 2^{-\omega_T(b,n,l)} \to \left| T(\underline{x}, \underline{y}, l) - T(\underline{\bar{x}}, \underline{\bar{y}}, l) \right| \leq_{\mathbb{R}} 2^{-n} \Big). \end{aligned} \tag{3}$$

This corresponds to the uniform continuity assumption made in positive bounded logic by which all L-terms denote uniformly continuous (on bounded subsets) functions (see [19], Definition 2.1 and p.27).

Lemma 6.12. *To each formula in prenex normal form with bounded quantifiers which is built up from formulas $r \leq_{\mathbb{R}} t$ and $t \leq_{\mathbb{R}} r$, viewed as prime formulas, by $\wedge, \vee$, we can construct a formula φ_0 in the class $\mathcal{PBL}$ such that $\mathcal{A}^\omega[X, \|\cdot\|] \vdash \varphi \leftrightarrow \varphi_0$. We construct φ_0 by induction on the complexity of φ (using implicitly the embedding of $\mathbb{Q}$ into $\mathbb{R}$ on the level of the representations):*

1. $r \leq_{\mathbb{R}} t$ *is replaced by* $\min\{r,t\} - r =_{\mathbb{R}} 0$,
2. $t \leq_{\mathbb{R}} r$ *is replaced by* $\min\{r,t\} - t =_{\mathbb{R}} 0$,
3. $\phi =_{\mathbb{R}} 0 \vee \psi =_{\mathbb{R}} 0$ *is replaced by* $\min\{|\phi|, |\psi|\} =_{\mathbb{R}} 0$,
4. $\phi =_{\mathbb{R}} 0 \wedge \psi =_{\mathbb{R}} 0$ *is replaced by* $\max\{|\phi|, |\psi|\} =_{\mathbb{R}} 0$.

Proof. The equivalence can be easily proven in $\mathcal{A}^\omega[X, \|\cdot\|]$. $\square$

The above lemma draws the connection between positive bounded logic and the class $\mathcal{PBL}$ which covers positive bounded logic. We do not want to go into more details, since one would need to define an explicit interpretation, add multiple base types X_i (which is possible, see [18]) and so forth.

Lemma 6.13 (Approximations as one formula)**.** *Let*

$$\Theta_m(T, \underline{r}, \underline{s}) :\equiv \forall l^{\mathbb{N}} \forall_{r_1} x_1^{\tilde{X}} \exists_{s_1} y_1^{\tilde{X}} \ldots \forall_{r_m} x_m^{\tilde{X}} \exists_{s_m} y_m^{\tilde{X}} \left(T(\underline{x}, \underline{y}, l) =_{\mathbb{R}} 0\right)$$

be a formula in the class $\mathcal{PBL}$. Then the following formula expresses the approximate truth of Θ_m:

$$\begin{aligned}\Theta_{m,\varepsilon}(T, \underline{r}, \underline{s}) :\equiv \forall k >_{\mathbb{N}} \max\{\lceil -\log_2(r_i)\rceil \mid i \in \{1, \ldots, n\}\}& \\ \forall l^{\mathbb{N}} \forall_{r_1 - 2^{-k}} x_1^{\tilde{X}} \exists_{s_1 + 2^{-k}} y_1^{\tilde{X}} \ldots \forall_{r_m - 2^{-k}} x_m^{\tilde{X}} \exists_{s_m + 2^{-k}} y_m^{\tilde{X}} &\left(\left|T(\underline{x}, \underline{y}, l)\right| \leq_{\mathbb{R}} 2^{-k}\right).\end{aligned}$$

Proof. As implicitly defined in Definition 6.5 an error parameter 2^{-k} is added to all prime formulas of positive bounded logic as follows: the formula $r - 2^{-k} \leq t$ is an approximation of $r \leq t$ for all $k \in \mathbb{N}$. If this error term is added to all prime formulas before applying Lemma 6.12 we obtain that the inner formula $T(\underline{x}, \underline{y}, l) =_{\mathbb{R}} 0$ is approximated by $\left|T(\underline{x}, \underline{y}, l)\right| \leq_{\mathbb{R}} 2^{-k}$. Then the range of the quantifiers is modified by the error parameter according to Definition 6.5. The final step is the universal closure since we want to express all approximations of Θ_m in one formula. $\square$

Next we introduce an abbreviation stating that a formula θ is extensional with respect to specified free variables:

Notation 6.14.

$$\mathrm{Ext}(\theta(x_1, \ldots, x_m)) :\equiv \forall x_1^{\tilde{X}}, \bar{x}_1^{\tilde{X}}, \ldots, x_m^{\tilde{X}}, \bar{x}_m^{\tilde{X}} \Big(\bigwedge_{i=1}^{m} x_i =_{\tilde{X}} \bar{x}_i \to (\theta(x_1, \ldots, x_m) \leftrightarrow \theta(\bar{x}_1, \ldots, \bar{x}_m)) \Big).$$

Definition 6.15.

$$\begin{aligned} \min\nolimits_1(x^1, y^1) &:= \lambda v^{\mathbb{N}}.\min\nolimits_{\mathbb{N}}(xv, yv) \\ \mathrm{retr}_{\tilde{X}}(x^{\tilde{X}}, y^{\tilde{X}}, n^{\mathbb{N}}) &:= \frac{\|y\|\, \check{x}}{\max\left\{\|\check{x}\|, \|y\|, 2^{-n}\right\}}, \\ \text{where } \check{x}^{\tilde{X}} &:= \begin{cases} x^X, & \text{if } \tilde{X} = X, \\ \min\nolimits_1(x, M(y(0)+1)), & \text{if } \tilde{X} = 1. \end{cases} \end{aligned}$$

The following lemma motivates the somewhat involved definition of $\mathrm{retr}_{\tilde{X}}(x, y, n)$.

Lemma 6.16.

$$\begin{aligned} \text{(i) } &\mathcal{A}^\omega[X, \|\cdot\|] \vdash \mathrm{Ext}(A(x)) \to (\forall n^{\mathbb{N}} \forall y^1 (|y| >_{\mathbb{R}} 2^{-n} \to (\exists x \preceq_{\tilde{X}} y A(x, y) \\ &\quad \leftrightarrow \exists x \preceq_1 M(y(0)+1) A(\mathrm{retr}_{\tilde{X}}(x, y, n), y) \leftrightarrow \exists x^1 A(\mathrm{retr}_{\tilde{X}}(x, y, n), y)))), \\ \text{(ii) } &\mathcal{A}^\omega[X, \|\cdot\|] \vdash \mathrm{Ext}(A(x)) \to (\forall n^{\mathbb{N}} \forall y^X (\|y\| >_{\mathbb{R}} 2^{-n} \to (\exists x \preceq_{\tilde{X}} y A(x, y) \\ &\quad \leftrightarrow \exists x^X A(\mathrm{retr}_{\tilde{X}}(x, y, n), y)))). \end{aligned}$$

Proof. (i): Since A is extensional in x w.r.t. $=_{\mathbb{R}}$ we are allowed to choose a small representation for x via $\tilde{(\cdot)}$ with $x =_{\mathbb{R}} \tilde{x}$ (see [31, p. 93]). Since $|y| \leq_{\mathbb{R}} (y(0)+1)_{\mathbb{R}}$ the type 1 bound follows from the Definition of $\tilde{(\cdot)}$ where $x \in [-m, m]$ with $m := y(0) + 1$. Moreover, $|y| >_{\mathbb{R}} 2^{-n}$ and $|\tilde{x}| \leq_{\mathbb{R}} |y|$ imply $\mathrm{retr}_{\tilde{X}}(\tilde{x}, y, n) =_{\mathbb{R}} \tilde{x}$. Reversely, from Definition 6.15 we know $|y| > 2^{-n} \wedge x \preceq_1 M(y(0)+1) \to |\mathrm{retr}_{\tilde{X}}(x, y, n)| \leq_{\mathbb{R}} |y|$. The second equivalence follows from Definition 6.15, here using that $\check{x}^1$ is applied to the first argument of $\mathrm{retr}_{\tilde{X}}$ and the fact that provably $\check{\check{x}} =_1 \check{x}$ together with QF-ER.
(ii): Definition 6.15 implies $\|y\| >_{\mathbb{R}} 2^{-n} \to \|\mathrm{retr}_X(x, y, n)\| \leq_{\mathbb{R}} \|y\|$ and $x \preceq_X y \wedge \|y\| >_{\mathbb{R}} 2^{-n} \to \mathrm{retr}_X(x, y, n) =_X x$. Thus, the equivalence follows from $\mathrm{Ext}(A(x))$. ☐

Proposition 6.17. *Let Θ be a formula in the class $\mathcal{PBL}$. Then there exists a formula Θ^*, which in the case where $\underline{r}, \underline{s}, T$ are given by closed terms is a sentence in Δ, such that*

$$\mathcal{A}^{\omega}[X, \|\cdot\|] \vdash \forall T^{1(\mathbb{N})(\tilde{X})\dots(\tilde{X})} \forall \omega_T^{\mathbb{N}(\mathbb{N})(\mathbb{N})(\mathbb{N})} \forall \underline{r}, \underline{s} \in (\mathbb{Q}_+^*)^{\mathbb{N}} \left(U_m(T, \omega_T) \to (\Theta^* \to \Theta)\right),$$

where $U_m(T, \omega_T)$ expresses the uniform continuity of T (see (3)*). In the presence of* b-AC$_X$ *also the converse implication $\Theta \to \Theta^*$ follows.*

Proof. Let Θ be a formula in $\mathcal{PBL}$:

$$\Theta \equiv \forall l^{\mathbb{N}} \forall_{r_1} x_1^{\tilde{X}} \exists_{s_1} y_1^{\tilde{X}} \dots \forall_{r_m} x_m^{\tilde{X}} \exists_{s_m} y_m^{\tilde{X}} (T(\underline{x}, \underline{y}, l) =_{\mathbb{R}} 0),$$

and assume that $U_m(T, \omega_T)$. First, we remove the universal premise due to the bounded universal quantifiers which is possible since (3) implies the extensionality of T.

$$\mathrm{Ext}(T(\underline{x}, \underline{y}) =_{\mathbb{R}} 0) \vdash \Big(\Theta \leftrightarrow \Theta_0 :\equiv \forall l^{\mathbb{N}} \forall x_1^{\tilde{X}} \exists_{s_1} y_1^{\tilde{X}} \dots \forall x_m^{\tilde{X}} \exists_{s_m} y_m^{\tilde{X}} (T(\underline{\tilde{x}}, \underline{y}, l) =_{\mathbb{R}} 0)\Big),$$
$$\tilde{x}_i := \mathrm{retr}_{\tilde{X}}(x_i, r_i, \lceil -\log_2 r_i \rceil + 1).$$

Then, using AC, we obtain the Skolem normal form (cp. [31, p. 142])

$$\Theta_0 \leftrightarrow \Theta^* :\equiv \exists_{\underline{s}} \underline{Y} \forall l^{\mathbb{N}} \forall \underline{x} (T(\underline{\tilde{x}}, \underline{Yxl}, l) =_{\mathbb{R}} 0),$$

where Θ^* is spelled out as follows.

$$\Theta^* \equiv \exists \underline{Y} \left(\bigwedge_{i=1}^{m} Y_i \preceq_{\tilde{X}(\tilde{X})(\mathbb{N})} \lambda y, l.s_i \cdot 1_{\tilde{X}} \wedge \forall l^{\mathbb{N}} \forall \underline{x} (T(\underline{\tilde{x}}, \underline{Y\tilde{x}l}, l) =_{\mathbb{R}} 0) \right).$$

Since $T(\underline{\tilde{x}}, \underline{Y\tilde{x}l}, l) =_{\mathbb{R}} 0$ is a universal formula and we can bound Y_i w.r.t. $\preceq_{1(1)(\mathbb{N})}$ and $\preceq_{X(X)(\mathbb{N})}$, which follows from the extensionality of T together with Lemma 6.16, we conclude that Θ^* can be written as a sentence Δ (and so the above use of AC can be re-casted as a use of b-AC$_X$) for closed terms $\underline{r}, \underline{s}, T$. ☐

Theorem 6.18 (Logical Metatheorem for the class $\mathcal{PBL}$)**.** *Let $\rho \in \boldsymbol{T}^X$ be an admissible finite type and Θ be a set of sentences of the class $\mathcal{PBL}$ such that for each $\varphi_T \in \Theta$ we have provably $U_m(T, \omega_T)$ (see* (3) *on p. 15) for some closed terms ω_T, T defined in the language of $\mathcal{A}^{\omega}[X, \|\cdot\|]$. Let $B_{\forall}(x, u)$, resp. $C_{\exists}(x, v)$, be $\forall$- resp. $\exists$-formulas that contain only the variables x, u resp. x, v free. Assume*

$$\mathcal{A}^{\omega}[X, \|\cdot\|] + \Theta \vdash \forall x^{\rho} \left(\forall u^{\mathbb{N}} B_{\forall}(x, u) \to \exists v^{\mathbb{N}} C_{\exists}(x, v)\right) \tag{4}$$

then one can extract a partial functional $\Phi : S_{\widehat{\rho}} \rightharpoonup \mathbb{N}$ whose restriction to the strongly majorizable elements of $S_{\widehat{\rho}}$ is a (bar recursive) computable functional of $\mathcal{M}^{\omega}$ and the following holds in all nontrivial normed spaces X s.t. $\mathcal{S}^{\omega,X} \models \Theta$: for all $x \in S_{\rho}$, $x^* \in S_{\widehat{\rho}}$ if $x^* \gtrsim_{\rho} x$ then

$$\forall u \leq \Phi(x^*) B_{\forall}(x, u) \to \exists v \leq \Phi(x^*) C_{\exists}(x, v).$$

The list (1)-(4) *of Theorem 5.13 holds analogously.*

Proof. We have to add the following arguments to the proof of Theorem 5.13: In Proposition 6.17 we have shown that all axioms in the class $\mathcal{PBL}$ (which covers all closed instances of positive bounded formulas in our theory) can be expressed by axioms $\Theta^* \in \Delta$, which are covered by Corollary 5.14. Thus, the only step is to use Θ^* instead of the original set Θ (since Θ^* provably implies Θ). Since with b-AC$_X$ conversely Θ also implies Θ^*, the validity of Θ in $\mathcal{S}^{\omega,X}$ implies that of Θ^* (and by Lemma 5.11 we also have the validity of Θ^* in $\mathcal{M}^{\omega,X}$). $\square$

Remark 6.19. The above metatheorem can be generalized to the setting where we do not require $\mathcal{A}^\omega[X, \|\cdot\|] + \Theta \vdash \bigwedge\!\!\!\bigwedge_{\varphi_T \in \Theta} U_m(T, \omega_T)$ and only assume $U_m(T, \omega_T)$ implicatively. Note that Θ is w.l.o.g. finite since the proof of (4) can only involve finitely many axioms. Then the functional φ would additionally dependent on ω_T (which allows one to construct a majorant of T).
Since the class Δ covers both regular and approximate positive bounded formulas (see Lemma 6.13 and the proof of Proposition 6.17) as the latter are again in the class $\mathcal{PBL}$, it is up to the user which variant to take as an axiom. However, as we will show in Theorem 6.36, one can assume the full non-approximative axiom for the proof of (4), which is equivalent using uniform boundedness to the approximate version, but still conclude that the extracted uniform bound will be valid in all structures satisfying the approximate version only.

Remark 6.20. Above we only considered $\mathcal{PBL}$-axioms in the signature of Banach lattices (possibly with a unit). However, the approach also applies to other signatures in the framework of positive bounded logic as the uniform continuity requirement on bounded sets for the constants made in positive bounded logic can always be stated as a universal axiom (once a modulus of continuity is added to the language) from which one then construct a majorant (see the proof of Corollary 17.71.4 in [31]).

6.3. Uniform boundedness principle

In this section we study a uniform boundedness principle that will be shown to serve as a proof-theoretic substitute to many uses of ultrapowers in the model theory of normed spaces. The starting point is the axiom scheme $\exists\text{-UB}^X$ from [30, Definition 3.1]:

$$\exists\text{-UB}^X :\equiv \begin{cases} \forall y^{\alpha(\mathbb{N})} (\forall k^{\mathbb{N}}, x^\alpha, \underline{z}^{\underline{\beta}} \exists n^{\mathbb{N}} A_\exists(y, k, \min_\alpha(x, yk), \underline{z}, n) \to \\ \exists \chi^1 \forall k^{\mathbb{N}}, x^\alpha, \underline{z}^{\underline{\beta}} \exists n \preceq_{\mathbb{N}} \chi k A_\exists(y, k, \min_\alpha(x, yk), \underline{z}, n)), \end{cases}$$

where $\alpha = \mathbb{N}(\sigma_k) \dots (\sigma_1)$, $\beta = X(\tau_m) \dots (\tau_1)$ (with τ_i,σ_i arbitrary finite types) and $A_\exists$ is an $\exists$-formula (see Definition 5.12). It is important to remark that $\exists\text{-UB}^X$ only makes sense when considering bounded metric spaces. Since in a bounded metric space all elements of type X are trivially majorized, the types in β can be very complex which is not possible in the case of normed spaces.
The axiom (and also our variations of it) is in general invalid, since one of its immediate consequences is the uniform continuity of all functions $f : B_1(0) \to X$ in the context of normed spaces. For simplicity reasons we will restrict ourselves to the case of points $y^{\tilde{X}}$ (and finite tuples $\underline{y}^{\underline{\tilde{X}}}$) instead of sequences $y^{\tilde{X}(\mathbb{N})}$. The sequential version of uniform boundedness has the advantage of proving e.g. from strict convexity the existence of a modulus of uniform convexity, whereas one would need choice in the pointwise version (see Proposition 6.41) to obtain a modulus. Since we work in a strong theory with DC the pointwise version suffices. We will need rather technical (intensional) uniform boundedness principles, having a subscript-minus at their names, and application oriented variants for extensional formulas which are applied in Section 6.5.

$$\Sigma^0_1\text{-UB}^X_-(A_\exists) : \begin{cases} \forall y^{\tilde{X}} \forall n^{\mathbb{N}} (\|y\| >_{\mathbb{R}} 2^{-n} \wedge \forall x^{\tilde{X}} \exists z^{\mathbb{N}} A_\exists(\mathrm{retr}_{\tilde{X}}(x, y, n), y, z) \\ \quad \to \exists z^* \forall x^{\tilde{X}} \exists z \preceq_{\mathbb{N}} z^* A_\exists(\mathrm{retr}_{\tilde{X}}(x, y, n), y, z), \end{cases}$$

where $A_\exists$ is an $\exists$-formula and $\mathrm{retr}_{\tilde{X}}(x,y,n)$ as defined in Definition 6.15.
When we refer to the above principle by Σ^0_1-UB^X_- we allow all instances of $A_\exists$; if we want to refer to the inner formula $A_\exists$, we use the notation above (compare Lemma 6.23 with Lemma 6.25).
Now consider the following axiom of type Δ (cp. [26] for type 1 and [31, Definition 17.99] for arbitrary types in the context of bounded metric structures)

$$\mathrm{F}^X_- :\equiv \begin{cases} \forall \Phi^{\mathbb{N}(X)} \forall y^X \forall n^{\mathbb{N}} (\|y\| >_{\mathbb{R}} 2^{-n} \to \exists y' \preceq_X y \forall x^X (\Phi(\mathrm{retr}_X(x,y,n)) \preceq_{\mathbb{N}} \Phi(y'))) \wedge \\ \forall \Phi^{\mathbb{N}(1)} \forall v^1 \exists v' \preceq_1 v \forall u^1 (\Phi(\min_1(u,v)) \preceq_{\mathbb{N}} \Phi(v')). \end{cases}$$

Proposition 6.21. $\mathcal{M}^{\omega,X} \models \mathrm{F}^X_-$.

Proof. The proof is similar to the proof of [31, Theorem 17.101]. □

Remark 6.22. Whenever the axiom F^X_- is in the theory we must use $\mathcal{M}^{\omega,X}$ as a model even if dependent choice is not used, simply because in $\mathcal{S}^{\omega,X}$ the axiom F^X_- is wrong (see Section 6.5).

Lemma 6.23. $\mathcal{A}^\omega[X, \|\cdot\|] + \mathrm{F}^X_- \vdash \Sigma^0_1$-$\mathrm{UB}^X_-$.

Proof. Suppose $\|y\| >_{\mathbb{R}} 2^{-n} \wedge \forall x^{\tilde{X}} \exists z^{\mathbb{N}} A_\exists(\mathrm{retr}_{\tilde{X}}(x,y,n),y,z)$. By applying AC-$\exists :\equiv \forall \underline{x} \exists \underline{y} A_\exists(\underline{x},\underline{y}) \to \exists \underline{f} \forall \underline{x} A_\exists(\underline{x}, \underline{f}\,\underline{x})$, (which is equivalent to QF-AC) to the second conjunct we get

$$\exists \Phi^{\mathbb{N}(\tilde{X})} \forall x^{\tilde{X}} A_\exists(\mathrm{retr}_{\tilde{X}}(x,y,n), y, \Phi(x)).$$

Now we distinguish the cases X and 1 and start with the former: Since provably

$$\|y\| > 2^{-n} \to \mathrm{retr}_X(\mathrm{retr}_X(x,y,n),y,n) =_X \mathrm{retr}_X(x,y,n) \tag{5}$$

by Definition 6.15 and (5) prenexing to a universal formula, we obtain with QF-ER

$$\|y\| > 2^{-n} \to \forall x^X A_\exists(\mathrm{retr}_X(x,y,n), y, \Phi(\mathrm{retr}_X(x,y,n))).$$

Using F^X_- we know that $\exists N^{\mathbb{N}} \forall x^X (\Phi(\mathrm{retr}_X(x,y,n)) \preceq_{\mathbb{N}} N)$.
For type 1 we use that provably $\check{\check{x}} =_1 \check{x}$ (with $\check{x}^1$ as in Definition 6.15) implying with QF-ER

$$|y| > 2^{-n} \to \forall x^1 A_\exists(\mathrm{retr}_{\tilde{X}}(x,y,n), \Phi(\check{x})).$$

Then we apply F^X_- (to Φ and $v := M(y(0)+1)$ from the definition of $\check{x}$) yielding $\exists N^{\mathbb{N}} \forall x^1 (\Phi(\check{x}) \preceq_{\mathbb{N}} N)$ and hence both cases together imply

$$\|y\| > 2^{-n} \to \exists z^* \forall x^{\tilde{X}} \exists z \preceq_{\mathbb{N}} z^* A_\exists(\mathrm{retr}_{\tilde{X}}(x,y,n), y, z).$$

□

From now on, whenever we want to use Σ^0_1-UB^X_- it is sufficient to have the theory $\mathcal{A}^\omega[X, \|\cdot\|] + \mathrm{F}^X_-$. This theory is suitable for a logical metatheorem (see Theorem 6.36) whereas the uniform boundedness principle does not have the right logical format. As we will see later, in most applications we have a (provably) extensional formula $A_\exists$ which will allow us to use the following uniform boundedness principle without having to deal with the $\mathrm{retr}_{\tilde{X}}$-operation:

Definition 6.24. We define the form of the uniform boundedness principle used in the applications (and which follows from Σ^0_1-UB^X_- for extensional formulas) with variables of type $\tilde{X}$ as follows.

$$\Sigma^0_1\text{-}\mathrm{UB}^X : \forall y^{\tilde{X}} (\forall x \preceq_{\tilde{X}} y \exists z^{\mathbb{N}} A_\exists(x,y,z) \to \exists z^* \forall x \preceq_{\tilde{X}} y \exists z \preceq_{\mathbb{N}} z^* A_\exists(x,y,z)),$$

where $A_\exists$ is an $\exists$-formula according to Definition 5.12. Again we may have tuples $\underline{x}, \underline{y}$ having the types X or 1.

Lemma 6.25. $\mathcal{A}^\omega[X, \|\cdot\|] + \Sigma^0_1\text{-UB}^X_-(A_\exists) \vdash \text{Ext}(A_\exists(x)) \to \Sigma^0_1\text{-UB}^X(A_\exists)$

Proof. Suppose that $A_\exists$ is extensional x w.r.t. $=_X$ resp. $=_\mathbb{R}$. If $y =_X 0$ then $x =_X 0$ and thus the premise and the conclusion are identical: $\exists z^\mathbb{N} A_\exists(0, 0, z)$. Similarly for $y =_\mathbb{R} 0$. If $\|y\| > 0$ there exists $n \in \mathbb{N}$ such that $\|y\| > 2^{-n}$. Then from $\forall x \preceq_{\tilde{X}} y \exists z^\mathbb{N} A_\exists(x, y, z)$ we get by applying Lemma 6.16 to the negated formula, that equivalently $\forall x^{\tilde{X}} \exists z^\mathbb{N} A_\exists(\text{retr}_{\tilde{X}}(x, y, n), y, z)$ holds. Now we apply $\Sigma^0_1\text{-UB}^X_-(A_\exists)$ resulting in

$$\exists z^* \forall x^{\tilde{X}} \exists z \preceq_\mathbb{N} z^* A_\exists(\text{retr}_{\tilde{X}}(x, y, n), y, z).$$

Again by $\text{Ext}(A_\exists(x))$ and Lemma 6.16 we have $\exists z^* \forall x \preceq_{\tilde{X}} y \exists z \preceq_\mathbb{N} z^* A_\exists(x, y, z)$. ☐

We now show how $\Sigma^0_1\text{-UB}^X_-$ (and with extensionality also $\Sigma^0_1\text{-UB}^X$) can be generalized to the situation where $A_\exists$ is not only an existential formula but of the format $\exists k^\mathbb{N} \forall a_1 \preceq_{\tilde{X}} y_1 \exists a_2 \preceq_{\tilde{X}} y_2 \ldots \theta_\text{qf}$.

To prove this generalized principle from $\Sigma^0_1\text{-UB}^X_-$ (and thus by F^X_-) we add two choice ("epsilon") operators ϕ for both types in $\tilde{X}$ to the language having roughly the following semantics: For the variables $y^{\tilde{X}}, z^{\mathbb{N}(\tilde{X})}$ (and $n^\mathbb{N}$) its output is an element $\phi(z, y) := x \preceq_{\tilde{X}} y$ such that $z(x) =_\mathbb{N} 0$. If such an element does not exist we set $\phi(z, y) := 0_X$ (or $\phi(z, y) := 0_\mathbb{R}$ respectively). To eliminate the hidden universal quantifier in $x \preceq_{\tilde{X}} y$ we use a technically involved axiom and also more involved semantics for which we refer to the proof of Proposition 6.27.

Definition 6.26. We define an extension of the theory $\mathcal{A}^\omega[X, \|\cdot\|]$ denoted by $\mathcal{A}^\omega[X, \|\cdot\|, \phi]$ by adding constants ϕ of type $X(\mathbb{N})(X)(\mathbb{N}X)$ and of type $1(\mathbb{N})(1)(\mathbb{N}1)$ and the following purely universal axioms

$$(\phi)\ \forall x^{\tilde{X}}, y^{\tilde{X}} \forall n^\mathbb{N} \forall z^{\mathbb{N}(\tilde{X})}$$
$$(\|y\| >_\mathbb{R} 2^{-n} \to (z(\text{retr}_{\tilde{X}}(x, y, n)) =_\mathbb{N} 0 \to z(\text{retr}_{\tilde{X}}(\phi(z, y, n), y, n)) =_\mathbb{N} 0)).$$

Proposition 6.27 (cp. [30, Definition 3.21])**.** *Let $(X, \|\cdot\|)$ be a nontrivial normed space. Then $\mathcal{S}^{\omega,X}$ becomes a model of $\mathcal{A}^\omega[X, \|\cdot\|, \phi]$ by letting the variables of type ρ range over S_ρ if all constants of $\mathcal{A}^\omega[X, \|\cdot\|]$ are interpreted as in Proposition 2.15 and ϕ is interpreted by any function with the semantics specified below. The same holds for all extensions of $\mathcal{A}^\omega[X, \|\cdot\|]$ and their respective models.*

Proof. The existence follows from the semantics of ϕ, which we define as follows (using AC on the metalevel):

$$\phi(z^{\mathbb{N}(X)}, y^X, n^\mathbb{N}) :=_X \begin{cases} \text{retr}_{\tilde{X}}(x, y, n) & \text{for } x^X \text{ with } z(\text{retr}_{\tilde{X}}(x, y, n)) = 0, \\ & \text{if } x^X \text{ exists,} \\ 0_X & \text{otherwise.} \end{cases}$$

$$\phi(z^{\mathbb{N}(1)}, y^1, n^\mathbb{N}) :=_1 \begin{cases} \min_1(x, M(y(0) + 1)) & \text{for } x^1 \text{ with } z(\text{retr}_{\tilde{X}}(x, y, n)) = 0, \\ & \text{if } x^1 \text{ exists,} \\ 0_\mathbb{R} & \text{otherwise.} \end{cases}$$

Since $z(\text{retr}_X(x, y, n)) =_\mathbb{N} z(\text{retr}_X(\text{retr}_X(x, y, n), y, n))$ and $z(\text{retr}_{\tilde{X}1}(x, y, n)) =_\mathbb{N} z(\text{retr}_{\tilde{X}1}(\check{x}, y, n))$ the axioms (ϕ) are fulfilled. We have to show that ϕ is majorizable in the proof of Theorem 6.36, which is the reason why the semantics involves the $\text{retr}_{\tilde{X}}(x, y, n)$ and $\min_1$ operations. ☐

We define a more general uniform boundedness principle

$$\Sigma^0_1\text{-UB}^X_{b,-} : \begin{cases} \forall \underline{y}^{\tilde X}\forall n^{\mathbb N}\Big(\bigwedge_{i=0}^{m}\|y_i\| >_{\mathbb R} 2^{-n} \wedge \forall x^{\tilde X}\exists z^{\mathbb N} A_{b,-}(\mathrm{retr}_{\tilde X}(x,y_0,n),\underline{y},z) \\ \quad \to \exists z^*\forall x^{\tilde X}\exists z \preceq_{\mathbb N} z^* A_{b,-}(\mathrm{retr}_{\tilde X}(x,y_0,n),\underline{y},z)\Big), \end{cases}$$

$$\text{where } A_{b,-} \equiv \exists k^{\mathbb N}\forall x_1^{\tilde X}\exists x_2^{\tilde X},\ldots\forall x_{m-1}^{\tilde X}\exists x_m^{\tilde X}$$
$$\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\ldots,\mathrm{retr}_{\tilde X}(x_m,y_m,n),k,\mathrm{retr}_{\tilde X}(x,y_0,n),\underline{y},z,\underline{a}),$$

where θ_{qf} is quantifier-free with arbitrary free parameters $\underline{a}$. First we show how the augmented theory $\mathcal{A}^\omega[X,\|\cdot\|,\phi]$ proves the more general uniform boundedness principle by $\Sigma^0_1\text{-UB}^X_-$ and thus by F^X_-.

Lemma 6.28. $\mathcal{A}^\omega[X,\|\cdot\|,\phi]+\Sigma^0_1\text{-UB}^X_- \vdash \Sigma^0_1\text{-UB}^X_{b,-}$.

Proof. Let $\theta_{\mathrm{qf}}(x^{\tilde X},y^{\tilde X},k^{\mathbb N},\underline{a})$ be a quantifier-free formula containing only the free variables indicated. Then there exists a closed term t_θ which provably satisfies

$$t_\theta(x,y,k,\underline{a}) =_{\mathbb N} 0 \leftrightarrow \theta_{\mathrm{qf}}(x,y,k,\underline{a}).$$

Now we apply ϕ to $z := \lambda x^{\tilde X}.t_\theta(x,y,k,\underline{a})$, y and n, implying (omitting all further arguments of t_θ for improved readability) under the assumption $\|y\| >_{\mathbb R} 2^{-n}$:

$$\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x,y,n),y,k,\underline{a}) \to \theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(\phi(\lambda x.t_\theta(x),y,n),y,n),y,k,\underline{a}). \tag{6}$$

By (6) we have that $\exists x^{\tilde X}\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x,y,n),y,k,\underline{a}) \leftrightarrow \theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(\phi(\lambda x.t_\theta(x),y,n),y,n),y,k,\underline{a})$. Analogously, this can be applied to $\exists x^{\tilde X}\neg\theta_{\mathrm{qf}}$ with the following outcome:

$$\forall x^{\tilde X}\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x,y,n),y,k,\underline{a}) \leftrightarrow \theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(\phi(\lambda x.t_{\neg\theta}(x),y,n),y,n),y,k,\underline{a}).$$

Iterating the procedure we obtain that $\Sigma^0_1\text{-UB}^X_-$ implies the more general case $\Sigma^0_1\text{-UB}^X_{b,-}$ where $A_{b,-}$ can be of the form

$$\exists k^{\mathbb N}\forall x_1^{\tilde X}\exists x_2^{\tilde X},\ldots\forall x_{m-1}^{\tilde X}\exists x_m^{\tilde X}\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\ldots,k,\mathrm{retr}_{\tilde X}(x,y,n),\underline{y},z,\underline{a}).$$

This is possible since by the previous algorithm one can transform $A_{b,-}$ to an equivalent existential formula and use $\Sigma^0_1\text{-UB}^X_-$. ☐

Corollary 6.29. $\mathcal{A}^\omega[X,\|\cdot\|,\phi]+\mathrm{F}^X_- \vdash \Sigma^0_1\text{-UB}^X_{b,-}$.

Proof. Follows from Lemmas 6.23 and 6.28. ☐

Lemma 6.30.

$$\begin{aligned}\mathcal{A}^\omega[X,\|\cdot\|]+\Sigma^0_1\text{-UB}^X_{b,-} \vdash \forall \underline{y}^{\tilde X}\forall n^{\mathbb N}\Big(&\bigwedge_{i=0}^{m}\|y_i\| >_{\mathbb R} 2^{-n}\\ &\wedge\forall x_1^{\tilde X}\exists x_2^{\tilde X}\ldots\forall x_{m-1}^{\tilde X}\exists x_m^{\tilde X}\exists z^{\mathbb N}\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\ldots,\mathrm{retr}_{\tilde X}(x_m,y_m,n),\underline{y},z,\underline{a})\\ &\quad\to\exists z^*\forall x_1^{\tilde X}\exists x_2^{\tilde X}\ldots\forall x_{m-1}^{\tilde X}\exists x_m^{\tilde X}\exists z \preceq_{\mathbb N} z^*\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\ldots,\mathrm{retr}_{\tilde X}(x_m,y_m,n),\underline{y},z,\underline{a})\Big)\end{aligned}$$

Proof. We apply the axiom $\Sigma^0_1\text{-UB}^X_{b,-}$ iteratively, first to $\forall x^{\tilde X}_{m-1}\exists z^{\mathbb N}\exists x^{\tilde X}_m\theta_{\mathrm{qf}}$ resulting in

$$\begin{aligned}&\forall x^{\tilde X}_1\exists x^{\tilde X}_2\dots\forall x^{\tilde X}_{m-3}\exists x^{\tilde X}_{m-2}\exists z^*\forall x^{\tilde X}_{m-1}\exists x^{\tilde X}_m\exists z\preceq_{\mathbb N} z^*\\ &\qquad\qquad\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\dots,\mathrm{retr}_{\tilde X}(x_{m-1},y_{m-1},n),\mathrm{retr}_{\tilde X}(x_m,y_m,n)\underline{y},z,\underline{a})\end{aligned}$$

until we obtain $\exists z^*\forall x^{\tilde X}_1\exists x^{\tilde X}_2\dots\exists z\preceq_{\mathbb N} z^*\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\mathrm{retr}_{\tilde X}(x_2,y_2,n)\dots,\underline{y},z,\underline{a}))$. □

Definition 6.31. We define the generalized uniform boundedness principle for extensional formulas $\exists v^{\mathbb N}\theta_{\mathrm{qf}}$:

$$\Sigma^0_1\text{-UB}^X_b(\exists v^{\mathbb N}\theta_{\mathrm{qf}}):\begin{cases}\forall\underline{y}^{\tilde X}(\forall x_1\preceq_{\tilde X} y_1\exists x_2\preceq_{\tilde X} y_2\dots\exists z^{\mathbb N},v^{\mathbb N}\theta_{\mathrm{qf}}\\ \to\exists z^*\forall x_1\preceq_{\tilde X} y_1\exists x_2\preceq_{\tilde X} y_2\dots\exists z\preceq_{\mathbb N} z^*\exists v^{\mathbb N}\theta_{\mathrm{qf}}(\dots)),\end{cases}$$

where we allow arbitrary free variables.

Proposition 6.32.
$\mathcal{A}^\omega[X,\|\cdot\|]+\Sigma^0_1\text{-UB}^X_{b,-}\vdash\mathrm{Ext}(\exists v^{\mathbb N}\theta_{\mathrm{qf}}(\underline{x}))\to\Sigma^0_1\text{-UB}^X_b(\exists v^{\mathbb N}\theta_{\mathrm{qf}})$.

Proof. Let $y_1,\dots,y_m\in X$ (or $y_1,\dots,y_m\in\mathbb R$) and $n\in\mathbb N$ such that all $\|y_i\|>2^{-n}$ and assume

$$\forall x_1\preceq_{\tilde X} y_1\exists x_2\preceq_{\tilde X} y_2\dots\exists x_m\preceq_{\tilde X} y_m\exists z^{\mathbb N},v^{\mathbb N}\theta_{\mathrm{qf}}(x_1,\dots,x_m,\underline{y},z,v,\underline{a}).$$

By $\mathrm{Ext}(\exists v^{\mathbb N}\theta_{\mathrm{qf}}(\underline{x}))$ together with Lemma 6.16 (applied m times) we have

$$\forall x^{\tilde X}_1\exists x^{\tilde X}_2\dots\exists x^{\tilde X}_m\exists z^{\mathbb N},v^{\mathbb N}\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\dots,\mathrm{retr}_{\tilde X}(x_m,y_m,n),\underline{y},z,v,\underline{a}).$$

We apply Lemma 6.30 (where the two existential number variables can be thought of coded into a single one) to obtain

$$\exists z^*,v^*\forall x^{\tilde X}_1\exists x^{\tilde X}_2\dots\exists z\preceq_{\mathbb N} z^*\exists v\preceq_{\mathbb N} v^*\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\mathrm{retr}_{\tilde X}(x_2,y_2,n)\dots,\underline{y},z,v,\underline{a}).$$

Then this implies the following weakening of the statement:

$$\exists z^*\forall x^{\tilde X}_1\exists x^{\tilde X}_2\dots\exists x^{\tilde X}_m\exists z\preceq_{\mathbb N} z^*\exists v^{\mathbb N}\theta_{\mathrm{qf}}(\mathrm{retr}_{\tilde X}(x_1,y_1,n),\mathrm{retr}_{\tilde X}(x_2,y_2,n)\dots,\underline{y},z,v,\underline{a}).$$

By the extensionality of $\exists v^{\mathbb N}\theta_{\mathrm{qf}}$ w.r.t. $\underline{x}$ we obtain with Lemma 6.16

$$\exists z^*\forall x_1\preceq_{\tilde X} y_1\exists x_2\preceq_{\tilde X} y_2\dots\exists x_m\preceq_{\tilde X} y_m\exists z\preceq_{\mathbb N} z^*\exists v^{\mathbb N}\theta_{\mathrm{qf}}(x_1,\dots,x_m,\underline{y},z,v,\underline{a}).$$

□

Theorem 6.33.

$$\mathcal{A}^\omega[X,\|\cdot\|,\phi]+\mathrm{F}^X_-\vdash\forall T^{1(\mathbb N)(\tilde X)\dots(\tilde X)}\forall\omega_T^{\mathbb N(\mathbb N)(\mathbb N)(\mathbb N)}\left(U_m(T,\omega_T)\to\left(\varphi(T)\leftrightarrow\forall k^{\mathbb N}\varphi_{2^{-k}}(T)\right)\right),$$

where $\varphi_{2^{-k}}$ *is the* 2^{-k}*-approximation of a formula* φ *of the class* $\mathcal{PBL}$ *according to Lemma 6.13 and* $U_m(T,\omega_T)$ *expresses the uniform continuity of* T *(see (3). Instead of* F^X_- *one can also use* $\Sigma^0_1\text{-UB}^X_-$.

Proof. Let $\varphi \in \mathcal{PBL}$. The direction $\varphi(T) \to \forall k^{\mathbb{N}} \varphi_{2^{-k}}(T)$ is trivial. For the converse direction we prove $\neg\varphi(T) \to \neg(\forall k^{\mathbb{N}}\ \varphi_{2^{-k}}(T))$. Let

$$\varphi(T) \equiv \Theta_m(T, \underline{r}, \underline{s}) :\equiv \forall l^{\mathbb{N}} \forall_{r_1} x_1^{\tilde{X}} \exists_{s_1} y_1^{\tilde{X}} \dots \forall_{r_m} x_m^{\tilde{X}} \exists_{s_m} y_m^{\tilde{X}} \left(T(\underline{x}, \underline{y}, l) =_{\mathbb{R}} 0\right)$$

be a formula in the class $\mathcal{PBL}$. Negating Θ_m gives

$$\exists l^{\mathbb{N}} \exists x_1 \preceq_{\tilde{X}} 1_{\tilde{X}} r_1 \forall y_1 \preceq_{\tilde{X}} 1_{\tilde{X}} s_1 \dots \exists k^{\mathbb{N}} (|T(\underline{x}, \underline{y}, l)| >_{\mathbb{R}} 2^{-k}).$$

Since $\left|T(\underline{x}, \underline{y}, l)\right| >_{\mathbb{R}} 2^{-k}$ is extensional (since T is uniformly continuous) and in Σ_1^0, we can apply Corollary 6.29 and Proposition 6.32, resulting in (using monotonicity w.r.t. k)

$$\exists k^{\mathbb{N}}, l^{\mathbb{N}} \exists x_1 \preceq_{\tilde{X}} 1_{\tilde{X}} r_1 \forall y_1 \preceq_{\tilde{X}} 1_{\tilde{X}} s_1 \dots (|T(\underline{x}, \underline{y}, l)| >_{\mathbb{R}} 2^{-k}).$$

Now we use the modulus of uniform continuity ω_T to prove the negated approximate formula according to Lemma 6.13:

$$\exists k^{\mathbb{N}}, l^{\mathbb{N}} \exists x_1 \preceq_{\tilde{X}} 1_{\tilde{X}} (r_1 - 2^{-k}) \forall y_1 \preceq_{\tilde{X}} 1_{\tilde{X}} (s_1 + 2^{-k}) \dots (|T(\underline{x}, \underline{y}, l)| >_{\mathbb{R}} 2^{-k}). \tag{7}$$

Since the modulus depends on the range of the bounded variables which we are about to modify we define a new modulus $\omega_T^*(b, k, l) := \max\{\omega_T(b+1, k+1, l), k+1\}$. Due to the uniform continuity with the new modulus ω_T^* we have

$$\begin{aligned} &\forall l^{\mathbb{N}} \forall x_1^{\tilde{X}}, \widetilde{x}_1^{\tilde{X}} \preceq_{\tilde{X}} 1_{\tilde{X}} (r_1 + 1) \forall y_1^{\tilde{X}}, \widetilde{y}_1^{\tilde{X}} \preceq_{\tilde{X}} 1_{\tilde{X}} (s_1 + 1) \dots \\ &\qquad \Big(\bigwedge_{i=1}^{m} \|\widetilde{x}_i - x_i\|, \|\widetilde{y}_i - y_i\| \leq_{\mathbb{R}} 2^{-\omega_T^*(b,k,l)} \to |T(\underline{x}, \underline{y}, l) - T(\underline{\widetilde{x}}, \underline{\widetilde{y}}, l)| \leq_{\mathbb{R}} 2^{-k-1} \Big), \end{aligned}$$

where $b := \lceil \max\{r_i, s_i \ \mid\ i, j \in \{1, \dots, m\}\} \rceil$.
Finally we need to argue why for any point $x \in B_r(0)$ there exists a point $x^* \in B_{r-2^{-n}}(0)$ (for all $n \in \mathbb{N}$ such that $r - 2^{-n} > 0$) such that $\|x^* - x\| \leq 2^{-n}$. Note that in a metric space this is not necessarily the case but in normed spaces this is always possible by setting $x^* := \dfrac{(r - 2^{-n})x}{\max\{\|x\|, r - 2^{-n}\}}$. Hence, we have shown

$$\exists k^{\mathbb{N}}, l^{\mathbb{N}} \exists x_1 \preceq_{\tilde{X}} 1_{\tilde{X}} (r_1 - 2^{-N}) \forall y_1 \preceq_{\tilde{X}} 1_{\tilde{X}} (s_1 + 2^{-N}) \dots |T(\underline{x}, \underline{y}, l)| >_{\mathbb{R}} 2^{-k-1},$$

where $b := \lceil \max\{r_i, s_i \ \mid\ i, j \in \{1, \dots, m\}\} \rceil$ and

$$N := \max\left\{\omega_T^*(b, k, l), \lceil -\log_2(r_i) \rceil + 1 \mid i \in \{1, \dots, m\}\right\}.$$

Due the fact that $N \geq k + 1$ we haven proven (7).
For the claim with Σ_1^0-UB_{-}^{X} one uses Lemma 6.28 instead of Corollary 6.29. ☐

Remark 6.34. There is a variant of the monotone functional interpretation, on which our metatheorems are based, due to [14] and extended to abstract spaces X in [13], which treats bounded quantifiers directly as computationally empty (thereby avoiding the need for an epsilon-operator) and which is particularly tailored towards conservation results for general uniform boundedness principles. However, this so-called 'bounded functional interpretation', is based on an intensional rule-based treatment of the bounding relation $\preceq_X$ which is not provably equivalent to the usual relation which we use (as in model theory).

It is interesting to note that in the presence of uniform boundedness, it would have been sufficient to assume that the function T is extensional, since uniform boundedness proves uniform continuity on bounded sets from extensionality (see [30, Proposition 4.3]). In model theory the assumption of extensionality is empty because in a model every function is extensional (because one has built-in equality). As a consequence of this, all function symbols are assumed to be uniformly continuous (on bounded sets) in model theory whereas in proof theory it is common to operate with partial forms of extensionality which only need weaker assumptions than full uniform continuity (see e.g. the treatment of functions satisfying the condition (E) used in fixed point theory in [35]).

Proposition 6.35 ([19, Proposition 9.26])**.** *Let $\mathcal{U}$ be a countably incomplete ultrafilter. For a normed space (L)-structure $\mathfrak{M}$ and any positive bounded formula φ, with elements $a_1, \ldots, a_n$ of $\mathfrak{M}$ of suitable sorts the following are equivalent:* $\mathfrak{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$ and $(\mathfrak{M})_{\mathcal{U}} \models \varphi[a_1, \ldots, a_n]$.

Discussion. In Theorem 6.33 we have shown that the uniform boundedness principle (via F^X_-) proves the equivalence of approximate truth of a positive bounded formula and the original formula (even allowing a more general class of formulas $\mathcal{PBL}$). Together with Proposition 6.35 this gives rise to the following analogy:

"Uniform boundedness in proof theory $\approx$ Ultrapower in model theory".

Theorem 6.36 (Logical Metatheorem for the uniform boundedness principle)**.** *Let $\rho \in \boldsymbol{T}^X$ be an admissible finite type and Θ be a set of sentences of the class $\mathcal{PBL}$, Θ_ε be the set of approximations of Θ in the sense of Lemma 6.13, such that for each $\varphi_T \in \Theta$ we have provably $U_m(T, \omega_T)$ (see* (3) *on p. 15) for some closed terms ω_T, T defined in the language of $\mathcal{A}^\omega[X, \|\cdot\|]$. Let $B_\forall(x,u)$, resp. $C_\exists(x,v)$, be $\forall$- resp. $\exists$-formulas that contain only the variables x, u resp. x, v free. Assume*

$$\mathcal{A}^\omega[X, \|\cdot\|, \phi] + \Theta + \mathrm{F}^X_- \vdash \forall x^\rho \left(\forall u^{\mathbb{N}} B_\forall(x,u) \to \exists v^{\mathbb{N}} C_\exists(x,v) \right) \tag{8}$$

then one can extract a partial functional $\Phi : S_{\widehat{\rho}} \rightharpoonup \mathbb{N}$ whose restriction to the strongly majorizable elements of $S_{\widehat{\rho}}$ is a (bar recursive) computable functional of $\mathcal{M}^\omega$ and the following holds in all nontrivial normed spaces X s.t. $\mathcal{S}^{\omega,X} \models \Theta_\varepsilon$: for all $x \in S_\rho$, $x^* \in S_{\widehat{\rho}}$ if $x^* \gtrsim_\rho x$ then

$$\forall u \leq \Phi(x^*) B_\forall(x,u) \to \exists v \leq \Phi(x^*) C_\exists(x,v).$$

Moreover,

1. *if $\widehat{\rho}$ is type 1, then $\Phi : S_{\widehat{\rho}} \to \mathbb{N}$ is a total computable functional (in the ordinary sense of type-2 recursion theory).*
2. *All variables may occur as finite tuples of the same type.*
3. *If the statement in* (8) *can be proven without the axiom of dependent choice, one does not need bar recursion. Then the functional $\Phi : S_{\widehat{\rho}} \to \mathbb{N}$ is primitive recursive (in the sense of Gödel).*

Proof. We have to add the following lines of reasoning to the proof of Theorem 5.13: In Proposition 6.27 the constants ϕ are interpreted in $\mathcal{S}^{\omega,X}$. Since the type of ϕ is (in case of type X) not admissible we have to argue that we can also interpret ϕ in $\mathcal{M}^{\omega,X}$ such that $[\phi]_{\mathcal{S}^{\omega,X}} \approx_\rho [\phi]_{\mathcal{M}^{\omega,X}}$, where $\approx_\rho$ is defined in [31, Proposition 3.71 and Lemma 17.84]. By restricting $[\phi]_{\mathcal{S}^{\omega,X}}$ to arguments of $\mathcal{M}^{\omega,X}$, *i.e.* $[\phi]_{\mathcal{M}^{\omega,X}} := [\phi]_{\mathcal{S}^{\omega,X}} \restriction_{\mathcal{M}^{\omega,X}}$, we obtain a suitable candidate for the interpretation of ϕ since all arguments have an admissible type and so Lemma 5.7 is applicable. Then we have to show that ϕ is majorizable, which is a straightforward computation if one uses the majorants for

type 1: $\lambda.z^2, y^1, n^{\mathbb{N}}.M(y(0)+1)$ and for type X: $\lambda z^1, y^{\mathbb{N}}, n^{\mathbb{N}}.y$ (see Definition 5.2 for the majorization types). Thus, we have proven $[\phi]_{\mathcal{S}^{\omega,X}} \approx_\rho [\phi]_{\mathcal{M}^{\omega,X}}$. Note that both majorants are simple, computable functions and thus do not contribute to the complexity of the extracted bounds.
The axiom F^X_- is an axiom Δ, hence we can apply Corollary 5.14. From Proposition 6.21 we know that $\mathcal{M}^{\omega,X} \models \mathrm{F}^X_-$ (but $\mathcal{S}^{\omega,X} \not\models \mathrm{F}^X_-$, see Section 6.5) and by Lemma 5.10 that $\mathcal{M}^{\omega,X} \models \tilde{\mathrm{F}}^X_-$.
Since we have shown in Theorem 6.33 that under F^X_- the sets Θ and Θ_ε are equivalent and Θ_ε can be written as an axiom Δ (see Lemma 6.13 and the proof of Proposition 6.17), we are free to assume the stronger version in the proof of (8), whereas we only have to demand the truth of the approximate version in the respective model. □

Remark 6.37. Observing the above proof shows why, even without using dependent choice in the proof of (8), the restrictions on the types cannot be relaxed, since we have to pass through the model $\mathcal{M}^{\omega,X}$ for F^X_- to hold.

Definition 6.38 (cp. [30, Definition 3.8]). The class $\mathcal{H}$ consists of all sentences (in the language of the theory in question) that have a prenex normal form

$$\forall a^\rho \forall b \preceq_\sigma ra \exists x_1^{\mathbb{N}} \forall y_1^{\tau_1} \ldots \exists x_n^{\mathbb{N}} \forall y_n^{\tau_n} F_\exists(a, b, \underline{x}, \underline{y}),$$

where $F_\exists$ is an $\exists$-formula according to Definition 5.12, the types τ_i, ρ are small and σ is admissible and bounded by a closed term r.

We present a conservation result for the class $\mathcal{H}$ for the uniform boundedness principle, which can be proven from F^X_- (see Lemma 6.23).

Corollary 6.39 (cp. [31, Corollary 17.49 and Corollary 17.104] and [30, Corollary 3.9]). *Let A be a sentence in the class $\mathcal{H}$. If*

$$\mathcal{A}^\omega[X, \|\cdot\|, \phi] + \mathrm{F}^X_- \vdash A,$$

then A holds in any nontrivial normed space X. Similarly for the extensions of $\mathcal{A}^\omega[X, \|\cdot\|, \phi]$.

Proof. The proof is similar to the proof of [31, Corollary 17.49]. Short summary: One applies Theorem 6.36 where the restrictions of types in the class $\mathcal{H}$ become apparent when bringing A to the right logical format (its Herbrand normal form) in order to be applicable. □

6.4. *Applications of the uniform boundedness principle*

In the following we will analyze some pairs of properties of normed spaces and their connection to uniform boundedness and forming ultrapowers.

Example 6.40 ([1, II, Theorem 4.5]). Let X be a Banach space and $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$. Then $(X)_\mathcal{U}$ is strictly convex $\Leftrightarrow$ $(X)_\mathcal{U}$ is uniformly convex $\Leftrightarrow$ X uniformly convex.

Proposition 6.41 (cp. [31, Proposition 17.110]).

$$\mathcal{A}^\omega[X, \|\cdot\|] + \Sigma^0_1\text{-UB}^X_- \vdash X \text{ is strictly convex} \to X \text{ is uniformly convex.}$$

Proof. Strict convexity can be formalized as follows

$$\forall k^{\mathbb{N}} \forall x_1, x_2 \preceq_X 1_X \exists n^{\mathbb{N}} \left(\left\| \tfrac{1}{2}(x_1 + x_2) \right\| \geq 1 - 2^{-n} \to \|x_1 - x_2\| < 2^{-k} \right) \tag{9}$$

The formula $\left\| \frac{1}{2}(x_1 + x_2) \right\| \geq 1 - 2^{-n} \to \|x_1 - x_2\| < 2^{-k}$ is of type $\exists v^{\mathbb{N}} \theta_{\mathrm{qf}}$ and is extensional allowing us to use Lemma 6.25 and apply $\Sigma^0_1\text{-UB}^X$ resulting in

$$\forall k^{\mathbb{N}} \exists n^{\mathbb{N}} \forall x_1, x_2 \preceq_X 1_X \left(\left\| \tfrac{1}{2}(x_1 + x_2) \right\| \geq 1 - 2^{-n} \to \|x_1 - x_2\| < 2^{-k} \right)$$

expressing uniform convexity. □

Using the conservation result from Corollary 6.39 we also show that adding the uniform boundedness principle does not invoke the provability of strict convexity (and further properties) of the normed space in question. To this end we only need to prove that the property in question can be equivalently formulated by a sentence in the class $\mathcal{H}$.

Proposition 6.42. $\mathcal{A}^\omega[X, \|\cdot\|] + \mathrm{F}^X_- \nvdash X$ is strictly convex.

Proof. The property of a space X to be strictly convex (see (9)) is in the class $\mathcal{H}$. The claim then follows from Corollary 6.39 and the fact that there exist Banach spaces which are not strictly convex (e.g. l_1, l_∞). ☐

Definition 6.43 ([22] and [46])**.** Let X be a Banach space. Let $B(X)$ denote the unit ball.

1. We call X *nonsquare* if $\forall x, y \in B(X) \left(\min\left\{\left\|\frac{x+y}{2}\right\|, \left\|\frac{x-y}{2}\right\|\right\} < 1\right)$.
2. We call X *uniformly nonsquare* if
$$\exists \delta > 0 \forall x, y \in B(X) \left(\min\left\{\left\|\tfrac{x+y}{2}\right\|, \left\|\tfrac{x-y}{2}\right\|\right\} < 1 - \delta\right).$$

Proposition 6.44. *Let X be a Banach space and $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$. Then the following are equivalent.*

1. *$(X)_\mathcal{U}$ is nonsquare;*
2. *$(X)_\mathcal{U}$ is uniformly nonsquare;*
3. *X is uniformly nonsquare.*

Proof. We only prove $1 \to 3$ the rest is rather trivial. Assume that $(X)_\mathcal{U}$ is nonsquare and for a contradiction that X is not uniformly nonsquare, *i.e.*

$$\forall k^\mathbb{N} \exists x, y \preceq_X 1_X \left(\left\|\frac{x-y}{2}\right\| \geq 1 - 2^{-k} \wedge \left\|\frac{x+y}{2}\right\| \geq 1 - 2^{-k}\right),$$

implying the existence of sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset B_1(0)$ such that

$$\left\|\frac{x_n - y_n}{2}\right\| \geq 1 - 2^{-n} \wedge \left\|\frac{x_n + y_n}{2}\right\| \geq 1 - 2^{-n}.$$

Now set $\widetilde{x} := \widetilde{(x_n)}$ and $\widetilde{y} := \widetilde{(y_n)}$ as elements of $(X)_\mathcal{U}$ having the following properties:

$$\lim_{n\to\infty} \|x_n - y_n\| = 2 = \|\widetilde{x} - \widetilde{y}\|, \qquad \lim_{n\to\infty} \|x_n + y_n\| = 2 = \|\widetilde{x} + \widetilde{y}\|,$$

contradicting the statement that $(X)_\mathcal{U}$ is nonsquare. ☐

Proposition 6.45.

$$\mathcal{A}^\omega[X, \|\cdot\|] + \Sigma^0_1\text{-UB}^X_- \vdash X \text{ is nonsquare} \to X \text{ is uniformly nonsquare}.$$

Proof. Follows by applying Σ^0_1-UB^X to the statement "X is nonsquare" which can be formalized as follows

$$\forall x, y \preceq_X 1_X \, \exists k \in \mathbb{N} \, \left(\min\left\{\left\|\tfrac{x+y}{2}\right\|, \left\|\tfrac{x-y}{2}\right\|\right\} < 1 - 2^{-k}\right), \tag{10}$$

Since $\min\left\{\left\|\frac{x+y}{2}\right\|, \left\|\frac{x-y}{2}\right\|\right\} < 1 - 2^{-k}$ is of the format $\exists v^\mathbb{N} \theta_{\mathrm{qf}}$ and is extensional, we can use Lemma 6.25. ☐

Proposition 6.46. $\mathcal{A}^\omega[X, \|\cdot\|] + \mathrm{F}^X_- \nvdash X$ is nonsquare.

Proof. The property of a space to be nonsquare (see (10)) is in the class $\mathcal{H}$. The claim then follows from Corollary 6.39 and the fact that there exist Banach spaces which are not nonsquare (e.g. l_1). ☐

The following result illustrates why forming ultrapowers can be seen as a form of completion (cp. [48, Remark 3]).

Proposition 6.47 ([31, cp. Proposition 17.105])**.**

$$\mathcal{A}^\omega[X, \|\cdot\|] + \Sigma^0_1\text{-UB}^X_- \vdash X \text{ is complete.}$$

Proof. By contradiction: Applying Σ^0_1-UB^X to the formula expressing the existence of a nonconvergent Cauchy sequence yields that the sequence cannot be Cauchy. ☐

Definition 6.48 ([36] and [43])**.** Let X be a Banach space, let $n \in \mathbb{N}$, and let $B(X)$ denote the unit ball and $S(X)$ the unit sphere.

1. X is called *p(n)*-convex if $\forall x_1, \ldots, x_n \in S(X) \exists 1 \leq i, j \leq n \, (i \neq j \wedge \|x_i - x_j\| < 2)$.
2. X is called *P(n)*-convex if $P(n) = \sup\{r > 0 \mid \ \exists n \text{ disjoint balls of radius } r \text{ in } B(X)\} < \dfrac{1}{2}$.

Example 6.49 ([43, cp. Theorem 3.8])**.** Let X be a Banach space and $n \in \mathbb{N}$ and let $\mathcal{U}$ be a nontrivial ultrafilter. Then $(X)_\mathcal{U}$ is $P(n)$-convex $\Leftrightarrow X$ is $P(n)$-convex $\Leftrightarrow (X)_\mathcal{U}$ is $p(n)$-convex.

Proposition 6.50. *For every fixed $n \in \mathbb{N}$*

$$\mathcal{A}^\omega[X, \|\cdot\|] + \Sigma^0_1\text{-UB}^X_- \vdash X \text{ is } p(n)\text{-convex} \to X \text{ is } P(n)\text{-convex}$$

Proof. One can formalize $p(n)$-convexity as follows (note that we can replace $S(X)$ by $B(X)$ as the property is trivial if one of the x_i has norm < 1)

$$\forall x_1, \ldots, x_n \preceq_X 1_X \exists k^{\mathbb{N}} \Big(\min\{\|x_i - x_j\| \mid i \neq j\} < 2 - 2^{-k}\Big). \tag{11}$$

Since $\min\{\|x_i - x_j\| \mid i \neq j\} < 2 - 2^{-k}$ is of the form $\exists v^{\mathbb{N}} \theta_{\mathrm{qf}}$ and is extensional we can use Lemma 6.25 and apply Σ^0_1-UB^X (to k only) yielding

$$\exists k^{\mathbb{N}} \forall x_1, \ldots, x_n \preceq_X 1_X \Big(\min\{\|x_i - x_j\| \mid i \neq j\} < 2 - 2^{-k}\Big)$$

which is equivalent to $P(n)$-convexity by [36, Remark 1.4]. ☐

Proposition 6.51. *For every fixed $n \in \mathbb{N}$: $\mathcal{A}^\omega[X, \|\cdot\|] + \mathrm{F}^X_- \nvdash X$ is p(n) convex.*

Proof. The property of a space X to be $p(n)$-convex (see (11)) is in the class $\mathcal{H}$. The claim then follows from Corollary 6.39 and the fact that there exist Banach spaces which are not $p(n)$-convex (e.g. $l_\infty, C[0,1]$ see [43, Example 3.4]). ☐

Definition 6.52 ([41, pp. 59-60] and [47])**.** Let X be a Banach space.

1. X is called *smooth* if the limit $\lim\limits_{t \to 0} \dfrac{\|x + ty\| - \|x\|}{t}$ exists for every $x, y \in X$ with $\|x\| = 1 = \|y\|$.

2. X is called *uniformly smooth* if for the modulus of smoothness $\rho_X(\tau)$, $\tau > 0$ it holds that $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} = 0$, where
$$\rho_X(\tau) := \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 \mid x, y \in X, \|x\| = 1, \|y\| = \tau \right\}.$$

Example 6.53 ([12])**.** Let X be a Banach space and let $\mathcal{U}$ be a nontrivial ultrafilter. Then X is uniformly smooth $\Leftrightarrow$ $(X)_\mathcal{U}$ is smooth $\Leftrightarrow$ $(X)_\mathcal{U}$ is uniformly smooth.

Proposition 6.54.
$$\mathcal{A}^\omega[X, \|\cdot\|, J] + \Sigma^0_1\text{-UB}^X_- \vdash X \text{ is smooth} \to X \text{ is uniformly smooth},$$
where $\mathcal{A}^\omega[X, \|\cdot\|, J]$ is an extension of $\mathcal{A}^\omega[X, \|\cdot\|]$ by a constant J for the normalized duality map and a universal axiom stating the properties of J (see [34]).

Proof. Consider the duality mapping J of X and let $f_x \in J(x)$ for some $x \in S(X)$. For each $\lambda > 0$ and $y \in S(X)$ it holds by [47, Proof of Theorem 4.3.1]
$$\frac{\|x\| - \|x - \lambda y\|}{\lambda} \leq \frac{f_x(y)}{\|x\|} \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \tag{12}$$
which is used to show that the single-valuedness of J is equivalent to smoothness. Now observe that smoothness implies
$$\forall m^\mathbb{N} \forall x, y \preceq_X 1_X \exists k^\mathbb{N} (\|x\|, \|y\| = 1 \to \left\|x + 2^{-k}y\right\| + \left\|x - 2^{-k}y\right\| < 2 + 2^{-k}2^{-m}) \tag{13}$$
which is a suitable format for the application of Lemma 6.25. Applying Σ^0_1-UB^X yields
$$\forall m^\mathbb{N} \exists k^* \forall x, y \preceq_X 1_X \exists k \preceq_\mathbb{N} k^* (\|x\|, \|y\| = 1 \to \left\|x + 2^{-k}y\right\| + \left\|x - 2^{-k}y\right\| < 2 + 2^{-k}2^{-m})$$
which implies with (12) the uniform norm-norm continuity of J which is equivalent to uniform smoothness (see [11, Theorems II.2.14 and II.2.16]). ☐

Proposition 6.55. $\mathcal{A}^\omega[X, \|\cdot\|] + \mathrm{F}^X_- \nvdash X$ is smooth.

Proof. Statement (13) is in the class $\mathcal{H}$. It can be shown that it is equivalent to smoothness. The claim then follows from Corollary 6.39 and the fact that there exist Banach spaces which are not smooth (e.g. l_1, l_∞). ☐

6.5. Applications of the uniform boundedness principle in current research

Definition 6.56 (cp. [15])**.** Let X be a real Banach space and X^* denote its dual space. Let $\phi : X \to [0, \infty)$ be a continuous function with $\phi(0) = 0$ and $x \neq 0 \to \phi(x) > 0$ satisfying:
For all sequences $(x_n)_{n\in\mathbb{N}}$ in X such that $(\|x_n\|)_{n\in\mathbb{N}}$ is non-increasing and $\lim\limits_{n\to\infty} \phi(x_n) = 0$ it holds that $\lim\limits_{n\to\infty} \|x_n\| = 0$.
An accretive operator $A : D(A) \to 2^X$ with $0 \in A(z)$ is called *ϕ-accretive at zero* if the following holds for all $(x, u) \in A$: $\langle u, x - z\rangle_+ \geq \phi(x - z)$.

The authors of [32] introduce a new definition which generalizes ϕ-accretivity at zero in the sense that the existence of the continuous function ϕ is not demanded but which has a stronger uniform requirement on the positivity of A at zero instead such that the distance from 0 only depends on the distance $\|x\|$ has from zero 0 but not on x itself.

Definition 6.57 ([32, Definition 10])**.** Let X be real Banach space. We call an accretive operator $A : D(A) \to 2^X$ *uniformly accretive at zero* if

$$\forall k, K \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall (x, u) \in A\ (\|x - z\| \in [2^{-k}, K + 1] \to \langle u, x - z\rangle_+ > 2^{-m}).$$

We now show that the uniform boundedness principle can be used to obtain uniform accretivity from ϕ-accretivity when considering only single valued operators A. Since in Definition 6.56 we have $x \neq 0 \to \phi(x) > 0$ it follows that (assuming $A(z) = 0$)

$$\forall x \in D(A)\, (\|x - z\| > 0 \to \langle A(x), x - z\rangle_+ > 0)$$

which is equivalent to

$$\forall k, K \in \mathbb{N}\, \forall x \in D(A)\, \exists m \in \mathbb{N}\ \left(\|x - z\| \geq 2^{-k} \wedge \|x - z\| \leq K + 1 \to \langle A(x), x - z\rangle_+ > 2^{-m}\right).$$

The variable K plays the role of y (which we could introduce as a dummy variable as well) bounding $x - z$ (and thus bounding x). Applying uniform boundedness and observing that the statement is monotone w.r.t. m yields

$$\forall k, K \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall x \in D(A)\ \left(\|x - z\| \geq 2^{-k} \wedge \|x - z\| \leq K + 1 \to \langle A(x), x - z\rangle_+ > 2^{-m}\right)$$

which is exactly Definition 6.57 when considering single valued maps.

Remark 6.58. Note that one has to add an additional predicate A to the language in order to formalize "$\forall (x, u) \in A$" in our framework. By adding the definition of accretivity as a universal axiom to the theory, the predicate A functions as an implicit quantification over all accretive operators. Of course, one also needs to formalize dual spaces and the normalized duality map in order to prove a logical metatheorem for the setting of (uniformly) accretive operators. In [34] the authors provide a formal representation of the normalized duality map, together with a continuous selection functional.

7. Logical Metatheorem for BL^pL^q-Banach lattices

In this section we recast the axiomatization of the BL^pL^q-Banach lattice from [20] in our proof-theoretic formal framework and explicitly write it as an axiom Δ so that Corollary 5.14 can be (suitably adapted) applied (see Theorem 7.13 below).

Definition 7.1. Let X be a lattice.

1. Two elements $x, y \in X$ are *disjoint* or *orthogonal* if $|x| \sqcap |y| = 0$, which is denoted by $x \perp y$.
2. For a subset $A \subseteq X$ we denote the set of all disjoint elements of A by

   $$A^{\perp} := \{x \in X \mid \forall a \in A (x \perp a)\}.$$

   $A^{\perp}$ is also called the *orthogonal complement* of A.

Definition 7.2 ([42, Definition 1.2.1])**.** Let X be a vector lattice.

1. A subspace U of X is called a *sublattice* of X if for all elements $x, y \in U$ both $x \sqcap y \in U$ and $x \sqcup y \in U$ hold.
2. A subspace I of X is called an *ideal* if for all $y \in I$ and $x \in X$ with $|x| \leq |y|$ also $x \in I$.
3. An ideal B of X is called a *band* if for every subset $A \subseteq B$ with $\sup(A) \in X$ also $\sup(A) \in B$.

In [20] the class of bands of $L^p(L^q)$-Banach lattices is considered, which is closed under ultrapowers, in contrast to the class of $L^p(L^q)$-Banach lattices (see [39]). Before discussing an axiomatization using Banach lattices we give a more analytical definition: An abstract $L^p(L^q)$-space is a Banach lattice X which, for some measure space (Ω, Σ, μ), can be equipped with the structure of an $L_\infty(\Omega, \Sigma, \mu)$-module and with a so-called random norm $N : X \to L^p(\Omega, \Sigma, \mu)_+$ with the following properties (see [20]). Note that all sentences have to read with the addition of "almost everywhere".

1. $\forall \varphi \in L_\infty(\Omega, \Sigma, \mu) \forall x \in X (\varphi \geq 0 \wedge x \geq 0 \to \varphi . x \geq 0)$.
2. $\forall x, y \in X (N(x + y) \leq N(x) + N(y))$.
3. $\forall \varphi \in L_\infty(\Omega, \Sigma, \mu) \forall x \in X (N(\varphi . x) = |\varphi| \, N(x))$.
4. $\forall x, y \in X (0 \leq |x| \leq |y| \to N(x) \leq N(y))$.
5. $\forall x, y \in X \, (x \perp y \to N(x + y)^q = N(x)^q + N(y)^q)$.
6. $\forall x \in X \, (\|x\|_X = \|N(x)\|_{L^p})$.

In the case which is most interesting for applications, N is explicitly defined by the map $f \mapsto \left(\int_0^\infty \|f(t)\|_{L^q}^p \, \mathrm{d}t\right)^{1/p}$. The multiplicative action of $L_\infty(\Omega, \Sigma, \mu)$ on $L^p([0, \infty), L^q(\Omega, \Sigma, \mu))$ is well-defined. If N is defined as above, the class of abstract $L^p(L^q)$-spaces coincides with that of bands in $L^p(L^q)$-Banach lattices (denoted by BL^pL^q-Banach lattices). Following the approach for the axiomatization of $L^p(\Omega, \Sigma, \mu)$-spaces the authors of [20] prove an axiomatization by Banach lattices, in this case relying on finite approximations.

Definition 7.3 (Banach-Mazur distance [38, p. 165])**.** Let X and Y be isomorphic Banach spaces. Define

$$d(X, Y) := \inf \left\{ \|L\| \left\|L^{-1}\right\| \mid L \text{ is a linear isomorphism of } X \text{ onto } Y \right\}$$

as the *Banach-Mazur distance* of X to Y.

The notion of $\mathcal{L}_p$-spaces due to ([40]) is applied to the setting of $L^p(L^q)$-Banach lattices by the authors of [20].

Definition 7.4 ([20, Definition 3.1])**.** A Banach lattice X is a $(\mathcal{L}_p\mathcal{L}_q)_\lambda$-lattice if for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ it holds: Let $x_1, \ldots, x_n$ be positive, pairwise disjoint elements of X. There exists a finite dimensional sublattice F of X which is isomorphic to a finite dimensional BL^pL^q-Banach lattice E with Banach-Mazur distance $d(F, E) \leq \lambda + \varepsilon$ and contains elements $x'_1, \ldots, x'_n$ such that $\|x'_i - x_i\| \leq \varepsilon$ for all $i = 1, \ldots, n$.

Proposition 7.5 ([20, Proposition 3.6])**.** *Let $1 \leq p, q < \infty$. A Banach lattice is a $(\mathcal{L}_p\mathcal{L}_q)_1$-lattice if and only if it is isometrically lattice isomorphic to a BL^pL^q-Banach lattice.*

Lemma 7.6. *Let X be a Banach lattice, $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots \alpha_n \in \mathbb{R}$.*

1. *For all pairwise disjoint positive elements $x_1, \ldots, x_n$ it holds that $\|\sum_{i=1}^n \alpha_i x_i\| \geq \max_{i \in \{1, \ldots, n\}} \{|\alpha_i| \, \|x_i\|\}$.*
2. *For all elements $x_1, \ldots, x_n$*

$$\left(\bigwedge_{i=1}^{n} (\|x_i\| \leq 1 \wedge x_i \geq 0) \to \bigwedge_{i=1}^{n} (\|x'_i\| \leq 1 \wedge x'_i \geq 0) \;\wedge \sum_{\substack{i,j=1,\ldots,n \\ i \neq j}} |x'_i| \sqcap |x'_j| = 0 \right),$$

where $x_i' := x_i - x_i \sqcap \sum_{k\neq i} x_k$ *for each* $i \in \{1, \dots, n\}$ *and*

$$\left(\bigwedge_{i=1}^{n} \|x_i\| \le 1 \wedge x_i \ge 0 \wedge \sum_{\substack{i,j=1,\dots,n\\ i\neq j}} |x_i| \sqcap |x_j| = 0\right) \to \bigwedge_{i=1}^{n} x_i = x_i'.$$

Proof. See Appendix A.7 and A.8. ☐

The following lemma is one of the main ingredients for the axiomatization of BL^pL^q-Banach lattices.

Lemma 7.7 ([20, Lemma 3.2])**.** *Let* X *be a* $(\mathcal{L}_p\mathcal{L}_q)_\lambda$*-lattice. Then for every* $\varepsilon > 0$ *and every finite dimensional sublattice* E *of* X *there exists a finite dimensional sublattice* F *of* X *and a vector lattice homomorphism* $T : E \to F$ *such that* F *is* $(\lambda + \varepsilon)$*-lattice isomorphic to a finite dimensional* BL^pL^q*-Banach lattice and for all* $x \in E$ *it holds:* $\|Tx - x\| \le \varepsilon \|x\|$*.*

Proof. See [20, Lemma 3.2], proving Lemma 7.6 is instructive for the proof. ☐

Now we discuss an axiomatization of BL^pL^q-Banach lattices in terms of finite dimensional subspaces in the language of Banach lattice due to Henson and Raynaud. They basically spell out Proposition 7.5 and the quantitative information from Lemma 7.7. Since it is not possible to formally speak about finite dimensional subspaces in the language of Banach lattices, a finite set $\underline{y}$ of generators of a subspace F is used instead. Such a subspace has the shape $\left(\oplus_{i=1}^m l_q^{d_i}\right)_p$, where $\underline{y} = (y_{ij})_{i,j}$ with $i = 1, \dots, m$ and $j = 1, \dots, d_i$.

Definition 7.8 ([20, p. 219])**.** The infinite list of axioms for BL^pL^q-Banach lattices $(A_{n,N})_{n,N\in\mathbb{N}}$ is built up as follows:

$$\psi'_{m,\underline{d},N}(\underline{y}) := \forall(\lambda_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,d_m}} \left(\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{d_i}|\lambda_{ij}|^q\right)^{p/q}\right)^{1/p} \le \left\|\sum_{i=1}^{m}\sum_{j=1}^{d_i}\lambda_{ij}y_{ij}\right\| \right.$$

$$\left. \le \left(1+\frac{1}{N}\right)\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{d_i}|\lambda_{ij}|^q\right)^{p/q}\right)^{1/p}\right)$$

$$\psi''_{m,\underline{d}}(\underline{y}) := \sum_{(i,j)\neq(i',j')} |y_{ij}| \sqcap |y_{i'j'}| = 0$$

$$\psi'''_{m,\underline{d}}(\underline{y}) := \sum_{i=1}^{m}\sum_{j=1}^{d_i} |y_{ij} - |y_{ij}|| = 0$$

$$\psi_{m,\underline{d},N}(\underline{y}) := \psi'_{m,\underline{d},N}(\underline{y}) \wedge \psi''_{m,\underline{d}}(\underline{y}) \wedge \psi'''_{m,\underline{d}}(\underline{y})$$

$$\varphi_{n,m,\underline{d},N}(\underline{x}) := \exists\underline{y}\left(\psi_{m,\underline{d},N}(\underline{y}) \wedge \bigwedge_{k=1}^{n} \exists\underline{\lambda}\left\|x_k - \sum_{i=1}^{m}\sum_{j=1}^{d_i}\lambda_{ij}y_{ij}\right\| \le \frac{1}{N}\right)$$

$$\phi_{n,N}(\underline{x}) := \bigvee_{m,\underline{d}} \varphi_{n,m,\underline{d},N}(\underline{x}), \qquad \text{where } m, d_1, \dots, d_m \in \mathbb{N} \text{ with } \sum_{i=1}^{m} d_i \le n2^{nN}$$

$$A_{n,N} := \forall x_1, \dots, x_n \left(\sum_{\substack{i,j=1,\dots,n\\ i\neq j}} |x_i| \sqcap |x_j| = 0 \to \phi_{n,N}(\underline{x})\right).$$

The formula $\psi'_{m,\underline{d},N}(\underline{y})$ expresses that the finite dimensional subspace generated by the elements y_{ij} has Banach-Mazur distance of at most $(1+\frac{1}{N})$ to $\left(\oplus_{i=1}^{m} l_q^{d_i}\right)_p$. The formulas $\psi''_{m,\underline{d}}(\underline{y})$ and $\psi'''_{m,\underline{d}}(\underline{y})$ express that the y_{ij} are positive and pairwise disjoint, which is necessary to show that their linear span is a sublattice. Then $\varphi_{n,m,\underline{d},N}(\underline{x})$ states that to given elements $x_1,\dots,x_n$ there exists points y_{ij} with the aforementioned properties such that a linear combination of those are an $\frac{1}{N}$-approximation of the x_i. Since in Definition 7.4 the existence of some finite dimensional subspace is required the formula $\phi_{n,N}(\underline{x})$ is a big disjunction over all possible dimensions m, d_m where the upper bound can be found in [20, Proposition 3.7]. In [20] it is indicated that one can translate the axioms $A_{n,N}$ into the language of positive bounded logic, which in turn can be translated into sentences Δ as shown in Proposition 6.17. One obstacle for the translation into positive bounded formulas are the unbounded quantifiers. First we can assume that the elements $x_1,\dots,x_n$ are positive, since Definition 7.4 is used in the axioms $A_{n,N}$. We can bound the norm of the elements x_i by 1, since we could renorm them which would lead to new coefficients λ_{ij} in $\varphi_{n,m,\underline{d},N}$ and larger error $\frac{\|x_k\|}{N}$ instead of $\frac{1}{N}$, which is of no harm since we implicitly quantify over all $N\in\mathbb{N}$. Setting all but one $\lambda_{ij}=0$ (and one to 1) we obtain from $\psi'_{m,\underline{d},N}(\underline{y})$ that $1\le\|y_{ij}\|\le 1+\frac{1}{N}\le 2$. The coefficients in $\varphi_{n,m,\underline{d},N}$ are in the interval $[-2,2]$ by the following reasoning: The y_{ij} are positive disjoint elements, and $\|x_k\|\le 1$ yielding $\left\|\sum_{i=1}^{m}\sum_{j=1}^{d_i}\lambda_{ij}y_{ij}\right\|\le 2$ which gives together with Lemma 7.6.A.7 that $\lambda_{ij}\in[-2,2]$. Finally, with the help of a construction $(x_1,\dots,x_n)\mapsto(x'_1,\dots,x'_n)$ from Lemma A.8 we can avoid the universal premise in $A_{n,N}$.
Using sequence types we can even avoid having an infinite list of axioms, in fact it is possible in our language to have only one axiom. To do so, we need some abbreviations:

Definition 7.9.

1. Set $1_{X(\mathbb{N})(\mathbb{N})} := \lambda n,m.1_X$ (constant 1_X-function of type $X(\mathbb{N})(\mathbb{N})$).
2. We set $\exists\lambda^{1(\mathbb{N})(\mathbb{N})(\mathbb{N})}\in[-2,2] := \exists\lambda \preceq_{1(\mathbb{N})(\mathbb{N})(\mathbb{N})} \lambda i,k,l.(\lambda n.j(2^{n+4}+1, 2^{n+2}-1))$.

We axiomatize BL^pL^q-Banach lattices in our language as follows.

Definition 7.10. We define the extension $\mathcal{A}^\omega[X,\|\cdot\|,\sqcup,p,q]$ of the theory $\mathcal{A}^\omega[X,\|\cdot\|,\sqcup]$ by adding the constants c_p, c_q of type 1 with the axioms $c_q \ge_{\mathbb{R}} 1_{\mathbb{R}}$, $c_p \ge_{\mathbb{R}} 1_{\mathbb{R}}$ and the axiom B:

$$B := \forall n^{\mathbb{N}}, N^{\mathbb{N}} \ge 1 \forall x^{X(\mathbb{N})} \exists y \preceq_{X(\mathbb{N})(\mathbb{N})} 2\cdot 1_{X(\mathbb{N})(\mathbb{N})} \exists \lambda^{1(\mathbb{N})(\mathbb{N})(\mathbb{N})} \in [-2,2]\left(\phi(n,N,x,y,\lambda)\right),$$

$$\phi(n,N,x,y,\lambda) := \exists m \preceq_{\mathbb{N}} n2^{nN} \exists d \preceq_{\mathbb{N}(\mathbb{N})} \lambda i.n2^{nN}\left(\sum_{i=1}^{m} d(i) \le_{\mathbb{N}} n2^{nN} \to \varphi(n,N,x,y,\lambda,m,d)\right),$$

where $\lambda i.n2^{nN}$is the λ-abstraction,

$$\varphi(n,N,x,y,\lambda,m,d) := \psi(N,y,m,d) \land \forall k \preceq_{\mathbb{N}} n \left(\left\|\tilde{x}(k)' - \sum_{i=1}^{m}\sum_{j=1}^{d(i)} \lambda(i)(j)(k) \cdot_X y(i)(j)\right\| \le_{\mathbb{R}} \frac{1}{N}\right),$$

$$\psi(N,y,m,d) := \psi'(N,y,m,d) \land \psi''(y,m,d) \land \psi'''(y,m,d),$$

$$\psi'(N,y,m,d) := \forall \lambda^{1(\mathbb{N})(\mathbb{N})} \left(\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{d(i)} |\lambda(i)(j)|_{\mathbb{R}}^{q}\right)^{p/q}\right)^{1/p} \le_{\mathbb{R}} \left\|\sum_{i=1}^{m}\sum_{j=1}^{d(i)} |\lambda(i)(j)|_{\mathbb{R}}\, y(i)(j)\right\| \right.$$
$$\left. \le_{\mathbb{R}} \left(1+\frac{1}{N}\right)\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{d(i)} |\lambda(i)(j)|_{\mathbb{R}}^{q}\right)^{p/q}\right)^{1/p}\right)$$

$$\psi''(y,m,d) := \forall i, i_0 \preceq_{\mathbb{N}} m \forall j \preceq_{\mathbb{N}} d(i) \forall j_0 \preceq_{\mathbb{N}} d(i_0)\left((i \ne i_0 \lor j \ne j_0) \to |y(i)(j)| \sqcap |y(i_0)(j_0)| =_X 0\right)$$

$$\psi'''(y,m,d) := \sum_{i=1}^{m} \sum_{j=1}^{d(i)} |y(i)(j) - |y(i)(j)|| =_X 0,$$

where $\tilde{x} := \dfrac{|x|}{\max\{\|x\|, 1\}}$ and $x(i)' := x(i) - x(i) \sqcap \left(\sum_{k \neq i} x(k)\right)$ for each $i \in \{1, \ldots, n\}$.

The axiom B can easily be written as an axiom Δ using the $\max\{\cdot\}$ operation for the bounded universal quantifiers and prenexing the two bounded existential quantifiers.

Remark 7.11. By having the universal closure instead of the infinite list of axioms we obtain a somewhat stronger theory. In terms of standard models ($\mathcal{S}^{\omega,X}$) both theories coincide - otherwise we would not have a correct axiomatization. However, the theory with the axiom B is stronger than the theory with an infinite list of axioms since more statements are provable (e.g. B).

Proposition 7.12 (cp. Proposition 3.5)**.** *Let $L^p(\Omega, U, \mu, L^q(\Omega', U', \mu'))$ be a band of a Bochner space (for $1 \leq p, q < \infty$). Then $\mathcal{S}^{\omega,X}$ becomes a model of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p, q]$ by letting the variables of type ρ range over S_ρ with the constants being interpreted as specified in Proposition 3.5 (interpreting c_q analogously to c_p). Conversely, any Banach lattice X such that $\mathcal{S}^{\omega,x}$ is a model of $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p, q]$ is isometrically isomorph to a band in some $L^p(\Omega, U, \mu, L^q(\Omega', U', \mu'))$-lattice.*

Proof. Follows from the discussion after Definition 7.8, Lemma 7.6.A.8, Remark 7.11 and the axiomatization of BL^pL^q-Banach lattices of [20]. □

Theorem 7.13 (Logical Metatheorem for BL^pL^q-Banach lattices)**.** *Let $\rho \in \boldsymbol{T}^X$ be an admissible finite type. Let $B_\forall(x,u)$, resp. $C_\exists(x,v)$, be $\forall$- resp. $\exists$-formulas that contain only the variables x, u resp. x, v free. Assume*

$$\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p, q] \vdash \forall x^\rho \left(\forall u^{\mathbb{N}} B_\forall(x,u) \rightarrow \exists v^{\mathbb{N}} C_\exists(x,v)\right)$$

then one can extract a partial functional $\Phi : S_{\widehat{\rho}} \rightharpoonup \mathbb{N}$ whose restriction to the strongly majorizable elements of $S_{\widehat{\rho}}$ is a (bar recursive) computable functional of $\mathcal{M}^\omega$ and the following holds for all bands of $L^p(\Omega, U, \mu, L^q(\Omega', U', \mu'))$ Bochner spaces (for $1 \leq p, q < \infty$): for all $x \in S_\rho$, $x^* \in S_{\widehat{\rho}}$ if $x^* \gtrsim_\rho x$ then

$$\forall u \leq \Phi(x^*) B_\forall(x,u) \rightarrow \exists v \leq \Phi(x^*) C_\exists(x,v).$$

Moreover, the supplements (1)-(4) *of Theorem 5.13 are also valid in this setting.*

Proof. The proof extends the proof of Theorem 5.13. The theory for BL^pL^q-Banach lattices $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup, p, q]$ has two new constant symbols c_p, c_q which are both majorizable (see Lemma 5.8) and is extending the theory $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$ by the axiom B which can be written as an axiom Δ (Definition 7.10). Thus everything follows from the proof of Theorem 5.13 and Corollary 5.14. □

Remark 7.14. Even more spaces can be added to be applicable for the above metatheorem. In [19, Example 8.3] the authors list spaces which can be axiomatized in positive bounded logic: normed algebras, C^*-algebras, dual pairs (X, X'), where X is a Banach space and X' is its dual space, triples (X, X', X'') and operator spaces. Proving and applying metatheorems for those spaces could be a natural sequel to this work.

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A. Appendix

Proposition A.1. *The axioms (B1)-(B10) are provable in* $\mathcal{A}^\omega[X, \|\cdot\|, \sqcup]$.

Proof. (B1): Let $x, y, z \in X$ such that $x \sqsubseteq y$ and $y \sqsubseteq z$. Then we have by (A3), extensionality (Proposition 2.11) and by assumption ($x \sqcup y =_X y$ and $y \sqcup z =_X z$)

$$x \sqcup z =_X x \sqcup (y \sqcup z) =_X (x \sqcup y) \sqcup z =_X y \sqcup z =_X z, \qquad \text{hence } x \sqsubseteq z.$$

The other axioms follow similarly (making free use of extensionality): (B2) is immediate from (A1). (B3) follows from (A2). (B4) follows from (A1) and (A3). (B5) is a consequence of (A4) and (A2). (B6) follows from (A3). (B7) follows from the definition of $\sqcap$ and axioms (A2),(A3),(A4). (B8) is a consequence of (A5). (B9) follows from (A6). The first conjunct of (B10) is immediate from (A7). The second conjunct follows from (A8). ☐

Proposition A.2. *The axioms from Definition 2.10 are true in any Banach lattice.*

Proof. 1. The law $x \sqcap y = -((-x) \sqcup (-y))$, which we used to define $\sqcap$, holds in any Banach lattice (see e.g. [42], Theorem 1.1.1).
2. The axioms (A1),(A2),(A3) and (A4) are fulfilled by [45, II, Section 1, p.48].
3. The truth of translation invariance (A5) in any Banach lattice follows from [45, p. 50].
4. Axiom (A6): Let $x, y \in X$ and $\lambda \in \mathbb{R}$. Then $|\lambda| \geq 0$. From $x \sqsubseteq x \sqcup y$ following from axiom (B4) we can use $(\mathrm{LO})_2$: $|\lambda| x \sqsubseteq |\lambda|(x \sqcup y)$ which is equivalent to $|\lambda| x \sqcup (|\lambda|(x \sqcup y)) =_X |\lambda|(x \sqcup y)$.
5. Axiom (A7) follows directly from axiom (B10).
6. Axiom (A8) can be inferred from axiom (B10) as follows. Let $x, y \in X$ and observe that $\forall u, v \in X(u \sqsubseteq u \sqcup v)$ is true. Thus we have $0_X \sqsubseteq 0_X \sqcup x \sqsubseteq (0_X \sqcup x) \sqcup y$ implying with (B10) $\|0_X \sqcup x\| \leq_{\mathbb{R}} \|(0_X \sqcup x) \sqcup y\|$.
7. Axiom (A9) can be proven from axiom (B10) together with [45, II, Prop. 1.4] as follows. We have for all $x_1, x_2, y_1, y_2 \in X$

$$\|x_1 \sqcup y_1 - x_2 \sqcup y_2\| =_{\mathbb{R}} \||x_1 \sqcup y_1 - x_2 \sqcup y_2|\|$$

and

$$|x_1 \sqcup y_1 - x_2 \sqcup y_2| \sqsubseteq |x_1 - x_2| + |y_1 - y_2|$$

which implies with the triangle inequality of the norm

$$\|x_1 \sqcup y_1 - x_2 \sqcup y_2\| \leq_{\mathbb{R}} \||x_1 - x_2| + |y_1 - y_2|\| \leq_{\mathbb{R}} \|x_1 - x_2\| + \|y_1 - y_2\|.$$

☐

Lemma A.3 ([42],Thm.1.1.1)**.** *Let a, b, c be elements of a Banach lattice X with $a, b, c \geq 0$. Then*

$$(a + c) \sqcap b \leq a \sqcap b + c \sqcap b.$$

Lemma A.4. *Let X be a Banach lattice and let $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_0 \neq 0$ and $x_1, x_2 \in X$ with $x_1, x_2 \geq 0$. Then*

1. $|\alpha_0 x_1| \sqcap x_2 = |\alpha_0| \left(x_1 \sqcap \frac{1}{|\alpha_0|} x_2 \right)$.

2. $|x_1| \sqcap |x_2| = 0 \to |\alpha_1 x_1| \sqcap |\alpha_2 x_2| = 0$.

Proof. 1. By [45, II, Prop. 1.4 and Cor. 1] we have

$$\begin{aligned} |\alpha_0| \left(x_1 \sqcap \frac{x_2}{|\alpha_0|} \right) &= \frac{1}{2} |\alpha_0| \left| \left| x_1 + \frac{x_2}{|\alpha_0|} \right| - \left| x_1 - \frac{x_2}{|\alpha_0|} \right| \right| \\ &= \frac{1}{2} \left| \Big| |\alpha_0|\, x_1 + x_2 \Big| - \Big| |\alpha_0|\, x_1 - x_2 \Big| \right| = |\alpha_0|\, x_1 \sqcap x_2 = |\alpha_0 x_1| \sqcap x_2. \end{aligned}$$

2. Suppose $|x_1| \sqcap |x_2| = 0$. If $\alpha_1 \cdot \alpha_2 = 0$ then (w.l.o.g. $\alpha_2 = 0$)

$$0 \leq |\alpha_1 x_1| \sqcap |\alpha_2 x_2| = |\alpha_1 x_1| \sqcap 0 \leq 0.$$

Otherwise, w.l.o.g. $0 < |\alpha_2| \leq |\alpha_1|$:

$$0 \leq |\alpha_1 x_1| \sqcap |\alpha_2 x_2| \overset{A.4.1}{=} |\alpha_1| \left(x_1 \sqcap \frac{|\alpha_2|}{|\alpha_1|} |x_2| \right) \leq |\alpha_1| \left(|x_1| \sqcap |x_2| \right) = 0.$$

☐

Lemma A.5. *Let $n, k \in \mathbb{N}$ with $n > k$ and $x_1, \ldots, x_n$ be pairwise disjoint positive elements of a Banach lattice X. Let $\alpha_1, \ldots \alpha_n \in \mathbb{R}$. Then $\sum_{i=1}^{k} \alpha_i x_i \perp \sum_{j=k+1}^{n} \alpha_j x_j$.*

Proof.

$$\begin{aligned} 0 \leq \left| \sum_{i=1}^{k} \alpha_i x_i \right| \sqcap \left| \sum_{j=k+1}^{n} \alpha_j x_j \right| &\overset{\text{[45, II, Prop. 1.4]}}{\leq} \sum_{i=1}^{k} |\alpha_i x_i| \sqcap \sum_{j=k+1}^{n} |\alpha_j x_j| \\ \overset{A.3}{\leq} \sum_{i=1}^{k} \left(|\alpha_i x_i| \sqcap \sum_{j=k+1}^{n} |\alpha_j x_j| \right) &\overset{A.3}{\leq} \sum_{i=1}^{k} \sum_{j=k+1}^{n} |\alpha_i x_i| \sqcap |\alpha_j x_j| \overset{A.4}{=} 0. \end{aligned}$$

☐

Lemma A.6. *Let X be a Banach lattice and let $n \in \mathbb{N}$. Then for all pairwise disjoint positive elements $x_1, \ldots, x_n$ it holds $\|\sum_{i=1}^{n} x_i\| \geq \max_{i \in \{1,\ldots,n\}} \{\|x_i\|\}$.*

Proof. By induction. For $n = 1$ the assertion is trivial. For the induction step we first note that $(\sum_{i=1}^{n} x_i) \perp x_{n+1}$ follows from Lemma A.5. Assume the statement holds for $n \in \mathbb{N}$, then

$$\begin{aligned} \left\| \sum_{i=1}^{n} x_i + x_{n+1} \right\| &\overset{\text{[45, II, Prop. 1.4]}}{=} \left\| \left(\sum_{i=1}^{n} x_i \right) \sqcup x_{n+1} + \left(\sum_{i=1}^{n} x_i \right) \sqcap x_{n+1} \right\| = \left\| \left(\sum_{i=1}^{n} x_i \right) \sqcup x_{n+1} \right\| \\ &\overset{(B4) \text{ and } (B10)}{\geq} \max \left\{ \left\| \sum_{i=1}^{n} x_i \right\|, \|x_{n+1}\| \right\} \overset{\text{IH}}{\geq} \max\{\max_{i \in \{1,\ldots,n\}} \{\|x_i\|\}, \|x_{n+1}\|\}. \end{aligned}$$

☐

Lemma A.7. *Let X be a Banach lattice and let $n \in \mathbb{N}$. Then for all pairwise disjoint positive elements $x_1, \ldots, x_n$ and all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ it holds $\|\sum_{i=1}^{n} \alpha_i x_i\| \geq \max_{i \in \{1,\ldots,n\}} \{|\alpha_i|\, \|x_i\|\}$.*

Proof. Suppose $\alpha_1, \dots, \alpha_k \geq 0$ and $\alpha_{k+1}, \dots, \alpha_n < 0$.

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| = \left\| \sum_{i=1}^{k} \alpha_i x_i - \sum_{j=k+1}^{n} |\alpha_j|\, x_j \right\| \overset{(B10)}{=} \left\| \left| \sum_{i=1}^{k} \alpha_i x_i - \sum_{j=k+1}^{n} |\alpha_j|\, x_j \right| \right\| \overset{\text{[45, II, Prop. 1.4]}}{=}$$

$$\left\| \sum_{i=1}^{k} \alpha_i x_i \sqcup \sum_{j=k+1}^{n} |\alpha_j|\, x_j - \underbrace{\sum_{i=1}^{k} \alpha_i x_i \sqcap \sum_{j=k+1}^{n} |\alpha_j|\, x_j}_{=0 \text{ by Lemma A.5}} \right\| \overset{\text{[45, II, Prop. 1.4]}}{=} \left\| \sum_{i=1}^{n} |\alpha_i|\, x_i \right\|$$

$$\overset{A.6, A.4}{\geq} \max_{i \in \{1,\dots,n\}} \{ |\alpha_i|\, \|x_i\| \}.$$

□

Lemma A.8. *Let X be a Banach lattice and let $n \in \mathbb{N}$. Then the following hold for all elements $x_1, \dots, x_n$:*

$$\left(\bigwedge_{i=1}^{n} (\|x_i\| \leq 1 \wedge x_i \geq 0) \to \bigwedge_{i=1}^{n} (\|x'_i\| \leq 1 \wedge x'_i \geq 0) \wedge \sum_{\substack{i,j=1,\dots,n \\ i \neq j}} |x'_i| \sqcap |x'_j| = 0 \right)$$

and

$$\left(\bigwedge_{i=1}^{n} \|x_i\| \leq 1 \wedge x_i \geq 0 \wedge \sum_{\substack{i,j=1,\dots,n \\ i \neq j}} |x_i| \sqcap |x_j| = 0 \right) \to \bigwedge_{i=1}^{n} x_i = x'_i,$$

where $x'_i := x_i - x_i \sqcap \left(\sum_{k \neq i} x_k \right)$ *for each* $i \in \{1, \dots, n\}$.

Proof. Let $x_1, \dots, x_n$ be positive elements of a Banach lattice X with norm at most 1. Positivity of x'_i follows from the positivity of x_i since $x_i \geq x_i \sqcap (\sum_{i \neq k} x_k)$. Also $\|x'_i\| \leq \|x_i\| \leq 1$ since $0 \leq x'_i \leq x_i$. The fact that the x'_j are disjoint can be checked as follows. For $i \neq j$:

$$\begin{aligned} |x'_i| \sqcap |x'_j| &= \left(x_i - x_i \sqcap \left(\sum_{k \neq i} x_k \right) \right) \sqcap \left(x_j - x_j \sqcap \left(\sum_{k \neq j} x_k \right) \right) \\ &\leq (x_i - x_i \sqcap x_j) \sqcap (x_j - x_j \sqcap x_i) = x_i \sqcap x_j - x_j \sqcap x_i = 0. \end{aligned}$$

For the second claim, let $x_1, \dots, x_n$ be positive disjoint elements with norm at most 1. Then

$$\|x_i - x'_i\| = \left\| x_i - \left(x_i - x_i \sqcap \left(\sum_{k \neq i} x_k \right) \right) \right\| = \left\| x_i \sqcap \left(\sum_{k \neq i} x_k \right) \right\| \overset{A.3}{\leq} \left\| \sum_{k \neq i} (x_k \sqcap x_i) \right\| = 0.$$

□