The Context of Inference

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"I am eager to examine together with you, Crito, whether this argument will appear in any way different to me in my present circumstances, or whether it remains the same, whether we are to abandon it or believe in it."—Plato *Crito*, 46d

1. Ambiguity

There is an ambiguity in the fundamental concept of deductive logic that went unnoticed until the middle of the 20th Century. Sorting it out has led to profound mathematical investigations with applications in complexity theory and computer science. The origins of this ambiguity and the history of its resolution deserve philosophical attention, because our understanding of logic stands to benefit from an appreciation of their details.

The ambiguity can be framed simply by asking what it is we seek in a rule of inference. Minimally, we say, it ought not lead us from truths to falsehoods. It must instead take us from truths only to other truths. Perhaps there are other conditions we should impose on an inference rule before calling it logically valid. Some suggest, for example, that it should be formal, meaning that its application can be triggered by the mere linguistic form of the data we are reasoning about irrespective of its content. But these sorts of considerations have to do with what makes the rule properly logical rather than what makes it deductively valid. Before specifying them, one might worry whether we've said enough about where the rules might lead us.

Suppose I am reasoning from premises that are false. Is there still a right way of going about things, or does the falsity of my starting point so spoil my project that it doesn't really matter what principles govern my reasoning? Nearly everyone agrees that a bad premise cannot get me off the hook so easily. I still can, and so in some sense ought to, reason rightly by adhering to valid rules of inference. But what are the valid rules of inference? Perhaps the same rules that lead from truths to truths are available to me even when my premises are false. But do they remain valid in my unfortunate circumstances? And are there others, rules that don't lead from truths to truths but still fare well when one starts with falsehoods? What if one of my initial assumptions is absurd or the collective of them are mutually contradictory? What distinguishes a valid inference from an invalid one in such circumstances?

The first corrective to this line of thought, the one nearly all logical theorists recommend, is to drop the preoccupation with which sentences are true and which false. The logical validity of an inference rule cannot depend in any way on which sentences happen to be true. Logic is the study of the relationships among sentences independent of all such contingencies. One cannot need to know what sentences are true in order to determine if an inference rule is valid, so it cannot do to define validity as the property of leading from truths to other truths.

What matters for logic is not truth but *logical truth*, the property of being true for reasons of logic alone. There are as many, or more, competing theories about what this notion amounts to as their are different logical systems, and we do not wish to accentuate any of their details. Most theorists, regardless of their affinities with one or another particular specification of what the logical truths are, recognize a class of privileged expressions—their logical system's *theorems*—that are, instead of expressions that just happen to be true, the objects that inference rules have a particular responsibility towards.

Here, then, is a possible specification of a logic's deductively valid inference rules: they are the rules that lead from logical truths to other logical truths. Indeed, the now storied tradition of defining a logic as nothing more than a set of theorems strongly recommends this specification. The valid inference rules are just the rules under which the logic is closed in the algebraic sense: by following them, one cannot start off within the set of theorems and wind up outside this set.

A possible problem with this account of deductive validity is that the theorems of a logical system are themselves often thought of as just those expressions that are generated by logically valid rules. Explaining what it is for a rule to be valid in terms of its action on the set of theorems then appears circular or at least puzzlingly recursive. We point this out now as a reference point but won't address the issue in detail until a later section.

Another possible problem with this account of deductive validity is that it doesn't directly speak to the question about reasoning properly from assumptions that might not be logical truths. Ought we think that the rules that reliably lead from theorems to other theorems are also the appropriate principles to govern inferences from arbitrary assumptions? Could some of them let us down in this more general activity? Alternatively, could there not be rules that fail to operate reliably on the set of theorems but manage to direct our reasoning appropriately when we reason from other sorts of assumptions?

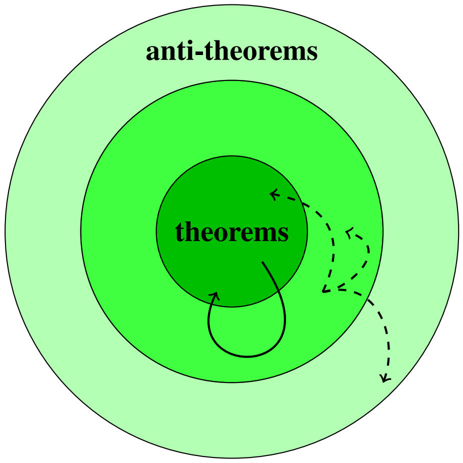

Figure 1. Arrows represent the action of a single premise inference rule on the formulas of a language. Given that the set of theorems is closed under this action (solid arrow), what can be said about the rule's behavior on the other formulas (broken arrows)? And does the concept of validity stipulate anything at all for the rule's action on unsatisfiable formulas?

However one answers these questions, to even ask them is to have in mind that there is something more to deductive validity than the set of theorems being closed under a rule's action. Often this something more is expressed as the condition that an instance of the rule's conclusion must be true under the assumption that the corresponding instances of its premises are true. This expression is problematic because it leaves undetermined what to say about rules, such as those with contradictory premises, whose premises cannot be assumed to have true instances.

There are several well-known responses to this issue. A common feature is to again prescind from questions of truth, formalize the notion of assumption by specifying a *context*—a set of assumed sentences—and say that a rule is valid if, whenever instances of its premises hold in a context, the corresponding instance of its conclusion holds in the same context. This is a straightforward generalization of the earlier notion because defining validity as the set of theorems being closed under a rule's action is just defining it in terms of the rule's behavior in empty contexts.

The ambiguity in the notion of deductive validity is just this. Shall we say that a rule $\frac{A \quad B}{C}$ is valid in case (1) instances of its conclusion are theorems when the corresponding instances of its premises are?

$$\frac{\vdash A \qquad \vdash B}{\vdash C}$$

Or shall we say that a rule is valid in case (2) instances of its conclusion hold in whatever contexts the corresponding instances of its premises hold?

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash C}$$

Socrates was right to ask the question he asked to Crito. If an inference rule is valid in the second sense, then it is good in all contexts. Clearly it is good, then, in the empty context: it is a reliable guide from premises to conclusion under no assumptions whatsoever—a reliable guide from theorems to other theorems. By contrast, a rule might be valid in the first sense in part due to structural features of the set of theorems. For all we know pre-theoretically, these features may not be preserved under contextualization. Socrates was notorious for reasoning in the clouds. On a day when his life is on the line, ideas of mortality, family, and the consequences of following through with his decisions might be unshakable. Even if these ideas don't figure directly into his reasoning, they very well might deform the space in which he is reasoning, so that arguments that were water-tight on previous occasions must now be abandoned.

2. History

Despite the fact that these two conceptions of deductive validity are different, many students of logic struggle to see that they are. In fact, it is not uncommon for professional scholars to conflate them in writing even today.[1] There is a theoretical explanation of this phenomenon that we will review later, but there is also a plausible genetic explanation that is important to understand first.

If one examines logical monographs from the early 20th Century, when the logicist style of presenting logical systems in terms of axioms and inference rules prevailed, one will typically find very little discussion of what the validity of an inference rule amounts to. In his (*1930*) completeness paper, for example, Gödel simply says that from the premises, the conclusion "follows."

Does Gödel mean that it follows, whenever the rule's premises are instantiated as theorems, that the corresponding instance of the rule's conclusion is a theorem as well? Or does he mean something possibly stronger? It is not easy to tell that he meant anything in particular. The first inference rule he specifies, *modus ponens*, is thought to be reliable not only when reasoning one's way from theorems but more generally when reasoning from arbitrary assumptions. The second, and only other, rule in Gödel's study—allowing one to infer from a formula any of its substitution instances—is known to fail in a general setting: While the act of taking a substitution instance preserves theorem-hood, it does not preserve satisfiability. There are reasons why one might not want to think of substitution as an inference rule at all. But setting those aside for now, it is clear that its conclusions do not follow from its premises in quite the same or all the same ways that *modus ponens*'s conclusions follow from its premises.

[1] See *Hakli and Negri 2012* for documentation of a remarkably sustained debate about the validity of the Deduction Theorem in certain modal logics that turned on just this point.

Does this suggest that Gödel was an exponent of the first concept of deductive validity? I think it is more properly understood as evidence that Gödel, like many other logicians of his time, simply had not disambiguated the two concepts. The fact that *modus ponens* can be used in formal derivations that start with arbitrary assumptions whereas substitution cannot is so apparent nowadays that it is hard to imagine a logical genius like Gödel categorizing them both as "rules of inference" whose conclusions "follow" from their premises without drawing attention to the serious difference in the senses in which they do. But Gödel never took up an investigation of the concept of derivation from assumptions in his early work. Even in stating his completeness theorem, he only says that all universally correct formulas are formally derivable (If $\models \mathrm{A}$, then $\vdash \mathrm{A}$), stopping short of the trivial extension to the claim that from a set of sentences all the semantic consequences can be formally derived (If $\Gamma \models \mathrm{A}$, then $\Gamma \vdash \mathrm{A}$).

Gödel was not alone. According to John Dawson (*1993*), it was not until *Robinson 1951* that the completeness of first-order quantification theory was expressed in this general way. If all one wants out of a completeness theorem is verification that from a set of inference rules, all and only logical truths can be derived (or, even more modestly, verification that a set of theorems is recursively enumerable—an attitude that persists among many logicians today[2])—it does not matter that the inference rules be, individually, anything more than transformations under which that set is closed. The historical record suggests that the way early 20th Century logicians conceived of their craft simply did not provide an occasion for disentangling the two senses in which an inference rule could be said to be valid.

3. Pre-history

Because it is so natural to expect an account of deductive validity to be applicable in arbitrary and even informal contexts, it is tempting to conclude that the entanglement of this notion with the more modest condition of leading from theorem to theorem is just the residue of the exclusive attention that Gödel and other mathematical logicians paid to formal logical systems in the early 20th Century. Did their scientific interest in mathematical characterizations of these systems in terms of completeness, decidability, and related properties blind them to any subtleties in the concept of validity that did not bear directly on these characterizations?

Actually, the entanglement has deeper roots. Some influential thinkers who resisted the meta-theoretical study of logic or even the whole project of formalizing logic mathematically

[2]I have heard Anand Pillay, Julia Knight, Yuri Gurevich, and other prominent logicians describe the "real meaning" of the completeness theorems in these terms, typically to emphasize their own computational or model theoretic approach to logic. Gurevich (*1984*) provides an early remark in this vein: "The calculus-independent meaning of this theorem is that first-order logic is recursively axiomatizable, which boils down to the fact that valid formulas are recursively enumerable" (p. 179).

also resisted subtleties in the notion of deductive validity. One can hardly find two attitudes about the nature of logic more at odds with one another than those of Frege and Brouwer. But they agreed about one fundamental thing. Both men emphasized that logical inference rules have no responsibility to operate in any particular way on arbitrary sentences or thoughts. If this seemingly common idea did not directly inspire later mathematical logicians' inattention to the behavior of inference rules in arbitrary contexts, it at least provided a comfortable environment for that inattention to go unnoticed and perpetuate with impunity.

3.1 Frege

Frege's stance on these matters apparently derives from his insistence that his logical system is meant "to express a content through written symbols in a more precise and perspicuous way than is possible with words" (*Frege 1883*, pp. 90–91). Fearing that his system would be misunderstood as an abstract formal calculus fit to accommodate a wide range of interpretations, Frege emphasized that he intended the formal precision of his (*1879*) "concept script" to do just the opposite: it is "a system of symbols," he said, "from which every ambiguity is banned"; its "strict logical form," far from accommodating a study of logical relationships independent of considerations of meaning, is something "from which content cannot escape" (*Frege 1882*, p. 86).

Among the great ironies in the history of logic is the fact that these very features paved the way for contemporary metatheory, the point of view from which logical formulas can be studied as mathematical objects, their form and content separable and independently adjustable. Frege objected to this way of studying logic, not only because he saw it as an abuse of his own innovations, but because he thought it rested on a fundamental misunderstanding of logical inference. When Hilbert suggested that the joint consistency of a collection of sentences could be established by concocting a reinterpretation of them all so that they could be read as simultaneously true, Frege pointed out that by reinterpreting a sentence you are left with a different sentence, so that this exercise cannot disclose anything about the sentences one began with (*Frege 1980*, pp. 39–40). To see that a collection of sentences are mutually consistent, one must attend to what they actually mean.

Further in this vein, and even more striking, is Frege's rejoinder to Hugo Dingler's proposal that one can determine that a collection of sentences are incompatible with one another by assuming them as premises and "inferring logically" a contradiction. Frege wrote[3]:

> Is this case at all possible? If we derive a proposition from true propositions according to an unexceptionable inference procedure, then the proposition is true. Now since at most one of two mutually contradictory propositions can be true,

[3] *Frege 1917*, pp. 16–17.

> it is impossible to infer mutually contradictory propositions from a group of true propositions in a logically unexceptionable way. On the other hand, we can only infer something from true propositions. Thus if a group of propositions contains a proposition whose truth is not yet known, or which is certainly false, then this proposition cannot be used for making inferences. If we want to draw conclusions from the propositions of a group, we must first exclude all propositions whose truth is doubtful. ... It is necessary to recognize the truth of the premises. When we infer, we recognize a truth on the basis of other previously recognized truths according to a logical law.

Frege's point in this passage is obscure and jarring, but it is not isolated to this one exchange. He emphasized in several other places that logical inference cannot begin with arbitrary assumptions, but only with premises which have been judged to be true:

> "... before recognizing its truth one cannot use a Thought as a premise of an inference, nor can one infer or conclude anything from it."[4]

> "An inference ... is the passing of a judgment made in accordance with logical laws on the basis of previously passed judgments. Each of the premises is a determinate Thought recognized as true; and in the conclusion too, a determinate Thought is recognized as true."[5]

> "Of course we cannot infer anything from a false Thought."[6]

> "From false premises nothing at all can be concluded. A mere Thought, which is not recognized as true, cannot be a premise. Only after a Thought has been recognized by me as true, can it be a premise for me. Mere hypotheses cannot be used as premises."[7]

Trying to make sense of these words, Frege's interpreters have suggested a wide range of things he might have meant. Perhaps he is stressing simply that inference is an activity performed on assertions that one puts forward in earnest rather than a mere relation binding the contents of these assertions. "[C]ertainly he admitted the possibility of inference from a thought which is mistakenly asserted, i.e. from a thought whose truth is mistakenly acknowledged" (*Stoothof 1963*, p. 407). Or perhaps Frege's remarks foreshadow the inferentialist doctrine of Dummett and Martin-Löf, according to which inference is to preserve, not truth, but justification, so that the grounds for believing the premises (which are absent in the case that the premises are mere hypotheses) are carried over by the inference to grounds for believing the conclusion (*Currie 1987*, *Smith 2009*).

[4] *Frege 1923*, p. 402.
[5] *Frege 1906*, p. 318.
[6] *Frege 1918*, p. 375.
[7] *Frege 1910*, p. 82.

Neither of these proposals are plausible reconstructions of what Frege has in mind in his note to Dingler. One could very well "recognize as true" some sentences that in fact are false. If this is all Frege insists one must do in order to be able to infer from those sentences, then inferences of contradictions would be not only possible but common. Frege's whole point is that inferring contradictions is impossible and therefore not an available method for discovering that a collection of sentences are incompatible. This can only be because inferences must have premises which are in fact true.

In saying that an inference's premises must be judgements, Frege therefore means that the premises must be rightly judged as true. Could this be because inference plays double duty, extracting from premises both what follows from them and also its justification? Surely not. What does it matter if Dingler's method fails to ensure that one's original reasons for endorsing an inference's premises provide, through that inference, reasons for endorsing a contradiction? So long as the inference is "unexceptional," the mere fact that the contradiction follows should be reason to rethink one's reasons for endorsing all the premises in the first place.

Instead, Frege is stressing that even though the form of an expression suffices to determine if it can stand in the premise position of a particular inference rule, logical inference is no mere transformation among the syntactic forms of expressions. It is a consideration of what things must be true given that some premises are. What if some putative premises are not in fact true? We are accustomed to asking what would have to be true under the counterfactual assumption that they were. Already this colloquial expression of deductive inference is problematic when the premises are not only false but incompatible with one another, for in that case what could it even mean to suppose they are all true and see what follows? The supposition itself is incoherent.

For Frege, this same problem just arises sooner. Inference is a way of deriving true, meaningful sentences from other true meaningful sentences. Consistent with his remarks to Hilbert, Frege's claim is that by hypothesizing the truth of a sentence that is really false in order to see what else must be true under that assumption, one loses contact with the meaningful sentence that one expected to be working with. What makes a rule valid is the fact that it leads reliably from truths to truths. Using a valid rule to "derive" a sentence from other sentences that are not true is not logical inference because such "pattern matching" uncovers, not what follows from meaningful sentences, only what is derivable from meaningless forms.

If it is controversial to read Frege this way, it is still not the whole story. For one thing, on plenty of occasions Frege used "unexceptional inference procedures" in just the way his remarks in the quoted passages seem to prohibit. In the same letter to Dingler, for example, Frege claims that from the thought that 2 is less than 1 and the thought that if something is less than 1 then it is greater than 2, one can derive that 2 is greater than 2. Even more alarmingly, Frege seems, on this reading, to be investing logic with a sensitivity to the contingencies of factual truth—something that logicians have long and universally agreed cannot be a relevant factor in the logicality of an inference.

These two puzzles are linked. When Frege does present derivations, following patterns of inference he recognizes as valid, with false or possibly false hypotheses in the premise position, he always insists, in keeping with his remarks quoted earlier, that these derivations do not exhibit genuine inference. Why present them at all, then? Frege has another purpose: using a valid rule to produce a derivation is a way of justifying the truth of a conditional statement with the derivation's "premises" in its antecedent position and the derivation's "conclusion" in its consequent position. Notice that in justifying a conditional statement in this manner, one does more than demonstrate that it is true. By deriving with valid laws its consequent from its antecedent, one has shown that the conditional is a *logical truth*.

Inference rules earn their status as deductively valid by operating appropriately on truths. But their application is broader. When they operate on false premises, the result is not an inference. Still, it is a verification that a certain conditional statement in which those premises occur as antecedent is itself a logical truth. And, in fact, all the conditional statements Frege recognizes as true judgements are verifiable either in this convenient fashion or directly with proofs from axioms in the concept script. To be a "true" conditional statement in the sense relevant to Frege is to be logically true. It is hardly a stretch to sense that Frege's apparent preoccupation with truth and falsity is an illusion. He does not distinguish logical truth from contingent truth in his logical writing, for the simple reason that he is studying logic. The sentences that he calls true are always true in the sense relevant to that study, i.e., not merely as a matter of contingent fact but demonstrably so. When Frege says that inferences must have as their premises, not mere hypotheses but judgements, he means that an inference's premises must be sentences determined to be true in the logically relevant sense—they must be logical theorems.

For Frege, deductively valid inference rules can be used generally, with arbitrary sentences in their premise positions, for the purpose of discovering (logically) true conditional statements. But to *infer* with such a rule is to operate with it on the space of logical truths. And *what makes it valid* is the fact that this space is closed under its action. Frege has arrived very near the first of our two senses of deductive validity. More, he has declared the second, stronger, sense of deductive validity incoherent. What makes an inference rule valid cannot be its behavior in arbitrary contexts, for its use in such contexts does not even qualify as inference.

Frege's two ideas—(1) that the validity of an inference rule depends only on its behavior on the space of true judgements and (2) that our license to use a rule in arbitrary contexts to form true conditional judgements derives from the fact that the space of true judgements is closed under its action—both arise from his deeply held view that logical expressions are inherently and unambiguously meaningful and that inference is fundamentally an operation on meaningful judgements. It is fascinating to see how Brouwer's rejection of these very tenets led him to conclusions about deductive validity strikingly similar to Frege's.

3.2 Brouwer

In some of his early writing, while railing against classical logic and specifically the law of excluded middle, Brouwer proposed that we stop thinking of "logical principles" as "*a priori* laws governing fetish-like concepts and their linkages" (*1928*, p. 1182). Think of them instead, he suggested, as "practically reliable" means of transitioning from one verifiable statement to another. Such principles, he suggested[8], are validated *a posteriori* by observing that

> [w]hen one applied these principles purely linguistically, i.e. derived linguistic expressions from other linguistic expressions with their help, without thinking about the mathematical contemplations indicated by these statements, it turned out that *the principles proved themselves*, i.e. it was found that every statement obtained in this way was capable of triggering an actual mathematical contemplation which turned out to be practically "identical" for all linguistically raised men . . . .

The critique of logic underlying this view is thoroughgoing. Whereas Frege had hoped that a more precise language would lock meaning in, Brouwer insisted that all language is meaningless on its own and serves only to trigger an agent to engage in meaningful activity. A statement of a mathematical result is successful, not because it expresses a fact, but because in practice the statement triggers an intended contemplation in persons habituated to the language. So too, a written proof is not a record of the sequence of mathematical contemplations leading to the verification of a mathematical fact, it is just an object which persons can use to inspire in other appropriately trained persons the same series of contemplations.

The chief culprit in what Brouwer perceived as a widespread insensitivity to the "natural" and "practical" role of language in our lives are the rules of logic. Often Brouwer is described as a defender of some logical laws as true as opposed to a select few laws, like the principle of excluded middle, which are false. This is a misrepresentation of the most significant feature of Brouwer's view of logic. For Brouwer, no logical laws are true, because all of them are meaningless. Their place in our lives amounts to no more than the fact that with some of them we can fairly reliably predict which expectations will be fulfilled and which frustrated.

For example, if I experience a mathematical construction, I might subsequently write

[8] *Brouwer 1928*, p. 1181.

down a linguistic expression that will later be useful to trigger in me the same experience. Because this inscription is a linguistic object, I can also transform it in various ways according to rule-governed manipulations. Some of these transformations will leave me with linguistic objects that will trigger in me other mathematical experiences. Others will not.

Call a linguistic expression that successfully triggers a series of contemplations that lead to the verification of a mathematical idea a proof. Among instructions for inscription manipulation, some reliably transform proofs into proofs. Others operate only on individual sentences that occur in proofs. Of special interest is a proof's characteristic expression, the individual sentence that corresponds to the mathematical contemplation of the idea that the series of experiences triggered by the proof ultimately verifies. Among instructions of this sort, some reliably transform a proof's characteristic expression into another expression for which there is a proof. These are what Brouwer calls valid logical rules.

Brouwer cautions against deciding prematurely that a transformation of this sort is a valid logical rule (p. 1182). His diagnosis of what he thinks is the fallacious acceptance of excluded middle illustrates this well. For much of human history, that rule proved to be as reliable as any other. But, Brouwer suggests, that is only because the sorts of problems we reasoned about were so simple. When our culture turned to more complex contemplations, especially those involving infinite collections or sequences, we continued using excluded middle out of habit, endorsing on its authority statements for which no proof (in Brouwer's sense) can be given.

When we discovered this about specific statements generated in this way, Brouwer claims, our habituated response was to say that the statement is nevertheless true because it follows from other true statements by a logical law. Brouwer's reply is that if the law leads from statements for which there is a proof to statements for which there is no proof, then the law is not valid after all. Because these laws "proved to be correct" throughout a long history of simple applications and always "seemed to be working if they were generally applied to the language of science or to events of other parts of practical life" and then "checked," we came to "accept and trust assertions derived by means of the logical principles even when these could not be subjected to direct check" (p. 1181). But if the correctness of a law amounts to nothing more than its reliable generation of statements that trigger appropriate experiences, there is no sense in maintaining its validity or the truth of statements attained with it when there is no experience to verify.

The details of this impasse are telling. The logicists understand linguistic expressions as meaningful and therefore true or false completely independently of what sorts of experiences they trigger. One could use a valid inference rule to discover definitively that a statement is true, just by feeding it premises that are known to be true, even if there is no independent verification to perform. The discovery of statements that can be derived in this way with excluded middle despite there being no direct way to experience their truth tells us that some things can only be discovered through logical inference. Rules of deduction become an essential part of the activity of proof.

For Brouwer, linguistic statements are meaningless and worthwhile only to the extent that in practice they trigger in us the right sorts of experiences. From this point of view, one cannot accept a statement *because* it follows logically from some other statement one accepts. In exactly the opposite way, an inference rule can be called valid *because*, and to the extent that, what one can derive with it from acceptable statements are reliably statements that are acceptable on independent grounds. Which rules do this and which do not cannot be determined by studying language and the meaning of parts of the rule. We only learn through experience that a rule is reliable by generating evidence that it is. Ample enough experience will be grounds to use the rule to make informed predictions about what experiences and mathematical constructions are possible, given that others are. But no amount of evidence can be conclusive, and on any future application the rule could let us down. Logical inference, therefore, can never be part of the activity of proof. It can only ever be a fallible way of transforming the linguistic inscriptions of proofs to generate new inscriptions that likely correspond with mathematical constructions that can be performed.

Frege claimed that "When we infer, we recognize a truth on the basis of other previously recognized truths according to a logical law." Brouwer denied that linguistic expressions can be true or false. He thought that logical inference was an operation performed on meaningless expressions. Brouwer further denied that any inference rule could be deemed "logically unexceptional"; there are only rules that have not yet been observed to transform expressions that successfully trigger mathematical contemplations to expressions that don't. In place of Frege's claim, Brouwer might have said, "When a logical law routinely transforms expressions that trigger verifiable expectations to other such expressions, one might justifiably use it in the future, expecting continued success."

Thus there is little common ground between Brouwer's conception of logic and Frege's. But like Frege, Brouwer apparently agrees that the business of a deductive inference rule is to lead from theorem to theorem or, in his terminology, from "expressions capable of triggering actual mathematical contemplations" to other such expressions. Setting aside the practical difficulty Brouwer stressed of ever determining that a rule will reliably do this, it is clear that what deductive validity would amount to, for him, is for the set of mathematical theorems to be closed under a rule's action. What becomes of other expressions transformed according to the rule is immaterial.

Frege and Brouwer are not the only philosophers who anticipated the first of the two senses of deductive validity we have distinguished. In fact their examples illustrate that conceiving of deductive validity as the closure under a rule of a set of theorems is not tied down to any one tenet about the nature or function of logic. For all the peculiarity of their ideas, Frege and Brouwer could not have been further apart on these matters, but they each arrived very close to the same concept of deductive validity. That concept may not often have been articulated as well as in their writing, and it may never have been the prevailing idea among logicians, but one can appreciate that at the time of the maturation of mathematical logic between 1920 and 1950 the philosophical currents did not lead readily to the more general, contextual, conception of validity and that even distinguishing the two ideas required paddling a bit upstream.

4. The emergence of admissibility

Although Brouwer's main gripe was not with any one logical principle but with what he took to be a confused understanding of the whole enterprise of formal logic, many of his followers took seriously a project that cannot be easily reconciled with his more critical and skeptical remarks: the specification of an "intuitionistic logic" whose principles each are valid in the anthropomorphic sense described above.

For nearly a century now philosophers have debated whether such a project makes sense. Some have pointed to Brouwer's own provisional attitude towards any hitherto reliable inference rule as evidence that a final intuitionistic logic could never be established. Others have indicated that developments in computer science and category theory have provided an environment where the correct principles of constructive inference can be identified. Still others have tried to let Brouwer's own dim remarks about language and formality, as well as his abiding disinterest in the development of intuitionsitc logic, disparage the enterprise. None of these considerations has been completely persuasive. In fact, even Brouwer did not have an obviously consistent opinion on the topic.[9]

Be that as it may, the study of intuitionistic logic is now a formidable scientific endeavor. Especially noteworthy for us is that its development has twice led to insights about the nature of deductive validity, particularly the precise distinction between the two senses of validity we have described. This occurred first in the debate about which principles to include in its formulation and later in observations about the immanent properties of the candidate formal systems.

The debate centered on the logical rule $\frac{\bot}{\mathrm{A}}$ countenancing any arbitrary sentence (A) under the assumption of a contradiction. Should this rule, *ex falso quodlibet*, count as valid on the intuitionistic understanding of logical inference?

[9] In his *2005*, van Dalen observes that Brouwer once expressed gratitude that Heyting produced his calculus so that Brouwer didn't have to do so himself (p. 676). Compare van Atten's observation (in his *2009*) that Brouwer thought Heyting's axiomatization of IPC was more important than Gödel's incompleteness theorem.

4.1 Kolmogorov

An early instance of a logician grappling with this question is Kolmogorov (*1925*). His assessment is puzzling. On the one hand, he flatly declared that *ex falso quodlibet* "does not have and cannot have any intuitive foundation since it asserts something about the consequences of something impossible." He therefore ruled the principle out "as an axiom." However, he maintained that, despite this, the principle could nevertheless "be proved on the basis of other [intuitionistically acceptable] axioms" (p. 421). He even declared that so long as the principle "is used only in a symbolic presentation of the logic of judgements," it "is not affected by Brouwer's critique" (p. 419).[10]

Ordinarily one expects a principle derivable from well-founded and acceptable principles to be deemed well-founded and acceptable itself. This leaves one wondering just what Kolmogorov had in mind. One pertinent consideration for sorting out his intention is the fact that he didn't directly impugn the intuitive validity of the principle. He only described his attempt to make sense of the rule as a non-starter. If, following Brouwer, the validity of an implication has to do with what things are intuitionistically provable given that some antecendent formula is intuitonistically provable, then the *ex falso quodlibet* principle is meaningless in a more pressing sense than the one in which Brouwer claims all logical principles are meaningless, for it asks us to consider what things are provable given the impossible: the provability of something unprovable. This passage suggests that Kolmogorov might have thought of intuitionistic logic as needing to be built up from "axioms" that can be given a direct intuitive meaning, from which might follow other logical laws that cannot be so understood.

Kolmogorov's position is close to what one should expect given the association of Brouwer's thought with the first of the two concepts of deductive validity. If a valid rule of inference is one that leads from theorems to theorems, then the validity of a rule explicitly about inference from something that cannot be true may not be directly evaluable.

Heyting, however, expressed no reservations about the *ex falso quodlibet* principle and included it as an axiom, in the form $\neg A \to (A \to B)$, in his formalization of the intuitionistic propositional calculus. Here[11] is his justification:

[10] The conclusion of this last sentence (Аксиома 5-ая употребляется только в символическом изложении логики суждений, поэтому критика Brouwer'а не коснулась ее, тем не менее она также не имеет интуитивных оснований) is mistranslated by van Heijenoort as "especially since it has no intuitive foundation either," suggesting that Kolmogorov thought that the acceptability of the principle in part *derives* from its lack of an intuitive foundation! The sentence is better translated as "Axiom 5 is used in only the symbolic rendering of the logic of judgments; therefore, Brouwer's criticism does not apply to it. Nevertheless, it, too [i.e., like the law of double negation], does not have intuitive foundations," expressing Kolmogorov's opinion that the principle could be used *despite* its lack of intuitive meaning. Thanks to Melissa Miller for verifying the translation.

[11] *Heyting 1956*, p.102.

> You remember that $\mathrm{A} \rightarrow \mathrm{B}$ can be asserted only if we possess a construction which, joined with the construction A, would prove B. Now suppose that $\vdash \neg \mathrm{A}$, that is, we have deduced a contradiction from the supposition that A were carried out. Then, in a sense this can be considered as a construction, which, joined to a proof of A (which cannot exist) leads to a proof of B. I shall interpret the implication in this wider sense.

If you expect this argument for the intuitionistic validity of *ex falso quodlibet* to be controversial, you will not be disappointed. The recorded attacks and defenses of the *ex falso quodlibet* rule form a vast literature. At the very least, one can say that the reasoning according to which constructions of both A and $\neg$A are supposed by Heyting to lead to a construction of B is somehow circular. For to the question, "How exactly am I supposed to produce a construction of B out of the hypothetical constructions of A and $\neg$A?" Heyting appears just to reply, "You aren't. But because you could never have constructions of both A and $\neg$A, it will plainly never be the case that you *both* have constructions of these and yet are unable to construct B." This transformation of the question "Can one reliably produce a construction of B out of a hypothetical construction of A?" to "Can one ever have a construction of A without being able to construct B?" departs from principles of constructivity, because the formulas $\mathrm{A} \rightarrow \mathrm{B}$ and $\neg(\mathrm{A} \wedge \neg \mathrm{B})$ are not obviously equivalent: whereas the derivation of $\neg(\mathrm{A} \wedge \neg \mathrm{B})$ from $\mathrm{A} \rightarrow \mathrm{B}$ is straightforward, the converse implication (the one Heyting invoked) is essentially classical and therefore depends on the validity of *ex falso quodlibet* (and more). It is unclear why an explanation of constructive implication could rely on non-constructive reasoning about vacuous cases, or why the defense of *ex falso quodlibet* in particular could rely on a reformulation of the intuitionistic concept of implication that is based on this very principle.

4.2 Johansson

In a series of letters to Heyting and a (*1937*) paper in *Compositio Mathematica*, the Norwegian mathmematician Ingebrigt Johansson clarified this issue substantially. In his clarification he provided what is likely the first articulation of the two types of deductive validity we have distinguished.[12]

In his paper, Johansson objected to Heyting's inclusion of *ex falso quodlibet* (in the form $\neg \mathrm{A} \rightarrow (\mathrm{A} \rightarrow \mathrm{B})$)[13] as an axiom of intuitionistic logic, questioning how it is that an implication, constructively understood, could follow from the absurdity of its antecedent. In his second letter to Heyting, he elaborated this objection with the observation that the axiom appears to state that "once $\neg$A has been proved, B follows from A, even if this had not been the

[12] These letters were brought to the author's attention by Tim van der Molen's analysis of them in his *2016*. The reader is encouraged to consult that paper for further details.

[13] In his first letter to Heyting, he focussed instead on the formula $(\mathrm{A} \wedge \neg \mathrm{A}) \rightarrow \mathrm{B}$.

case before." A restatement of Johansson's point in terms of construction might be to say that if I currently have no idea how to effectively transform a hypothetical construction of A into a construction of B, the demonstration that I could never be given a construction of A does not solve this problem in the sense that I am seeking it. After all, mathematics is full of meaningful ideas about how certain problems could be solved provided that a solution to some other problem is presented quite independently of any understanding of whether that other problem could even be found.

But Johansson pressed further, observing that the deletion of this principle leaves other principles underivable. In the paper, Johansson focussed on $((\mathrm{A} \wedge \neg\mathrm{A}) \vee \mathrm{B}) \to \mathrm{B}$; in his first letter to Heyting, he focussed on $((\mathrm{A} \vee \mathrm{B}) \wedge \neg\mathrm{A}) \to \mathrm{B}$ (*disjunctive syllogism*). In both cases the point is that the principles behind these formulas, unlike *ex falso quodlibet*, appear to be intuitionsitically meaningful and even desirable, and yet they are interderivable (modulo those axioms of Heyting's calculus that Johansson finds unobjectionable) with *ex falso quodlibet*.

This seems to place Johansson at an impasse: Either accept *disjunctive syllogism* as a principle of logic, based on its apparent unobjectionability, and accept with it the intuionistically problematic *ex falso quodlibet*, or maintain the opposition to *ex falso quodlibet* even at the cost of intuitionistically plausible principles like *disjunctive syllogism*. In retrospect, it seems remarkable that Kolmogorov could have foreseen, without any direct verification, that such a situation could arise. His remarks in *1925* ("This, of course, does not exclude the possibility that the axiom can be a formula proved on the basis of other axioms.") suggest that he might have chosen the first option, securing thereby the validity of a logical principle that could not, on account of its meaninglessness from the intuitionistic point of view, be directly established.

Johansson's approach is different. Unlike Kolmogorov, he did not attribute to *ex falso quodlibet* the status of being a meaningless formula that simply cannot be directly founded. He read the principle as a substantive claim about available constructions that is simply false according to the intuitionistic understanding of things. One might expect Johansson, then, to have preferred the second option. According to this view, the apparent validty of *disjunctive syllogism* is an illusion. The principle is not derivable from other intuitionistic axioms, and its interderivability (modulo those axioms) with *ex falso quodlibet* indicates that tucked away into its content is a thread of fallacious reasoning. From Johansson's point of view, the consequences of choosing this second option would be surprising, but perfectly legitimate—except that even upon reflection, the apparent validity to *disjunctive syllogism* didn't *seem* illusory to Johansson. If you've shown that one of two claims must be true and also that one of them is absurd, can you not proceed forthright to the conclusion that the first claim is true? And isn't this inference such that from constructions of $\mathrm{A} \vee \mathrm{B}$ and of $\neg\mathrm{B}$ one could be said to have constructed A?

In response to this question, Johansson presented a remarkable analysis of *disjunctive syllogism*—one that allowed him to transcend the impasse, confirming its intuitionistic validity despite persisting in his denial of the validity of *ex falso quodlibet*.

Johansson's proposal is to consider the inference rule $\frac{A \vee B \qquad \neg B}{A}$ literally in terms of the intuitionistic understanding of validity: "If we have proofs of $A \vee B$ and of $\neg B$, then we can construct from them a proof of A." Understood in this way, *disjunctive syllogism* is indeed a valid inference rule in the logical system whose principles he finds acceptable (i.e., in the "minimal calculus" (MPC) that results from deleting *ex falso quodlibet* from Heyting's axiomatization of intuitionistic logic (IPC)).[14] For this claim, Johansson argued as follows: The minimal calculus has the "disjunction property," the fact that only for formulas $A \vee B$ for which either $\vdash_{\text{MPC}} A$ or $\vdash_{\text{MPC}} B$ do we have $\vdash_{\text{MPC}} A \vee B$.[15] Thus if we have $\vdash_{\text{MPC}} A \vee B$, then either $\vdash_{\text{MPC}} A$ or $\vdash_{\text{MPC}} B$. But if we also have $\vdash_{\text{MPC}} \neg B$, then by the consistency of MPC, we can conclude that $\vdash_{\text{MPC}} A$.

Johansson pointed out that this understanding of *disjunctive syllogism* is of a property, like the disjunction property, *of* the minimal calculus, not of a property expressed by a formula *in* the minimal calculus. The inference rule tells us that a formula (A) will be provable whenever the formulas $A \vee B$ and $\neg B$ both are. Precisely this is what Johansson claimed the intuitionistic correctness of *disjunctive syllogism* amounts to.

Another thing one might mean by the intuitionistic correctness of disjunctive syllogism is that $A \vee B, \neg B \vdash A$, i.e., that from the hypothesis that $A \vee B$ and $\neg B$ are both true, it follows that A is true. In fact many writers have mistakenly conflated these two ideas. Johansson observed their distinctness: For $\vdash ((A \vee B) \wedge \neg B) \rightarrow A$ follows from $A \vee B, \neg B \vdash A$, and yet $((A \vee B) \wedge \neg B) \rightarrow A$ is demonstrably not a theorem of the minimal calculus.

By distinguishing these two senses in which a rule of inference like *disjunctive syllogism* can be said to be valid, Johansson was able to maintain his objection to Heyting's inclusion of *ex falso quodlibet* as an axiom of intutionistic logic while acknowledging the correctness of *disjunctive syllogism* as an intuitionistically valid rule of inference.

It is worth considering the *disjunctive syllogism* one last time in order to make Johans-

[14]Because they each put forward their logical systems as candidate formalizations of Brouwer's intuitionistic thought, it is anachronistic, in a discussion of the exchange between them, to refer to Johansson's system as "the minimal calculus" and to Heyting's as "the intuitionistic calculus." However, it is so standard nowadays to refer to Johansson's system, Heyting's system, and classical sentence logic respectively as MPC, IPC, and CPC, so that IPC = MPC + $\frac{\bot}{A}$ and CPC = MPC + $\frac{\neg\neg A}{A}$ = IPC + $\frac{\neg\neg A}{A}$ that for the sake of reference it makes sense to use these labels.

[15]The disjunction property was proved for Heyting's intuitionstic calculus independently by Gödel (*1932*) and Gentzen. Gentzen's (*1934–35*) proof is an elegant consequence of the cut-elimination theorem for his sequent calculus presentation of intuitionistic logic and the one that Johansson adapts for his minimal calculus.

son's point clear. Doesn't the intuitive appeal of this inference rule suggest that it should be valid in the stronger sense that A must be provable under the mere hypothesis that $A \vee B$ and $\neg B$ are? Well, if $A \vee B$ were not actually provable, then it would not be the case that either A or B is provable (by the disjunction property), so Johansson's reasoning for the validity of *disjunctive syllogism* cannot get off the ground. Similarly, even if $A \vee B$ is actually provable, if we are merely, and mistakenly, supposing the provability of $\neg B$, then it might be B, rather than A, which has a proof. The intuitionistic validity of this rule depends crucially on the actual existence of proofs. For how is it that we are supposed to construct a proof of A out of proofs of $A \vee B$ and $\neg B$? By taking the normalized proof (in natural deduction, say) of $A \vee B$, and simply deleting its last inference, leaving us with a proof of A. Such proof transformation techniques are surely not what Frege had in mind, but they are a remarkably transparent realization of his attitude that "mere hypotheses cannot be used as premises."

Thus for Johansson there are two distinct senses in which an inference rule can be valid or intuitionistically correct. The rule could be captured by an axiom, or theorem, in the system of intuitionistic logic (for Johansson, this is the minimal calculus), or it could be (merely) an expression of a property of that system. Faced with the apparent validity of *disjunctive syllogism* and invalidity of *ex falso quodlibet*, despite these principles' equivalence modulo the axioms of the minimal calculus, Johansson offered the following diagnosis: Both principles are in fact invalid in the first sense. The intuitive appeal of *disjuncitve syllogism* stems from the fact that it is valid in the second sense, which after all is the notion of validity more in line with Brouwer's attitude that the role of a logical rule is merely to be a reliable guide from theorem to theorem.

It is natural to ask why Johansson did not provide this analysis directly for *ex falso quodibet*. Is this rule not invalid in the first sense although valid in the second sense, just like *disjuncitve syllogism*? With this question in mind, consider again Johansson's remarks about this rule. He asked Heyting to explain how it is that an implication, on its intuitionistic understanding, could be proved given just that its antecedent is absurd. The question is about how to transform a proof of A into a proof of B given only the data that there is a proof of $\neg A$. Johansson appears to be questioning whether *ex falso quodlibet* is valid even in the second sense. The interderivability of this principle with *disjunctive syllogism* over minimal logic tells us only that the two principles must be either both valid or both invalid in the first sense. The validity of *disjunctive syllogism* in the second sense does not obviously tell us anything about *ex falso quodlibet*, because the antecedent of this rule can never be a theorem. This concept of deductive validity is coherent and considerably more precise than what one finds in the writing of Frege, Brouwer, Kolmogorov, or Heyting, but it was to undergo still another modification.

4.3 Lorenzen

In *1955* Paul Lorenzen introduced the concept of *admissibility* [*Zulässigkeit*] of an inference rule. In his explanation of this concept, Lorenzen considered collections of inference rules as logical systems, and said that a rule r is admissible in the system K if for every theorem A of $K + r$, the proof of A in $K + r$ can be transformed into a proof of A that does not involve r. It is natural to to describe this property by saying that the derivability of A in $K + r$ implies the derivability of A in K, but, as Schroeder-Heister points out in *2008*, Lorenzen avoids putting things this way. Because he considers admissibility to be the fundamental notion of logic upon which the concept of (intuitionistic) implication is based, he prefers not to invoke the concept of implication to define the property and uses the proof transformation description instead.

There is a risk in describing admissibility either in the more "natural" way or in terms of proof transformation in that these descriptions import the very ambiguity that Lorenzen meant to abolish. For example, if K is the set of rules of Johansson's minimal calculus, presented in natural deduction form (an introduction and elimination rule for each logical particle) and r is the rule for *disjunctive syllogism*, then it is certainly true that more is provable in $K + r$ than in K if rules are understood to apply to arbitrary sentences occurring anywhere in a proof. For example, the formula $\bot \to \mathrm{A}$ will in that case be a theorem of $K + r$ but not of K. However, Lorenzen means for his rules to apply only when theorems are in the premise locations. And on this understanding, the admissibility of r in K is precisely what Johansson had shown.

For this reason a better description of Lorenzen's concept is in terms of the set of theorems generated by a system of rules. If Thm_K denotes the set of theorems generated by the rules K, a rule r is admissible for K if its application to members of Thm_K does not generate any "new theorems," i.e., if the set Thm_K is closed under the action of r.

However, Lorenzen's preference to speak of the admissibility of a rule in terms of its eliminability from arbitrary derivations is doubly illustrative of his conception of logic. It showcases, for one thing, his agreement with Johansson and Brouwer that the whole point of logical inference is to reason from theorem to theorem. He saw no need to specify that he was restricting his attention to the behavior of a rule on the set of theorems, because he understood it as inherent in the concept of logical inference that one is proceeding from established truths to other established truths.

Lorenzen's choice of definition also indicates his belief that constructive proof transformation is the proper way to understand intuitionistic inference. The admissibility of a rule not only ensures, given the provability of some formulas, that some other formula is provable. It provides a reliable procedure for transforming the given proofs into a proof of that formula.

Lorenzen sometimes emphasized this by pointing out that by shifting one's focus from mere derivability in K to the more general notion of admissibility for K, one could recognize the correctness of rules of inference based on evidence or insight that extended the methods of the formalists but, because of its adherence to constructive proof techniques, still had a "definite" meaning.[16] This is a particularly clear and elegant expression of the familiar but often vague intuitionist conviction that implication is somehow stronger and less mechanistic than the formalists concede and yet not idealistic or transcendental as classical realists have maintained.

Because admissibility typically outstrips the often simplistic, mechanizable concept of derivability, Lorenzen observed that we are left in need of new techniques for determining when rules are admissible.[17] To that end, Lorenzen offered five "protological" principles for conveniently identifying the admissible rules of a given logical framework.

Lorenzen's five principles of *deduction*, *induction*, *inversion*, *equality*, and *underivability* are all used in various ways and in various contexts to discover, given some basic initial framework or mathematical system, that an inference rule is admissible. For example, the first principle simply states the obvious fact that any derivable rule is admissible (although the converse is not true in most frameworks, for example not in Johansson's minimal calculus, as we have seen). The other principles are then routes to the discovery of those admissible rules that might not simply be derivable.

These principles are not exhaustive. In some settings the problem of identifying admissible rules is highly complex and not so easily systemized. The principles do, however, indicate that Lorenzen's concept of admissibility, although largely based on a distinction first drawn by Johansson, differs in some important ways from what might have seemed like a more natural way to think of deductive validity.

Of special interest is the principle of *underivability*. The principle states that any rule with an underivable premise is admissible, i.e., if A is not a theorem of K, then the rule $\frac{\mathrm{A}}{\mathrm{B}}$ is admissible for K. It follows immediately from this principle that $\frac{\bot}{\mathrm{A}}$, i.e., *ex falso quodlibet*, is admissible in ordinary systems of propositional logic such as Johansson's minimal calculus. This is interesting because it is a departure from Johansson's own attitude about the "second sense" of validity he recognized, a property he attributed to *disjunctive syllogism* (for just the reasons Lorenzen would give for the rule's admissibility) but not to *ex falso quodlibet*.

Lorenzen's rationale for the principle of underivability is simple: Because the rule can never be triggered by the theorems of K, it will idle and therefore trivially will not produce any formulas that aren't already theorems. Thus, according to Lorenzen, even though the *ex*

[16]See *Schroeder-Heister 2008* p. 217–18 for discussion of this point.

[17]In a later section we will have occasion to observe more precise measures of the complexity of admissibility relative to derivability.

falso quodlibet is underivable, it is valid in its "operational definition"—i.e., it is admissible and so valid in the sense pertinent to intuitionism. This "cheap route" to validity is just the one that Kolmogorov and Johansson had questioned. Lorenzen's unceremonious acceptance of it indicates how, through his systematic treatment of admissibility, a concept of logical validity can both be based on concrete proof transformations and still tend towards abstraction. Admissibility, as Lorenzen formulated it and as logicians have inherited the concept, is precisely an algebraic closure principle.

In section 4.3 it was suggested that Kolmogorov's idea—that *ex falso quodlibet* could be established on the basis of intuitionistic axioms despite having no direct foundation itself—might have led him to conclude, from the principle's interderivability with *disjunctive syllogism*, that the rule is derivable after all. An alternative way to make sense of his remarks is to align him more closely with Lorenzen, for whom the derivable rules of the minimal calculus might be said to have a direct intuitionistic foundation, while rules like *disjunctive syllogism* and *ex falso quodlibet* are shown to be acceptable despite lacking such a foundation, because the theorems of intuitionistic logic are closed under their operation.

Before turning to an analysis of the modern theory of admissibility, it is worthwhile to reflect on the concept's origins. As has been mentioned, the ambiguity in the concept of deductive validity—what we can now identify as an ambiguity between derivability and admissibility—persists to this day. Not surprisingly, there are today logicians who identify one concept in contrast with the other as what it truly is for a rule of inference to be valid. What has not been emphasized in these discussions, so far as I am aware, is the extent to which the concept of admissibility emerged as distinct from derivability largely in the context of implementing Brouwer's intuitionistic philosophy. We have seen that there were options along the way. It is unclear whether Kolmogorov's remarks indicate that he was prepared to accommodate two separate notions of validity simultaneously. Heyting evidently was not. He insisted that all intuitionistically well-founded rules be included as derivable rules in the proper intuitionistic logic. Johansson clearly thought it was appropriate both to distinguish two senses of deductive validity and to consider each of them legitimate, although his "second sense" of validity differs in one important way from admissibility as we know it today. Lorenzen was very clear about the difference between admissibility and derivability, but he considered the latter to be a mere formal property of logical systems, one among many ways for a rule to be admissible but of no conceptual significance of its own.

5. The Frege-Brouwer dynamics

Recall that according to Frege, although the validity of a rule amounts to its role in inference, and although inference is always an act leading from established truths to other established truths, valid inference rules do have a legitimate use on arbitrary sentences. The result of applying such rules to arbitrary sentences in the premise location is the confirmation of the truth of a conditional statement. Thus, if r is a rule which, whenever applied to a theorem of K yields another theorem of K, and A is not a theorem of K, then one could still apply r to A purely formally, yielding perhaps B. According to Frege, even though one could not say that one had "inferred" B from the assumption A, one could still say that one had established the conditional $\mathrm{A} \rightarrow \mathrm{B}$ as (logically) true.

Frege's conviction seems natural to many readers today, but it is not obviously justified. For arbitrary logical systems K, a rule r could be admissible without the conditional $\mathrm{A} \rightarrow \mathrm{B}$ generated as above being a theorem. MPC illustrates this, for as Johansson observed *disjunctive syllogism* is admissible in MPC although $(\mathrm{A} \vee \mathrm{B}) \wedge \neg \mathrm{B}) \rightarrow \mathrm{A}$ is not a theorem of MPC. Therefore if, following Frege, one thinks that what makes a rule valid is just its admissibility *and also* that the validity of $\frac{\mathrm{A}}{\mathrm{B}}$ induces the theoremhood of $\mathrm{A} \rightarrow \mathrm{B}$, then MPC could not be a complete systematization of logical reasoning. Contemporary logicians refer to systems like MPC that have admissible rules that aren't derivable as "structurally incomplete," typically without providing a compelling description of the sense in which the system is supposed to be inadequate. Such a description can perhaps be found in Frege. But we have seen that many logicians, including Johansson and Lorenzen, did not see anything wrong with this sort of incompleteness.

These considerations suggest the following dynamical perspective. One begins by identifying some inference rules that one finds unobjectionable. Perhaps the natural deduction rules associated with individual logical operators are recommended on an inferentialist account of meaning. Following Lorenzen, one then observes that the set of theorems generated by these rules is closed under some transformation rules neither included among them nor derivable from them. Following Frege, one concludes that the original set of rules is incomplete: There are rules $\frac{\mathrm{A}}{\mathrm{B}}$ that can be shown to be valid based on these rules and therefore used to show that conditionals of the form $\mathrm{A} \rightarrow \mathrm{B}$ are logically true despite not being theorems of the original system. One then "corrects" this deficiency by adding the new rule to the system. Notice that these dynamics are reminiscent of Brouwer's own provisional attitude towards logical rules. Could such a perspective recommend collapsing the distinction drawn by Johansson and Lorenzen, updating logical systems by projecting their admissible rules, as they are discovered, into the system itself?

The first thing to observe about this Frege-Brouwer dynamical perspective is that it cannot be used to justify Heyting's preference of IPC over MPC. It is true that IPC differs from MPC precisely by explicitly including the class of MPC-admissible rules interderivable (over MPC) with *disjunctive syllogism* and *ex falso quodlibet*, so that IPC can be seen as an "update" of MPC. But IPC cannot be seen as a "structural completion" of MPC, for it has its own non-derivable admissible rules. An important[18] example is Harrop's rule (*1956*), $\dfrac{\neg \mathrm{A} \to (\mathrm{B} \lor \mathrm{C})}{(\neg \mathrm{A} \to \mathrm{B}) \lor (\neg \mathrm{A} \to \mathrm{C})}$, whose IPC-admissibility can be verified with reasoning analogous to a verification of the disjunction property.[19]

IPC can thus be seen at best as a first step towards "the complete logic" induced by MPC. The system KP = IPC + Harrop's rule would be a step further. A natural question to ask is whether, and at what point, the Frege-Brouwer dynamics converges. Can we continue to update intuitionistic logic indefinitely? If not, what system do we wind up with?

The impetus behind this question is the hope of identifying the true, complete intuitionistic logic. Again, the idea is that a logical system with any underivable admissible rules is inadequate, that—as the expression "structurally incomplete" suggests—such systems are lacking. Rybakov (*1997*), for example, suggests that "there is a sense in which a derivation inside a [structurally incomplete] logical system corresponds to conscious reasoning [and] a derivation using [its] admissible rules corresponds to subconscious reasoning." He faults such systems for having rules that are "valid in reality" yet "invalid from the viewpoint of the deductive system itself" (pp. 10–11). Structurally complete systems, by contrast, are said to be "self contained" in the sense that they have the "very desirable property" of being conscious of all the rules that are reliable tools for discovering their own theorems (p. 476).

The best way to understand why this attitude about structurally incomplete logical systems is mistaken is to pursue the question it inspires. Doing so exposes a series of intuitive but erroneous preconceptions which should prompt a renewed understanding and appreciation of the role that two distinct senses of validity can play.

6. Consequence relations

So far we have used the notation $\vdash$ to denote specific realizations of logical relationships. For example, $\vdash_{\mathrm{CPC}} \mathrm{A}$ has been used to express the claim that A is derivable in some formal system of classical propositional logic, whereas $\models_{\mathrm{CPC}} \mathrm{A}$ has been used to express the claim

[18] This rule was identified by Kreisel and Putnam (*1957*) as the characteristic axiom of a system that extends IPC but still has the disjunction property. Łukasiewicz (*1952*) had conjectured that the disjunction property characterizes IPC in the sense that it fails in any stronger logic. In light of Kriesel and Putnam's refutation of Łukasiewicz's conjecture and the association of the disjunction property with the BHK interpretation of intuitionistic logic, de Jongh (*1968*) indicated a related property that does so characterize IPC. Iemhoff (*2001*, p. 137), indicated another that is close to the heart of the present paper: IPC is the only intermediate logic with the disjunction property for which all the Visser rules $\mathcal{V}$ are admissible (see section 7.5).

[19] This is not to say that the admissibility of Harrop's rule is in any sense equivalent to the disjunction property. Harrop's rule is derivable, and therefore admissible, in CPC, although the disjunction property obviously fails in CPC.

that A is a classical propositional validity, i.e., that A comes out true under every assignment of truth values to atomic sentences.

When studying the relationship between derivability and admissibility, one focusses instead on logical relationships themselves. Thus the notation $\vdash$ is used to express a certain type of relation that sentences might bear to one another, and the question whether there are convenient proof systems or semantic frameworks that realize these relations is set aside. By abstracting in this way from the details of the mechanisms that generate logical structure and attending to that structure directly, one is able to consider extensions, transformations, and sequences of these relations that result from simple operations, even without knowing whether the relations that result are simple enough to have familiar sorts of realizations (e.g., one might not know if one is dealing with a recursively enumerable relation and therefore whether anything like a syntax-based proof system realizes the relation.)

This approach to logic originated in the 1930's in the work of Tarski and Gentzen. Tarski's aim was to provide a fully general characterization of what it means for a sentence A to follow logically from a set Γ of sentences. Sometimes he expressed this in terms of models, so that A follows logically from Γ if every model of Γ is a model of A. In other places he wrote that every relation $\mathcal{R}$ on a set S that is reflexive ($\mathcal{R}(\{\mathrm{A}\}, \{\mathrm{A}\})$ for every $\mathrm{A} \in S$), transitive (if $\mathcal{R}(\Gamma, \{\mathrm{A}\})$ and $\mathcal{R}(\{\mathrm{A}\} \cup \Delta, \{\mathrm{B}\})$ then $\mathcal{R}(\Gamma \cup \Delta, \{\mathrm{B}\})$), and monotone (if $\mathcal{R}(\Gamma, \{\mathrm{A}\})$ then $\mathcal{R}(\{\mathrm{B}\} \cup \Gamma, \{\mathrm{A}\})$) is a consequence relation.

Tarski used these two definitions interchangeably from 1935 onward, but the first observation of their equivalence is in Gentzen's *1932*. Although Gentzen worked with an informal notion of substitution instead of a robust theory of set models and restricted his attention to finite sets Γ, he seems to have made the original discovery that the general, intuitive concept of logical consequence ("There's no way for all of these sentences to be true without this other sentence also being true") can be captured exactly by a set of algebraic conditions.[20]

In contemporary studies, one finds the concept of a (single-conclusion) finitary consequence relation defined as a relation $\vdash$ on a set S of formulas[21] that satisfies, for all finite subsets $\Gamma, \Delta \subseteq S$ and all formulas $\mathrm{A}, \mathrm{B} \in S$:

1. $\mathrm{A} \vdash \mathrm{A}$ (reflexivity)

[20]For Gentzen, these conditions provided the structural framework of the sequent calculus, with transitivity called "cut," monotonicity called "thinning," and reflexivity expressed as a condition on "initial formulas." A detailed analysis of Gentzen's result and its relevance both to his unique conception of logic and to his more well-known logical achievements can be found in *Franks 2010*.

[21]One also finds studies about consequence relations whose elements are, rather than formulas, sequents—a framework in which one can observe, depending on how one formulates the relevant calculi, that the CUT rule is admissible but not derivable. A discussion of the several additional complexities presented by such a framework can be found in *Iemhoff 2016*. We consider only the simplest case of single-conclusion consequence relations, whose elements are formulas.

2. if $\Gamma \mid\!\!- A$ then $B, \Gamma \mid\!\!- A$ (monotonicity)

3. if $\Gamma \mid\!\!- A$ and $A, \Delta \mid\!\!- B$ then $\Gamma, \Delta \mid\!\!- B$ (transitivity)

with the convention that a serial list of formulas be understood as their union and a single formula standing for its own "singleton set." It is uncommon[22] to find this definition of abstract consequence relations promoted as an appropriate general definition of logical consequence—typically modern studies of abstract consequence relations do not mention the definition's adequacy with respect to the informal substitutional concept of logical consequence. But one should ideally have in mind its origins in the work of Tarski and Gentzen, their point having been that although these conditions do not pick out any one particular realization of logical consequence, for that very reason they do capture the full intention behind the general, intuitive idea of something following logically from some other things.

This framework allows for succinct definitions of several key notions:

- If $\mid\!\!-$ is a consequence relation, then $\mathrm{Thm}(\mid\!\!-) = \{A : \mid\!\!- A\}$, i.e. $\mathrm{Thm}(\mid\!\!-)$, "the theorems of $\mid\!\!-$," denotes the set of all formulas that stand in the $\mid\!\!-$ relation to the empty set.
- If $\mid\!\!-$ is a consequence relation, and $r = \dfrac{\Gamma}{A}$ is an inference rule, then $\mid\!\!\overset{r}{-}$ is the smallest consequence relation $\vdash$ extending $\mid\!\!-$ for which $\Gamma \vdash A$.
- More generally, if R is a set of rules, then $\mid\!\!\overset{R}{-}$ is the smallest consequence relation $\vdash$ extending $\mid\!\!-$ for which $\Gamma \vdash A$ for each $\dfrac{\Gamma}{A} \in R$.
- A rule $\dfrac{\Gamma}{A}$ is *derivable* in $\mid\!\!-$ if $\Gamma \mid\!\!- A$.
- A rule $r = \dfrac{\Gamma}{A}$ is *admissible* in $\mid\!\!-$ if $\mathrm{Thm}(\mid\!\!-) = \mathrm{Thm}(\mid\!\!\overset{r}{-})$. The notation $\Gamma \mid\!\!\sim A$ is used to express the fact that $\dfrac{\Gamma}{A}$ is *admissible* in $\mid\!\!-$.

Notice, in particular, how these last two definitions disambiguate Lorenzen's expression of admissiblity in terms of being able to prove all the same formulas in K as in $K + r$. Using as an example the proof system MPC and the rule $\dfrac{\bot}{A}$, one might mean by "$K + r$" simply IPC. In that case Lorenzen's claim would be that

$$\mathrm{Thm}(\mid\!\!-_{\mathrm{IPC}}) = \mathrm{Thm}(\mid\!\!-_{\mathrm{MPC}})$$

which is obviously false. IPC has, for example, the theorem $\bot \to A$. What Lorenzen had in

[22] For an elegant exception, see *Koslow 1992*.

mind can be expressed in our notation as

$$\mathrm{Thm}(\vdash^{\perp/\mathrm{A}}_{\mathrm{MPC}}) = \mathrm{Thm}(\vdash_{\mathrm{MPC}}).$$

This claim is true. The smallest consequence relation extending $\mathrm{Thm}(\vdash_{\mathrm{MPC}})$ and including $\frac{\perp}{\mathrm{A}}$, although a proper extension of $\mathrm{Thm}(\vdash_{\mathrm{MPC}})$, has exactly the same theorems.

It should not be immediately obvious that $\vdash^{\perp/\mathrm{A}}_{\mathrm{MPC}}$ has the same theorems as $\vdash_{\mathrm{MPC}}$. Simply adding $\frac{\perp}{\mathrm{A}}$ to $\vdash_{\mathrm{MPC}}$ does not necessarily result in a consequence relation. One must check that when one "rounds out" the relation one gets in order to satisfy the definition of a consequence relation, no new theorems are generated. We will not verify this here, but we make note of the general strategy for determining, given $\vdash$ and a set of rules R, the relation $\vdash^{R}$, which is to take the union of the rules of $\vdash$ and R, close this set under monotonicity, and then close the resulting set under transitivity.[23]

In this way it becomes clear that Lorenzen's two definitions of admissibility—that adding a rule to a logical system results in no new theorems and that the set of theorems of a logical system is closed under the action of that rule—amount to the same thing.

Finally, we remark that if $\vdash$ is a consequence relation, then $\mathrel{|\!\!\sim}$ is also a consequence relation.[24] Of course these two relations have the same theorems. More generally, if $\vdash_1$ and $\vdash_2$ are consequence relations such that $\mathrm{Thm}(\vdash_1) = \mathrm{Thm}(\vdash_2)$, then $\mathrel{|\!\!\sim}_1 = \mathrel{|\!\!\sim}_2$. This situation has led some logicians to remark that, because they depend only on the set of theorems, a logic's true characterization is in terms of its admissible rules. The derivable rules, by contrast, depend on a "design choice," i.e., on which proof system or semantic framework one chooses to generate the theorems.[25]

7. The dynamical fallacy

With the theory of abstract consequence relations in hand, we can survey some of the remarkable behaviors exhibited by a logical system's admissible rules. These phenomena were all discovered in the last half of the 20th Century. Although the Frege-Brouwer dynamics introduced in section 5 are not, to my knowledge, explicitly endorsed by any logician today, the way in which they are undermined by the phenomena showcased here is instructive.

[23]For precise definitions and a verification that this strategy works, see *Iemhoff 2016*, proposition 3.6.

[24]See, for example, *Iemhoff 2016*, corollary 4.3.

[25]See especially *Rybakov 1997*. Similarly *Iemhoff 2016* remarks that whether or not a logic is structurally complete "depends very much on the particular consequence relation one uses for a logic," whereas "admissibility solely depends on the [logic's] theorems" which is invariant for all such consequence relations.

What many theoreticians do endorse, and what seems like plain common sense, is the idea that for any field of inquiry with a definite logical structure there is a single correct notion of deductive validity. Even so-called "logical pluralists," who admit that each field of inquiry might have its own fully legitimate "logic," often suggest as much. What the failure of the Frege-Brouwer dynamics suggests is that some fields of inquiry come equipped with a pair of concepts of deductive validity, and that debate about which of them is correct is misguided. Such inquiries are *context sensitive* in the sense that reasoning in arbitrary contexts and reasoning strictly among established truths are governed by different sets of laws.

7.1 Structurally complete logics

In section 2 we explored a genetic explanation of the persistent tendency to conflate derivability and admissibility. There is also a theoretical explanation.

The most familiar propositional logic, CPC, has no admissible rules that are not derivable.

Theorem 1. *For all* $\frac{\Gamma}{\mathrm{A}}$, $\Gamma \vdash_{\mathrm{CPC}} \mathrm{A}$ *if, and only if,* $\Gamma \mathrel{|\!\!\sim}_{\mathrm{CPC}} \mathrm{A}$.

Proof. In general, a logic's derivable rules are admissible. To verify the other direction for CPC suppose that $\frac{\Gamma}{\mathrm{A}}$ is not derivable. Then the truth functional semantics for CPC tells us that there is an assignment v of truth values to the atoms in Γ and A under which each of the formulas in Γ evaluates as TRUE and A evaluates as FALSE. For each $\gamma \in \Gamma$, let γ' be the result of replacing every atom in γ assigned TRUE by v with $(p \to p)$ and every atom in γ assigned FALSE by v with $(p \wedge \neg p)$. Clearly each γ' is a classical tautotology and therefore a theorem of CPC. Similarly let A' be the result of replacing every atom in A assigned TRUE by v with $(p \to p)$ and every atom in A assigned FALSE by v with $(p \wedge \neg p)$. Clearly A' is not a theorem of CPC—in fact its negation is a theorem. Thus $\frac{\Gamma}{\mathrm{A}}$ is not admissible, as it can lead from theorems to nontheorems. □

Because CPC is structurally complete, it is possible when studying this system to work loosely with the idea of logical validity, never specifying whether one intends for inference rules to be truth preserving or validity preserving, without running into any difficulties. Many people have spent only a little time thinking about logics other than CPC, so that they have not been habituated to distinguish admissibility from derivability.

The semantic argument for the structural completeness of CPC just presented tells us more than that every underivable rule will have instances whose premises are theorems but whose conclusion is not a theorem—it tells us that there would always be instances whose conclusions are anti-theorems. This suggests that structural completeness can arise in broader contexts. Indeed, such logics as IPC + $\neg A \vee \neg\neg A$ and IPC + $(A \to B) \vee (B \to A)$ are weaker than CPC but structurally complete.

Earlier we questioned how Frege could have conceived of logical validity only in terms of a rule's behavior on the space of logical truths and yet used inference rules that are valid in this sense on arbitrary premises in order to discover logically true conditional statements. In general, the admissibility of a rule does not assure us that conditional statements formed in this way will be logically true. One should keep in mind that Frege's logic was classical, so that for him the divide between admissibility and derivability collapses. Thus he could define validity in terms we would recognize today as admissibility but be guaranteed that all such rules are in fact derivable.

7.2 Structural completion

We have observed that one could mean one of two things by "adding a rule to a logical system." One could add the rule, in the ordinary sense, to the logical calculus. For example, if one considers MPC as a natural deduction system and adds to it a rule such as $\dfrac{A \vee B \qquad \neg B}{A}$, $\dfrac{A \wedge \neg A \qquad B}{B}$, or $\dfrac{\bot}{A}$, then one gets a natural deduction system for IPC. But if one considers the consequence relation $\vdash_{\mathrm{MPC}}$ realized by MPC and adds one of these rules in Lorenzen's sense, one gets a new relation such as $\vdash^{\bot/A}_{\mathrm{MPC}}$, where $\mathrm{Thm}(\vdash^{\bot/A}_{\mathrm{MPC}}) = \mathrm{Thm}(\vdash_{\mathrm{MPC}}) \neq \mathrm{Thm}(\vdash_{\mathrm{IPC}})$.

Generalizing from this, there are two different things one could mean by the "structural completion" of a logic. If one adds all the rules admissible but not derivable in a logic L to the consequence relation that L realizes, then one will get a new consequence relation with the same theorems as L, all the admissible rules of which are derivable. This is the construction that *Rybakov 1997* for example calls the structural completion of L.

However, this consequence relation may have some undesirable features. For example, if L is a propositional logic with a connective like $\to$ characterized by familiar inference rules $\dfrac{A \to B \qquad A}{B}$ ("$\to$-elimination") and $\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \to B}$ ("$\to$-introduction"), then these rules will have to be restricted when the new rules are added. Specifically, the "$\to$-introduction" rule will no longer apply when A and B are the premise and conclusion of a rule $\dfrac{A}{B}$ that was admissible but not derivable in L. This is easier to see when one rewrites the the elimination and introduction rules as filter and minimality conditions on the relation $\vdash_{L}$:

$$A \to B, A \mathrel{|\!\!\frac{\quad}{L}} B \tag{1}$$

$$\text{For all } C, \text{if } C, A \mathrel{|\!\!\frac{\quad}{L}} B, \text{then } C \mathrel{|\!\!\frac{\quad}{L}} A \to B \tag{2}$$

The first condition tells us that $A \to B$ is an object which, combined with A, allows the inference to B. The second condition tells us that $A \to B$ is the least such object: If any other object C allows this same inference, this can only be because C itself is stronger (in the sense of $\vdash_L$) than $A \to B$.

A compelling way to think of logical operators is in terms of such filter and minimality conditions on consequence relations.[26] Thus, given a consequence relation $\vdash$, the conditions 1 and 2 define the concept of a "conditional" or "hypothetical" over $\vdash$. Not all consequence relations will have, for every A, B, a conditional object. In our imagined example, $\vdash_L$ does have a conditional, namely the sentence $A \to B$ for every A, B, but its "structural completion" $\vdash^R_L$, where R is the set of rules $\frac{\Gamma}{A}$ for which $\Gamma \mathrel{|\!\!\sim_L} A$, does not. Even though the symbol $\to$ appears in the formulas of this consequence relation, it no longer picks out the conditional of A and B for all objects A, B. It is natural to say that it no longer has the same meaning that it had before.

Thus if one begins with IPC and generates in this way its "structural completion," the resulting consequence relation will have the same theorems as IPC but will not have a proof system that is a natural extension of the proof system of IPC. In particular, the Deduction Theorem will fail for this consequence relation, which is to say that the rule "$\to$-elimination" will not be a rule for any proof system that realizes this consequence relation, which is to say, again, that the symbol $\to$ will not mean the same thing in this setting that it means in IPC.

For this reason, another thing one might mean by the "structural completion" of a logic L is the logic that results when one adds all the rules $\frac{\Gamma}{A}$ for which $\Gamma \mathrel{|\!\!\sim_L} A$ and then "updates" the Deduction Theorem so that the connective ($\to$, $\supset$, etc.) that corresponds to the conditional operator in L continues to do so in the new logic. The idea behind this is that Quine's (*1970*) dictum, "change the logic, change the meaning," is too crude. Whereas changing a logic will typically garble the meanings of its connectives, so that they no longer correspond to the logical operators they previously corresponded to, an attentive change can result in an extension of one's original logic in which the original meanings of the connectives are preserved.

This second understanding of a "structural completion" of a logic is the one pursued in the Frege-Brouwer dynamics. According to this framework, one cannot abide the realization

[26]Koslow (*1992*) provides a full elaboration of this point of view, which is also implicit in the category-theoretic work on topoi in for example *Lawvere 1964*.

that the theorems of one's logic are closed under a rule of inference that is itself not part of the logic. But it doesn't suffice just to add that rule to the conseqeunce relation and have it be valid in the sense that $\Gamma \vdash A$ if it doesn't follow from this that $\vdash (\bigwedge \Gamma) \rightarrow A$, because the failure of the Deduction Theorem impugns the very meaning of $\rightarrow$. One must also add the formula $(\bigwedge \Gamma) \rightarrow A$ as a theorem.

7.3 Inconsistency

Observing the admissibility of, say $\dfrac{A \vee B \qquad \neg B}{A}$ in MPC and responding by first adding this rule to the relation $\vdash_{\mathrm{MPC}}$ to get $\vdash^{< A \vee B, \neg B >/A}_{\mathrm{MPC}}$ and then updating the Deduction Theorem so that $\rightarrow$ again corresponds with the conditional operator and $((A \vee B) \wedge (\neg B)) \rightarrow A$ is a theorem is a natural progression from MPC to IPC. But in general things are more complicated.

The modal logic GL provides a dramatic example of how extending logics by adding admissible rules can go badly. In general, modal logics are good frameworks for bringing the distinction between derivablity and admissibility into relief. For example, although the decidablity of admissibility implies the decidability of derivability, Chagrov (*1992*) identified a modal logic with decidable derivable rules but undecidable admissible rules. GL illustrates how projecting admissibility across this complexity gap can lead to unexpected results.

GL is the modal logic characterized by the axioms $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ and $\Box(\Box A \rightarrow A) \rightarrow \Box A$ and the rules of inference *modus ponens* and *necessitation* (from $\vdash_{\mathrm{GL}} A$ infer $\vdash_{\mathrm{GL}} \Box A$). (The axiom $\Box A \rightarrow \Box\Box A$ follows from the others so needn't be included.) The importance of GL is due to Solovay's (*1976*) arithmetical completeness theorem which states that if every arithmetical realization of A is a theorem of PA, then $\vdash_{\mathrm{GL}} A$, i.e., that GL is the logic of PA-provability.

It is easy to verify that Löb's rule $\dfrac{\Box A \rightarrow A}{A}$ is derivable in GL and that the rule $\dfrac{\Box A}{A}$ is admissible in GL (as it is in many modal logics such as K, D, K4, and S4). Suppose we "complete" the structurally incomplete GL by adding $\dfrac{\Box A}{A}$ to $\vdash_{\mathrm{GL}}$, yielding $\vdash^{\Box A/A}_{\mathrm{GL}}$, and then updating the Deduction Theorem in order to preserve the meaning of $\rightarrow$, so that $\Box A \rightarrow A$ is a theorem. It follows from the derivability of $\dfrac{\Box A \rightarrow A}{A}$ and *modus ponens* that every formula A is a theorem of this logic, i.e., that the logic is inconsistent.

Recalling Rybakov's assessment of structural incompleteness, we can verify that the admissible rule $\dfrac{\Box A}{A}$, according to the arithmetical interpretation, is indeed "valid in reality" even though its validity is not acknowledged by GL itself: One cannot prove, in a consistent framework like PA, that A is provable unless one can in fact prove A. But PA only recognizes the inference from the fact of A's provability to A when "it has to," i.e., when A is actually a theorem. Reflecting on this situation, Rohit Parikh has joked that "Peano arithmetic could not be more modest about its own veracity." We have observed that attempting to "correct" this excessive modesty, by projecting the admissible rules of GL into GL itself, only leads to inconsistency.

7.4 Non-monotonicity

The fact that the Frege-Brouwer dynamics founders in modal logic is good reason to doubt that projecting admissible rules into a calculus is a sensible corrective to structural incompleteness. The situation with GL captures the fact that some phenomena of great scientific interest, such as formal provability in arithmetic, cannot be completely systematized: their logic is such that only some inferential relationships can be represented within a formal system while the others are exhibited as properties of that system. In fact, the project of formalizing valid inferences about arithmetical provability and discovering this rift between those inferences that can be given explicit expression within the system and those that arise only as properties of the formal apparatus is directly connected to deep insights about the nature of those inferences, e.g., Gödel's discovery that the reasoning required to prove that an arithmetical system is consistent cannot be carried out within that same system.

For propositional logic, however, the inconsistency problem does not arise.

Theorem 2. *For any intermediate logic L, i.e.,* $\mathrm{Thm}(\vdash_{\text{IPC}}) \subseteq \mathrm{Thm}(\vdash_{L}) \subseteq \mathrm{Thm}(\vdash_{\text{CPC}})$, *if* $\Gamma \mathrel{|}\!\sim_{L} \mathrm{A}$ *then* $\Gamma \vdash_{\text{CPC}} \mathrm{A}$.

Proof. A consequence of Glivenko's (*1929*) proof that, if $\vdash_{\text{CPC}} \mathrm{A}$, then $\vdash_{\text{IPC}} \neg\neg\mathrm{A}$, is the fact that: If A is any formula and $\sigma(\mathrm{A})$ is created by substituting $p \supset p$ for some of the sentence letters in A and substituting $p \wedge \neg p$ for the others, then $\vdash_{\text{IPC}} \sigma(\mathrm{A})$ if $\vdash_{\text{CPC}} \sigma(\mathrm{A})$, and because any intermediate logic L has strictly more theorems than IPC also $\vdash_{L} \sigma(\mathrm{A})$ (see *Franks 2017*).

Assume now that $\Gamma \mathrel{|}\!\sim_{L} \mathrm{A}$, i.e., if $\vdash_{L} \bigwedge\Gamma$, then $\vdash_{L} \mathrm{A}$, and let σ be a substitution instance as defined above. If $\vdash_{\text{CPC}} \sigma(\bigwedge\Gamma)$, then $\vdash_{L} \sigma(\bigwedge\Gamma)$, and so $\vdash_{L} \sigma(\mathrm{A})$, from which it follows that $\vdash_{\text{CPC}} \sigma(\mathrm{A})$. Therefore by the completeness of CPC with respect to the truth functional semantics, it follows that $\Gamma \vdash_{\text{CPC}} \mathrm{A}$.

□

Thus for any intermediate logic L and rule $\frac{\Gamma}{\mathrm{A}}$ such that $\Gamma \mathrel{|}\!\sim_{\mathrm{L}} \mathrm{A}$, $L + \frac{\Gamma}{\mathrm{A}}$ is another intermediate logic. Because CPC is consistent, so too is $L + \frac{\Gamma}{\mathrm{A}}$. This suggests that it still might be worth considering the prospects for the Frege-Brouwer dynamics for IPC.

Let us observe first that the set of admissible rules of IPC is quite complicated. Whereas IPC-derivability is coPSPACE-complete, IPC-admissibility is coNEXP-complete (*Citkin 1977*). Furthermore, although IPC-admissibility is decidable (*Rybakov 1992*) there is no finite set R of rules such that $\mathrel{|\!\!\widetilde{}}_{\mathrm{IPC}} = \vdash^{R}_{\mathrm{IPC}}$ (i.e., there is no finite *basis* R for IPC-admissibility) (*Rybakov 1997*). Examples of admissible but underivable rules of IPC include, in addition to the above-mentioned

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)} \text{ Harrop's rule}$$

other well-analyzed rules such as

$$\frac{(A \to B) \to (A \vee C)}{((A \to B) \to A) \vee ((A \to B) \to C)} \text{ Mints's rule}$$

$$\frac{(\neg\neg A \to A) \to (\neg\neg A \vee \neg A)}{\neg\neg A \vee \neg A} \text{ Rose's rule}$$

$$\frac{(A \to B) \to (C \vee D)}{((A \to B) \to A) \vee ((A \to B) \to C) \vee ((A \to B) \to D)} \text{ Visser's rule}$$

We know from Theorem 2 that all of these rules can be consistently added to IPC, individually or collectively. What is more surprising is that if we pursue the Frege-Brouwer dynamics, it matters significantly both where we start and in what order we add rules.

Theorem 3. *Visser's rule is not admissible in* IPC + *Harrop's rule (*KP*).*[27]

Proof. Suppose

$$(A \to B) \to (C \vee D) \mathrel{|\!\!\widetilde{}}_{\mathrm{KP}} ((A \to B) \to A) \vee ((A \to B) \to C) \vee ((A \to B) \to D).$$

Then because

$$\vdash_{\mathrm{KP}} (\neg A \to (B \vee C)) \to (\neg A \to B) \vee (\neg A \to C)),$$

[27]This theorem is an instance of Iemhoff's (*2001*) general observation that IPC is the maximal logic that both has the disjunction property and admits Visser's rules, but the direct verification of this particular instance is instructive. The customary label "KP" used here refers to Kreisel and Putnam, whose *1957* featured the rule (see note 18).

it follows, by replacing A with $\neg\mathrm{A}$, B with $\mathrm{B} \vee \mathrm{C}$, C with $\neg\mathrm{A} \to \mathrm{B}$, and D with $\neg\mathrm{A} \to \mathrm{C}$ that

$$\vdash_{\mathrm{KP}} ((\neg\mathrm{A} \to (\mathrm{B} \vee \mathrm{C})) \to \neg A) \vee ((\neg\mathrm{A} \to (\mathrm{B} \vee \mathrm{C})) \to (\neg\mathrm{A} \to \mathrm{B})) \vee ((\neg\mathrm{A} \to (\mathrm{B} \vee \mathrm{C})) \to (\neg\mathrm{A} \to \mathrm{C})).$$

But KP has the disjunction property (*Kreisel and Putnam 1957*) and therefore must prove one of $((\neg\mathrm{A} \to (\mathrm{B}\vee\mathrm{C})) \to (\neg\mathrm{A} \to \mathrm{B}))$, $((\neg\mathrm{A} \to (\mathrm{B}\vee\mathrm{C})) \to (\neg\mathrm{A} \to \mathrm{C}))$, or $((\neg\mathrm{A} \to (\mathrm{B}\vee\mathrm{C})) \to \neg A)$. However, none of these three formuals is even classically valid. □

Thus if one begins, say with MPC, adds the rule *disjunctive syllogism* yielding IPC, and then adds Harrop's rule, one will no longer have occasion to add Visser's rule. One could just as easily have added Visser's rule first, in which case the Frege-Brouwer dynamics proceed in a different direction. In other words, these dynamics are not well defined. If the reason one wants to update one's logic is an appreciation of the validity of admissible rules and the belief that valid rules applied to arbitrary formulas should yield theorems of conditional form, then it should be unacceptable that updating in this way can result in a rule no longer being admissible.

It is a well-known but quite remarkable feature of inference rules that their admissibility is *non-monotonic*: It can happen that for L_1, L_2 for which $\mathrm{Thm}(\vdash_{L_1}) \subseteq \mathrm{Thm}(\vdash_{L_2})$, $\Gamma \mid\!\sim_{L_1} \mathrm{A}$ but not $\Gamma \mid\!\sim_{L_2} \mathrm{A}$. This is well-illustrated by the modal logics K and its extension K4, as Löb's rule is admissible in K but not in K4. Theorem 3 tells us even more, namely that monotonicity can be violated even when the extending theory L_2 is the result of adding to L_1 a rule admissible in L_1.

Theorem 2 deserves another look in light of the non-monotonicity of admissibility. For we now know that a rule can be admissible for one intermediate logic but not in one of its extensions. Considering that the reason that rules like $\dfrac{\neg\mathrm{A} \to (\mathrm{B} \vee \mathrm{C})}{(\neg\mathrm{A} \to \mathrm{B}) \vee (\neg\mathrm{A} \to \mathrm{C})}$ are admissible in IPC is that IPC has the disjunction property, one might not expect it to be admissible again in CPC, for which the disjunction property fails. Here it is helpful to emphasize the "only if" reading of the fact of admissibility: "The only way this logic can ever prove an expression of the form $\neg\mathrm{A} \to (\mathrm{B} \vee \mathrm{C})$ is by proving already $(\neg\mathrm{A} \to \mathrm{B}) \vee (\neg\mathrm{A} \to \mathrm{C})$, but a stronger logic might have resources for proving expressions of the form $\neg\mathrm{A} \to (\mathrm{B} \vee \mathrm{C})$ even when the corresponding instance of $(\neg\mathrm{A} \to \mathrm{B}) \vee (\neg\mathrm{A} \to \mathrm{C})$ is not a theorem." Theorem 2 tells us that, on the contrary, all the rules that are admissible for any intermediate logic, even if they fall out of view in some extensions, return again as admissible (and in fact derivable) for CPC—but for a different reason.

7.5 Oversimplification

Recall that there is no finite basis for IPC-admissibility, i.e., no finite set R of rules such that $\mathrel{|\!\!\sim}_{\mathrm{IPC}} = \vdash^{R}_{\mathrm{IPC}}$. This does not preclude there being some basis R that can be conveniently described and that gives us some orientation around the space of IPC's admissible rules.

In fact there is an inductively defined sequence of rules, "Visser's rules,"[28] $\{\mathcal{V}_i\}_{i=1}^{\infty}$, that does just this:

$$\mathcal{V}_n = \frac{\bigwedge_{i=1}^{n} (\mathrm{A}_i \to \mathrm{B}_i) \to (\mathrm{A}_{\mathrm{n}+1} \vee \mathrm{A}_{n+2})}{\bigvee_{j=1}^{n+2} (\bigwedge_{i=1}^{n} (\mathrm{A}_i \to \mathrm{B}_i) \to \mathrm{A}_j)}$$

Thus,

$$\mathcal{V}_1 = \frac{(\mathrm{A}_1 \to \mathrm{B}) \to (\mathrm{A}_2 \vee \mathrm{A}_3)}{((\mathrm{A}_1 \to \mathrm{B}) \to \mathrm{A}_1) \vee ((\mathrm{A}_1 \to \mathrm{B}) \to \mathrm{A}_2) \vee ((\mathrm{A}_1 \to \mathrm{B}) \to \mathrm{A}_3)}$$

which is just Visser's rule from the previous subsection with its atoms relabeled. The next rule in the sequence,

$$\mathcal{V}_2 = \frac{\chi \to (\mathrm{A}_3 \vee \mathrm{A}_4)}{(\chi \to \mathrm{A}_1) \vee (\chi \to \mathrm{A}_2) \vee (\chi \to \mathrm{A}_3) \vee (\chi \to \mathrm{A}_4)}$$

where $\chi = (\mathrm{A}_1 \to \mathrm{B}_1) \wedge (\mathrm{A}_2 \to \mathrm{B}_2)$, conveys a general sense of how each of the rules expands from its predecessor.

In *2001*, Iemhoff proved that for all n, $\vdash^{\mathcal{V}_n}_{\mathrm{IPC}} \subset \vdash^{\mathcal{V}_{n+1}}_{\mathrm{IPC}}$, so the rules $\{\mathcal{V}_i\}_{i=1}^{\infty}$ are independent of their predecessors as generators of consequence relations over $\vdash_{\mathrm{IPC}}$. Thus although $\{\mathcal{V}_i\}_{i=1}^{\infty}$ is a basis for IPC-admissibility, no finite collection of Visser's rules is.

One possible attempt at executing the Frege-Brouwer dynamics for IPC—one meant to bypass the non-monotonicity problem—might be simply to add each $\mathcal{V}_n$ to IPC as a theorem. It is instructive to see why this idea is misconceived.

Theorem 4. *Each rule in the sequence* $\{\mathcal{V}_i\}_{i=1}^{\infty}$ *is derivable in* IPC + $\mathcal{V}_1$.

[28] The rules are named after Albert Visser who devised them and conjectured that they form a basis, over IPC, for IPC-admissibility. Iemhoff verified this in *2001*.

Proof. It suffices to show that $\mathcal{V}_2$ is derivable in IPC + $\mathcal{V}_1$ and to observe that the same argument can be used to show that $\mathcal{V}_{n+1}$ is derivable in IPC + $\mathcal{V}_n$.

Assume $(\mathrm{A}_1 \to \mathrm{B}_1) \wedge (\mathrm{A}_2 \to \mathrm{B}_2) \to (\mathrm{A}_3 \vee \mathrm{A}_4)$ and reason inside IPC + $\mathcal{V}_1$. From this, $(\mathrm{A}_1 \to \mathrm{B}_1) \to ((\mathrm{A}_2 \to \mathrm{B}_2) \to (\mathrm{A}_3 \vee \mathrm{A}_4))$ follows. Now the right hand side of this formula is an instance of the left hand side of $\mathcal{V}_1$. Therefore

$$(\mathrm{A}_1 \to \mathrm{B}_1) \to (((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_2) \vee ((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_3) \vee ((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_4))$$

follows. But now this whole formula is an instance of $\mathcal{V}_1$ and so leads to

$$\begin{gathered}((\mathrm{A}_1 \to \mathrm{B}_1) \to \mathrm{A}_1) \vee \\ ((\mathrm{A}_1 \to \mathrm{B}_1) \to (((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_2) \vee ((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_3))) \vee \\ ((\mathrm{A}_1 \to \mathrm{B}_1) \to ((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_4)).\end{gathered}$$

Now let $\chi = (\mathrm{A}_1 \to \mathrm{B}_1) \wedge (\mathrm{A}_2 \to \mathrm{B}_2)$ and observe that from the first disjunct of this formula one can infer $\chi \to \mathrm{A}_1$. Meanwhile the second disjunct is another instance of $\mathcal{V}_1$, from which $((\mathrm{A}_1 \to \mathrm{B}_1) \to \mathrm{A}_1) \vee ((\mathrm{A}_1 \to \mathrm{B}_1) \to ((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_2)) \vee ((\mathrm{A}_1 \to \mathrm{B}_1) \to ((\mathrm{A}_2 \to \mathrm{B}_2) \to \mathrm{A}_3))$ follows. The same reasoning we have been using can be used again to reason from this to $(\chi \to \mathrm{A}_1) \vee (\chi \to \mathrm{A}_2) \vee (\chi \to \mathrm{A}_3)$. Finally, from the third disjunct we infer $\chi \to \mathrm{A}_4$. Repeated applications of the $\vee$-introduction rule allows us to infer from each of these disjuncts the same formula $(\chi \to \mathrm{A}_1) \vee (\chi \to \mathrm{A}_2) \vee (\chi \to \mathrm{A}_3) \vee (\chi \to \mathrm{A}_4)$. It follows by $\vee$-elimination that $\mathcal{V}_2$ is derivable. □

It seems almost paradoxical that the Visser rules could be independent in the sense that $\mathcal{V}_{n+1}$ is not derivable in $\left|\frac{\mathcal{V}_n}{\mathrm{IPC}}\right.$ and yet completely redundant in the sense that every $\mathcal{V}_n$ is derivable in IPC + $\mathcal{V}_1$. This result underscores a profound difference between admissibility and derivability that is easy to overlook.

That difference is evident again in the fact that admissibility in IPC is not accurately recovered as provability in IPC + $\mathcal{V}_1$. To illustrate this,[29] let us use v_1 as an abbreviation of the conditional corresponding to $\mathcal{V}_1$, i.e., $\mathrm{v}_1 = ((\mathrm{A} \to \mathrm{B}) \to (\mathrm{C} \vee \mathrm{D})) \to (((\mathrm{A} \to \mathrm{B}) \to \mathrm{A}) \vee ((\mathrm{A} \to \mathrm{B}) \to \mathrm{C}) \vee ((\mathrm{A} \to \mathrm{B}) \to \mathrm{D}))$. Then the rule $\dfrac{\mathrm{A} \to \mathrm{A}}{\mathrm{v}_1}$ is derivable in IPC + $\mathcal{V}_1$ although not admissible in IPC. The theorem structure of IPC + $\mathcal{V}_1$ oversimplifies IPC-admissibility by including *too much*.

As Lorenzen emphasized, some intuition that outstrips mere formal derivability is needed in order to discover the "validity" of some inference rules, and in this sense deductive validity is a subtler affair than what some have claimed it to be. If one's reaction upon discovering that

[29]This simple example was suggested by Rosalie Iemhoff, discussions with whom have been instructive and inspirational for many aspects of this project.

a rule is valid in this way is to lump it together with the rules than *are* formally derivable, to change one's logic so that this rule will be represented within the formal system, the result is not a new formal system with all the subtlety of the previous system's admissible rules built into its theorem structure but a formal system in which that subtlety has been exchanged for a coarser notion of formal derivability.

8. Context sensitivity

Probably the first example of an underivable but admissible rule for IPC discoverd is Rose's rule,

$$\frac{(\neg\neg A \to A) \to (\neg\neg A \lor \neg A)}{\neg\neg A \lor \neg A}.$$

In response to Kleene's *1945* "realizability" interpretation of the formulas of IPC, G. F. Rose (*1953*) put forward this rule, every instance of which is realizable yet which is not derivable in IPC. By the time (*1957*) Kreisel and Putnam refuted Łukasiewicz's conjecture with the observation that IPC + Harrop's rule has the disjunction property, they were able to remark that Dana Scott had already made the same observation about IPC + Rose's rule and even about a generalization of this rule (with $\neg A \lor A$ standing in place of $\neg\neg A \lor \neg A$ in the consequent position of the premise). It is unclear from the publication record exactly who observed what and when, but the evidence suggests that these investigations were prompted in part by the inadequacy of IPC with respect to realizability.

It is an important open problem today to find a formal logic that is sound and complete with respect to realizability. But it is a mistake to think that IPC is somehow inadequate in virtue of not filling this role. Systems like IPC + Rose's rule are departures from IPC, not improvements of the same idea. It is fine to depart from IPC because what one is interested in is, instead of the properties of IPC itself, an analysis of realizability. But it is fine as well to remain interested in IPC, perhaps even especially so because of how phenomena like realizability spread across its entire collective of derivable and underivable admissible rules.

What we have learned from the case of IPC is that some inferential frameworks are context sensitive. They have some rules that apply in arbitrary contexts and another, larger set of rules that apply only to their provable formulas. We have considered two different proposals for closing this gap between derivability and admissibility.

One approach is just to expand the logic's consequence relation so that the distinction between admissibility and derivability vanishes. One is left with the same theorems one began with, so from one perspective one has only made a change in "design choice" and left the logic as one found it. However, one's logical connectives no longer correspond with natural logical operators on the new consequence relation. The Deduction Theorem fails, the harmony between introduction and elimination rules is lost, and the logical vocabulary no longer means what it meant before. From a broader view of the value and intrigue of logic, much has been lost.

The other approach is to build admissible rules directly into the logic as new theorems. Here there are plenty of pitfalls awaiting. In some settings one risks arriving at an inconsistent system. Even in IPC the result is highly contingent on the order in which theorems are added, and there is no guarantee that by starting at IPC instead of one of its fragments one hasn't already closed off some promising possible avenues. More serious than these specific traps, though, is the fact that however one proceeds, one will be abandoning a rich terrain for a comparably simple one.

We are compelled to study IPC for several reasons. It is a leading attempt to systematize Brouwer's unique philosophical perspective. It is the product of the inferentialist analysis of the meanings of ordinary logical particles. It is the "inner logic" of topoi and of the generalized concept of logical consequence. This framework, which appears again and again in unrelated places, gives rise to a bifurcated notion of deductive validity, one immanent, relatively simple, and available in all contexts, the other transcendent, intriguingly complex, and descriptive of the special relationships in which its provable formulas stand. More, it mysteriously evades all of our attempts to provide for it a well-defined interpretive scheme.[30] Far from things to take measures to correct, these are all the more reason to pay attention.

Context is plainly relevant for logical inference. One's circumstances determine what can and cannot be invoked as premises. What no one imagined until the last century is that for some purposes context is relevant even when idling in the background—that which rules are even available for inferring can depend on circumstances that don't figure directly into one's reasoning. That a series of incremental moves leading from Frege, through Gödel, the intuitionist logicians, and finally solidifying in the work of Lorenzen could reveal that this is the case is surely a significant philosophical development.

Yet it has a history not only of being unappreciated but of going unnoticed. In his *1957* review of *Lorenzen 1955*, Thoralf Skolem does not indicate that Lorenzen's definition of admissibility is novel and in fact does not even mention the concept. Skolem even declares that "apart from a subjective terminology the contents of Part I are in agreement with the usual points of view in mathematical logic" (p. 289). Even in recent decades, authors of textbooks in modal logic have been divided about whether the Deduction Theorem holds in standard systems like S4 and S5, apparently the result of some of the authors overlooking the fact that the necessitation rule for these systems is admissible but not derivable (*Hakli and Negri 2010*).

Some inferential structures are context sensitive. Our understanding of deductive validity can only be enriched by attending to this remarkable fact.

[30]*Terwijn 2004* provides a vivid history of these evasions.

References

van Atten, M. 2009. "The development of intuitionistic logic," *Stanford Encyclopedia of Philosophy*.

Brouwer, L. E. J. 1928. "Mathematics, science, and language," in W. Ewald *From Kant to Hilbert: a sourcebook in the foundations of mathematics* vol. II (1996), 1175–85.

Bynum, T. W. (ed.) 1972. *Conceptual Notation and Related Articles*. New York: Oxford University Press.

Chagrov A. V. 1992. "A decidable modal logic with the undecidable admissibility problem for inference rules," *Algebra and Logic*, **31**, 53–55.

Citkin, A. I. 1977. "On admissible rules for intuitionistic propositional logic," *Sbornik: Mathematics*, **31**, 279–88.

Currie, G. 1987. "Remarks on Frege's conception of inference," *Notre Dame Journal of Formal Logic*, **28** (1), 55–68.

van Dalen, D. 2005. *Mystic, Geometer, and Intuitionist: the life of L. E. J. Brouwer, vol II. Hope and Disillusion*. New York: Oxford University Press.

Dawson, J. W. 1993. "The compactness of first-order logic: from Gödel to Lindström," *History and Philosophy of Logic*, **14** (1), 15–37.

Feferman, S., J. W. Dawson, Jr., S.C. Kleene, G. H. Moore, R. M. Solovay, and J. van Heijenoort (eds.) 1986. *Kurt Gödel, Collected Works, Vol I: Publications 1929–1936*. New York: Oxford University Press.

Franks, C. 2010, "Cut as consequence," *History and Philosophy of Logic*, **31** (4), 349–79.

Franks, C. 2017, "Glivenko's theorem," http://www.nd.edu/ ˜ cfranks/writing/glivenko.pdf

Frege, G. 1879. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle: L. Nebert. Translated by T. W. Bynum as *Conceptual Notation: a formula language of pure thought modeled upon the formula language of arithmetic* in *Bynum 1972*, 101–208.

Frege, G. 1882. "Über die wissenschaftliche Berechtigung einer Begriffsschrift," *Zeitschrift für Philosophie und philosophische Kritik*, **81**, 48–56. Translated by T. W. Bynum as "On the scientific justification of a conceptual notation' in *Bynum 1972*, 83–9.

Frege, G. 1883. "Über den Zweck der Begriffsschrift," *Sitzungsberichte der Jenaischen Gesellschaft für Medizin und Naturwissenschaft für das Jahr 1882*, **16**, 1–10. Translated by T. W. Bynum as "On the aim of the "Conceptual Notation' ' in *Bynum 1972*, 90–100.

Frege, G. 1906. “Foundations of geometry: second series,” translated and reprinted in *Frege 1984*, 293–340.

Frege, G. 1910. “Letter to Jourdain,” translated and reprinted in *Frege 1980*.

Frege, G. 1917. “Letter to Dingler,” translated and reprinted in *Frege 1980*.

Frege, G. 1918. “Negation,” translated and reprinted in *Frege 1984*.

Frege, G. 1923. “Compound thoughts,” translated and reprinted in *Frege 1984*.

Frege, G. 1980. *Philosophical and Mathematical Correspondence*. G. Gabriel, et al. (eds.). Chicago: University of Chicago Press.

Frege, G. 1984. *Collected Papers*. B. F. McGuiness (ed.). Oxford: Blackwell.

Gentzen, G. 1932. “Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen,” *Mathematische Annalen*, **107**, 329–50. Translated as “On the existence of independent axiomsystems for infinite sentence systems’ in *Szabo 1969*, 29–52.

Gentzen, G. 1934–35, “Untersuchungen über das logische Schliessen.” Gentzen’s doctoral thesis at the University of Göttingen, translated as “Investigations into logical deduction’ in *Szabo 1969*, 68–131.

Glivenko, V. 1929. “Sur quelques points de la logique de M. Brouwer,” *Bulletins de la classe des sciences*, **15** (5), 183–88.

Gödel, K. 1930. “Die Vollständigkeit der Axiome des logischen Functionenkalküls,” *Monatshefte für Mathematik und Physik*, **37**, 349–60. Translated by S. Bauer-Mengelberg as “The completeness of the axioms of the functional calculus of logic’ reprinted in *Feferman et al. 1986*, 102–23.

Gödel, K. 1932, “Zum intuitionistischen Aussagenkalkül,” *Anzeiger der Akademie der Wissenschaftischen in Wien* **69**: 65–66. Translated by J. Dawson as “On the intuitionistic propositional calculus’ in *Feferman et al. 1986*, 223–25.

Gurevich, Y. 1984 “Toward logic tailored for computational complexity,” in *Computation and Proof Theory* M. Richter, et al. (eds.). Springer Lecture Notes in Mathematics. **1104**: 175–216.

Hakli, R. and S. Negri 2012. “Does the deduction theorem fail for modal logic?,” *Synthese*, **187** (3), 849–67.

Harrop, R. 1956. “On disjunctions and existential statements in intuitionistic systems of logic,” *Mathematische Annalen*, **132**, 347–61.

van Heijenoort, J. 1967. *From Frege to Gödel: A Source Book in Mathematical Logic: 1879–1931*. Cambridge: Harvard University Press.

Heyting, A. 1956. *Intuitionism: an introduction*. Amsterdam: North Holland.

Iemhoff, 2001. "Provability Logic and Admissible Rules," PhD. thesis, Institute for Logic, Language, and Computation, Universiteit van Amsterdam.

Iemhoff, 2016. "Consequence relations and admissible rules," *Journal of Philosophical Logic*, **45** (3), 327–48.

Johansson, I. 1937. "Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus," *Compositio Mathematica*, **4**, 119–136.

de Jongh, D. H. J. 1968. "Investigations on the intuitionistic propositional calculus," PhD. thesis, University of Wisconsin, Madison.

Kleene, S. C. 1945, "On the interpretation of intuitionistic number theory," *Journal of Symbolic Logic*, **10**: 109–24.

Kolmogorov, A. 1925. "О принципе tertium non datur," *Sbornik: Mathematics*, **32**(4), 646-67. Translated as "On the principle of the excluded middle' by J. van Heijenoort in *van Heijenoort 1967*, 416–37.

Koslow, A. 1992. *A structuralist theory of logic*. Cambridge University Press.

Kreisel, G. and H. Putnam 1957. "Eine Unableitbarkeitsbeweism ethode für den Intuitionistischen Aussagenkalkül," *Zeitschrift für Mathematische Logik and Grundlagen der Mathematik* **3**, 74–78.

Lawvere, F. W. 1964. "Elementary Theory of the Category of Sets," *Proceedings of the National Academy of Science*, **52** (6), 1506–11.

Lorenzen, P. 1955. *Einführung in die operative Logik und Mathematik*. Berlin: Springer.

Łukasiewicz, J. 1952. "On the intuitionistic theory of deduction," *Koninklijke Nederlandse Akademie van Wetenschappen* Proceedings, series A, **55**, 202–12.

van der Molen, T. 2016. "The Johansson/Heyting letters and the birth of minimal logic," Institute for Logic, Language, and Computation, Amsterdam.

Plato. "Crito," trans. by G. M. A. Grube, in *Plato: Complete Works* (1997), ed. by J. M. Cooper, Indianapolis: Hackett.

Quine, W. V. O. 1970. *Philosophy of Logic*. Minneapolis: Prentice-Hall.

Robinson, A. 1951. *On the Metamathematics of Algebra*. Amsterdam: North Holland.

Rose, G. F. 1953. “Propositional calculus and realizability,” *Transactions of the American Mathematical Society*, **75**, 1–19.

Rybakov, V. 1992. “Rules of inference with parameters for intuitionistic logic,” *Journal Symbolic Logic*, **57** (3), 912–23.

Rybakov, V. 1997. *Admissibility of logical inference rules*. Amsterdam: Elsevier.

Schroeder-Heister, P. 2008. “Lorenzen’s operative justification of intuitionistic logic,” in M. van Atten, P. Boldini, M. Bourdeau, G. Heinzmann (eds.), *One Hundred Years of Intuitionism (1907–2007)*, Basel: Birkhäuser.

Skolem, T. 1957. “Review: Paul Lorenzen, *Einfuhrung in die Operative Logik und Mathematik*,” *Journal of Symbolic Logic*, **22** (3), 289–90.

Smith, N. J. J. 2009. “Frege’s judgement stroke and the conception of logic as the study of inference not consequence,” *Philosophy Compass*, **4** (4), 639–65.

Solovay, R. M. 1976. “Provability interpretations of modal logic.” *Israel Journal of Mathematics* **25**: 287–304.

Stoothoff, R. H. 1963. “Note on a doctrine of Frege,” *Mind*, **72** (287), 406–8.

Szabo, M. E. 1969. *The collected papers of Gerhard Gentzen*, London: North Holland.

Terwijn, S. A. 2004, “Intuitionistic logic and computation,” *Vriendenboek ofwel Liber Amicorum ter gelegenheid van het afscheid van Dick de Jongh*, Institute for Logic, Language, and Computation. Amsterdam.