The Fate of the Syllogism
in the Göttingen school

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A quiet revolution

One often hears of a radical transformation in our conception of logic ushered in by the work of Gottlob Frege, a transformation away from thinking of logic in terms of the theory of syllogism and towards the axiomatic propositional and predicate calculi, later pursued by Russell, Tarksi, Heyting, Łukasiewicz, and others. Under this transformation, the assorted patterns of syllogistic inference, which for centuries had been thought of as primitive inference schemes from which the cogency of rational discourse could be evaluated, became objects of study and analysis from a more fundamental and more fine-grained vantage point.

A roughly simultaneous transformation took place that one hears about less often. Even if several of the technical results it brought about are familiar individually, this second transformation is not well understood as a whole. What is perhaps most characteristic of this quiet revolution is that rather than being dislodged from their place as first principles and resituated as complex patterns of reasoning analyzable from some other more basic starting point, syllogistic inference was simply cast alongside other patterns of inference. Relieved of the burden of isolating "the real first principles of reasoning," a tradition of logicians were able to identify other questions about the role of syllogistic reasoning in terms of its relationship with non-syllogistic inference.

The culmination of each transformation can be seen as a sort of exorcism of the syllogism from formal logic. In the more familiar scheme, the syllogism has disappeared from the foundations of logic entirely. Its trace is its complex recovery as a derived object from the what are considered logic's proper foundations. In the quiet revolution, one finds such cryptic and dramatic announcements about the fate of the old logic as the following remark of Jacques Herbrand (1908–31): "the theorem [just presented] permits us to show [...] that the rule of the syllogism, the basis of Aristotelean logic, is of no use in any mathematical argument."[1] But here again the exorcism leaves behind a trace. In his thesis, just after declaring again that his work "shows [...] that the rule of implication, whose origin, after all, is in the classical syllogism, is not necessary in building logic," Herbrand adds: "The rule remains necessary, however, in mathematical theories, which contain hypotheses."[2]

The eliminability result Herbrand refers to in the passages just cited is familiar and well understood. So from one point of view, what Herbrand means is uncontroversial. What is less familiar and not so well understood is the context of Herbrand's discovery. What sort of ideas about the syllogism inspired efforts by Herbrand and Gerhard Gentzen (1909–45) to demonstrate its "eliminability" and to study the trace it leaves behind in theories containing hypotheses? In answering this question, we will see that certain classical ideas about the syllogism that have been alleged to be irrecoverably lost in the transition to modern mathematical logic are in fact very much alive, inspiring profound discoveries and informing our modern conceptions of such central modern notions as logical completeness and deductive validity.

The capsule statement of the view promoted here is that the celebrated theorems of Gentzen and Herbrand, because of how they are informed by Aristotle's own analytic/synthetic distinction as well as his concept of logical completeness, underline the centrality of syllogistic reasoning precisely by "eliminating" it in the now familiar way. Those theorems can and should be read as completeness results. But as I've already suggested, because it depends on understanding the context of discovery, the full meaning of the view is not easily encapsulated. Therefore we will proceed by establishing the context in which the Göttingen logicians worked through a sequence of historical moments, beginning with Aristotle himself.

Except as noted in occasional remarks, the picture that emerges will not be of a culmination of a historical progression of thought. We will not be concerned to establish—nor even to speculate about—direct lines of influences. We turn to historical moments instead because they present conceptual frameworks alternative to those that prevail today. These frameworks then provide the key to understanding the syllogism's role in the developments in Göttingen in the early 20th century.

Aristotle's *Prior Analytics*

In *Prior Analytics* I23, Aristotle wrote:

> It is clear from what has been said that the deductions in these figures are completed by means of the universal deductions in the first figure and are reduced to them.
>
> That every deduction without qualification can be so treated will be clear presently when it has been proved that every deduction is formed through one or another of these figures. [...]
>
> But if this is true, every demonstration and every deduction must be formed be means of the three figures mentioned above.
>
> But when this has been shown it is clear that every deduction is completed by means of the first figure and is reducible to the universal deductions in this figure.[3]

The naive reading of this passage certainly suggests that Aristotle is putting forward a claim of logical completeness, with "these figures" serving as a trunnion: "Every deduction without qualification" will be shown to be formed through them, and they will in turn be "completed" and "reduced" to the universal deductions in the first figure.

But before 1994, Aristotle's readers resisted this naive reading, and it is interesting to consider why. Informed by an understanding of how logical completeness has been conceived as well as how it has been established since Bernays, Gödel, and Henkin, several scholars noticed that several key features of that conception and style of demonstration were absent from Aristotle's account. They concluded that we are misled by Aristotle's language and our contemporary interests to think that he was after logical completeness—that instead he was looking for something more particularly related to the idiosyncratic features of his framework with no vivid parallel in contemporary logic.

Timothy Smiley explained:

> The natural inference from the [passages extracted from I23] is that the intervening material represents a completeness proof. Corcoran made this very point, but he was deterred from following it up by two objections. One was that the text did not fit his picture of a completeness proof. The other was that Aristotle was not "clear enough about his own semantics to understand the problem" of completeness. Corcoran therefore fell back on seeing the chapter, not as finishing off a completeness proof for Aristotle's chosen rules of inference, but as supplying a proof of the equivalence between them and a second set of rules [. . .] .[4]

He ascribed a similar resistance to this natural inference to himself and then added:

> Lear made the point that "it would be anachronistic to attribute to Aristotle the ability to raise the question of completeness" because, unlike a modern logician, "Aristotle had a unified notion of logical consequence—not the bifurcated notion of semantics and syntactic consequence."[5]

But Smiley concluded that the very thing that informed these scholars also blinded them: "As to one's picture of a completeness proof, it is quite true that Aristotle's proof is unlike anything one would expect [. . .] but in emphasizing the difference between Aristotle's project and the modern one there is a danger of overlooking their similarity; a similarity that seems to me to be more significant than their difference" (27). True, Aristotle did not have anything like clearly demarcated realms of semantics and syntax that a completeness theorem would tether to one another as we are accustomed to doing. But, Smiley observed, "Aristotle was conscious of the distinction between what follows and what can be shown to follow"—a distinction that does not depend on distinctly articulated semantic and syntactic categories—and his proof in I23 is precisely a demonstration that everything that in fact follows can be shown to do so.

In the later sections I will argue that the "similarity between Aristotle's project and the modern one" runs much deeper even than Smiley pointed out. A corollary to this is that our being informed by the prevailing conception of logical completeness blinds us even to the conceptual significance of portions of our modern project. But the first step towards seeing the ripples of the Aristotelean conception in 20th century Göttingen is just to appreciate Smiley's reconstruction of Aristotle's own proof, which will comprise the remainder of this section.

Aristotle wrote:

> If one wants to deduce that A belongs or does not belong to B, one must assume something of something. If no A should be assumed of B, the proposition originally in question will have been assumed. But if A should be assumed of C, but C should not be assumed of anything, nor anything of it, nor anything else of A, no deduction will be possible. [...]
>
> So we must take a middle term relating to both, which will connect the predications, if we are to have a deduction relating this to that.

The proof begins by arguing that a certain sort of linkage is a necessary condition for the validity of arguments in general. Smiley formulates this "chain condition," as he calls it, as a stipulation that any valid argument must be of the form "AC, CD, DE, EF, ..., GH, HB; therefore AB," where expressions XY represent propositions of one of the Aristotelean forms with terms X and Y (either one being in the subject position). So as with modern formulations of "relevance" constraints on deductive validity, the chain condition serves to ensure relevance by stipulating that expressions will have common terms.

He continued:

> If then we must take something in common in relation to both, and this is possible in three ways (either by predicating A of C and C of B, or C of both, or both of C), and these are the figures of which we have spoken, it is clear that every deduction must be made in one or other of these figures.
>
> The argument is the same if several middle terms should be necessary to establish the relation to B; for the figure will be the same whether there is one middle term or many.

The first of these sentences is about the case where there is a single middle term. This, Smiley discovered, plays the role of a base step in an inductive argument. The next sentence is Aristotle's presentation of the general case, where there is an arbitrary number of middle terms. Although it is compressed, Smiley cited textual clues that led him to interpret it as an inductive step.

Here is the expansion he suggested: "The argument is the same if several middle terms should be necessary to establish the relation of A to B. For suppose C is next to another middle term D. Then C stands as a middle term between A and D, and, as we have just said, this is possible in three ways. If D is next to another middle term E we apply the same argument to

A D E, and so on as many times as required, until we have eventually related A and B through a series of deductions each of which is in one or another of the three figures" (31).

In Smiley's reconstruction one can see the chain condition at work, allowing a linear progression through the arbitrary finite number of middle terms C, D, E, etc. separating A from B in the premises. One can also see, as Smiley himself pointed out, two glaring problems with the proof:

1. It assumes without justification that AC and CD are a "conclusive pair," i.e. that some conclusion of the form AD follows from them. In Smiley's memorable quip, this is analogous to an assumption "that each pair of ingredients in a high explosive has to be explosive" (33).

2. It assumes without justification, using the second step in the inductive argument as an example of the general problem, that AB follows from AD, DE, EF, . . . , GH, HB. In general, AD will be a weaker sentence than the combination of AC and CD, so replacing the latter combination with the former sentence will potentially (and indeed will typically) weaken the sequence of premises so that not everything (including possibly AB) that followed from them originally still does so.

Smiley attended to these gaps in Aristotle's reasoning and suggests various considerations drawn from the broader context of his oeuvre that might excuse them. But as we are only interested in the style of argument Aristotle provides and not its success, we will not consider these matters. Here, then, is Aristotle's full, though possibly erroneous, completeness proof as reconstructed by Smiley:[6]

Proof. We are to show that a deduction with several premises can always be reduced to a series of syllogisms in one or other of the four specified figures.

Suppose therefore that AB follows from premises AC, CD, DE, . . . , GH, HB.

First we show that AC and CD are a conclusive pair. Do this by showing that if not, terms can be chosen so as to make AB false but all the premises true. Simply choose terms to make AB false and then appropriate terms to make each of HB, GH, . . . , DE true. Having done that, enumerate all combinations of AC and CD that give rise to an inconclusive pair (64 of these!). In the 37 inconclusive ones, it is possible to choose a term for C that makes both AC and CD true.

Now choose some strongest conclusion of AC and CD, AD. An inductive argument shows that AD and DE are a conclusive pair, and so on until AH and BH. At this last stage, one gets not only that AH and BH are conclusive, but in fact that AB follows from them (for otherwise we could make AB false and these two premises true, contradicting the induction hypothesis.)

It follows that the deduction of AB can be effected through a series of syllogisms, AD following from AC and CD, AE from AD and DE, and so on until finally AB follows from AH and HB. □

Smiley described Aristotle as recognizing the difference between what follows and what can be shown to follow. We can now elaborate on that description. Aristotle asked whether every categorical statement that follows from a finite set of premises, in the sense that no substitution of categorical terms could simultaneously make those premises all true while falsifying the candidate conclusion, could be formally deduced from those premises with a predesignated stock of inference rules. Both his notion of categorical term substitution and his derivational apparatus were meager: The former had no stock of permitted moves so precisely designated as to strike us as well-defined. That is, Aristotle had no full fledged semantic theory. The latter governed such a small number of inferential patterns as to leave untouched many logical relationships one might hope to formalize. Still, Aristotle showed that the two were perfectly suited to one another, and we observed that, in fact, the curious features of each are just what allow one to get away with such rough and ready notions of logical consequence.

So one can ask if there is anything missing from Aristotle's deduction system and get some concrete answers. Crucially, if we accept the relevance constraints Aristotle places on argumentative validity, and if we overlook the two problems we noted in the execution of the proof procedure, the inductive argument establishes that there are no valid consequence relationships among categorical statements that his deductive system cannot accommodate in full. Pried free from concerns of relevance, this is the same conceptualization that we will see featured in the work of Gentzen and his Göttingen colleague Paul Hertz (1881–1940).

Bolzano's proto-proof theory

There is no doubt about the influence of Aristotle on Bernard Bolzano's (1781–1848) logical investigations, as presented in *Beyträge* and *Wissenschaftslehre*.[7] Bolzano explicitly appealed to Aristotle's distinction between analytic and synthetic reasoning and he applied this distinction to a theoretical treatment of Aristotle's doctrine of topical purity according to which a properly scientific exposition of a topic must not involve any essential use of concepts external to the terms of that topic. According to Bolzano, the culprit in every "impure" exposition is the use of synthetic inference. A proper exposition is comprised solely of analysis. Aristotle himself was sensitive to this issue, but hoped to steer away from impurity by requiring certain conceptual "connections" between the terms in each syllogism:

> Because accidents are not necessary one does not necessarily have reasoned knowledge of a conclusion drawn from them (this is so even if the accidental premises are invariable but not essential, as in proofs through signs; for though the conclusion be actually essential, one will not know it as essential nor know its reason); but to have reasoned knowledge of a conclusion is to know it through its cause. We may conclude that the middle must be consequentially connected with the minor, and the major with the middle. (*Posterior Analytics* I6)

Not content with Aristotle's vague idea about connections among concepts that appear in them, Bolzano advocated instead for an all out ban on syllogisms.

Bolzano's development of this line of thought in the context of formal logic was driven by a metaphysical doctrine: He believed that the distinction between axioms and theorems was absolute and objective, that there were basic truths for which no proof can be given and that all other truths are grounded in these. He insisted, as well, that for non-basic truths exactly one proof exists, one that discloses the objective grounds behind its truth. These convictions are obviously hard to reconcile with the actual development of mathematics, in which the framework of inquiry shifts continuously without any evident appeal to absolutism or objective grounds. And this point was hardly lost on Bolzano, who sustained a radical critique of mathematics as it was practiced in his day precisely because of its single-minded focus on solving problems, uncovering relationships, enhancing human understanding, and discovering truths without attending to questions about proper axiom-hood and objective grounds.[8]

It is easy to see the connection between the Aristotelean themes that Bolzano sought to develop and his metaphysical program. Synthetic reasoning abounds in human inquiry and especially so in mathematical thought. But synthesis is impure in the sense that it courses through terms and concepts topically isolated from those that figure in the conclusion of a line of thought. Therefore, though it may serve us well in convincing us of the truth of a claim, it cannot be expected to uncover the objective reasons behind those truths. As Aristotle had maintained, the truths of arithmetic cannot rest on those of geometry, nor vice versa.[9] Bolzano agreed, and he saw in the syllogistic forms an inevitable source for such impurity: These forms are characterized by "middle terms" that appear among the premises but not in the concluding statement. Because the concepts that appear in a syllogism's conclusion put no constraints on what those middle terms can be, the syllogism's premises cannot be guaranteed to be among the grounds of its conclusion, and so the syllogism itself cannot be part of a properly scientific exposition: "I must point out that I believed I could not be satisfied with a completely strict proof *if it were not even derived from concepts* which the thesis proved contained, but rather made use of some fortuitous alien, *intermediate concept* [*Mittelbegriff*], which is always an erroneous μετάβασις εἰς ἄλλο γένος[10]" (1810, Preface, par. 4).

Inspired by this network of ideas, Bolzano set out to design a framework of logic that would lead to the discovery of "real proofs" that uncover a theorem's objective grounds. Such proofs had to meet the standard of analyticity, and so syllogistic theory could no longer be the central component of logic. Accordingly, the first step in Bolzano's development of logic was to *supplement* syllogistic theory with analytic rules: "I believe that there are some *simple kinds of inference* apart from the syllogism" (§12). This, however, only provided the framework. A grid for transcribing mathematical reasoning into explicit formal derivations that highlight and distinguish synthetic and analytic inferences will only lead to the discovery of "real proofs" if occasional bits of reasoning turn up to be purely analytic under the transcription. But Bolzano's critical attitude about mathematics as it was practiced was just a skepticism about the chances of this ever happening. He wanted his logic to be an instrument, not just for detecting real proofs for what they are, but for constructing them.

The second step in Bolzano's development of logic, then, was a theory of proof transformation. He described ways not just to transcribe mathematical reasoning, but to take such transcriptions and manipulate them so that their synthetic components could be systematically removed. Comparing his notion of analytic and synthetic proof, Bolzano wrote in (1810) §31, "the whole difference between these two kinds of proof is based simply on the *order and sequence* of the propositions in the exposition." In §27 he elaborated: "If several propositions appearing in a scientific system have the same subject, then the proposition with the more compound predicate must follow that with the simpler predicate and not conversely." It has been suggested that the origin of the tree representation of formal proofs can be found in the work of Hertz and Gentzen at Göttingen.[11] But a century earlier, in his explication of the theory of grounding in §220 of (1937), Bolzano described proof trees explicitly, indicating two distinct presentations: one in which tokens of a single proposition can appear at multiple nodes, serving as premises for more than one inference, and one which "avoids this" feature. Dissatisfied with both representational schemes, he wrote: "But it is obvious that in cases where one and the same truth is repeated several times the representation becomes very confusing. Let others find a better method." Unfortunately, Bolzano never described actual operations on such trees, presumably because his logic was too informal to admit a precise enough articulation of either structure. But he successfully argued informally for transformation procedures that could conceivably be translated into such operations, and the ambition behind this was clear: A technique for rendering logical derivations into a certain normal form would yield the topically pure, analytic proofs that Bolzano thought would disclose ultimate grounds. The normalization he sought was syllogism elimination.

Although Bolzano's strides in the theory of proof transformation were only partial, a few successes are noteworthy. There was, significantly, some feedback from the second step in his development back to the first. The shape of the logical framework Bolzano ultimately settled on was a radical departure from previous systems of logic not only in its inclusion of analytic rules of inference alongside the syllogistic forms: Bolzano (1810) refined those forms themselves, concluding because of discoveries made possible by proof transformations in his hybrid framework that instead of a vast array of syllogistic forms only a single synthetic inference rule was needed: "It is not without hesitation that I proceed to put forward my opinion, which is so very different from the usual one. Firstly, concerning the *syllogism*, I believe there is only a single, simple form of this, namely *Barbara* or Γράμματα in the *first* figure" (§12). Beginning with Aristotle himself, the systematization of the 24 (or so) valid syllogistic forms into those that are only conditionally valid and those that are unconditionally valid, those that are essential and those that are derivative, etc., had been a central component of the study of logic. The radical departure from traditional investigations of this kind made possible by Bolzano's hybrid framework is evident in the fact that in it this entire enterprise is obviated by rudimentary proof transformations. The first step towards the elimination of syllogism was its reduction to a single form.

A second notable success of Bolzano's is ironic because it is a discovery that he took to be a failure of his program. Bolzano claimed in §20 that with proper transformations, one could eliminate syllogism from that portion of a proof that involved sentences with compound subjects and predicates. But, as modern researchers will have anticipated, he observed that in doing so one will only push the syllogisms further up in the proof tree, towards its leafs. If the result of this process left one with some sentences further up in the tree that are not "axioms" in Bolzano's strong sense, despite having only simple subjects and predicates, then *their* derivation would have to be synthetic. "On the other hand," he wrote, "how propositions with simple concepts could be proved other than through a syllogism, I really do not know." This eventuality is obviously unsatisfactory for someone like Bolzano, whose whole purpose is to disclose objective grounds. But from another point of view, one can just be interested in the conditions under which syllogism is eliminable, and the specific role that it plays in situations where it cannot be. As we will soon see, the elimination theorems of the Göttingen school were pursued from just such a point of view, and the lessons to be learned from Bolzano's observation, which those theorems ratify and clarify, come into focus there precisely because of how the Göttingen logicians were unencumbered by Bolzano's ideology.

Another thread in Bolzano's thought that will inform our reading of the elimination theorems involves a parallel development of two distinct notions of logical consequence. One notion we have already seen. A purely analytic proof of the sort that Bolzano wanted normalization procedures to uncover would, he contended, display a grounding relationship. This is a metaphysically substantial sort of logical consequence because, beyond simple modal claims about which things must be true given the truth of other things, for a sentence to be a consequence in this sense is for those other truths to be the objective reasons for that sentence's truth.

But Bolzano also recognized a metaphysically thin consequence relation, under which a sentence follows from some other sentences simply because the their truth suffices for the truth of the former. Bolzano's (1837) specification of this second notion of consequence has its peculiarities that are foreign from the modern perspective: a fixed universe of discourse from which terms can be chosen, a "compatibility" condition on the premises that precludes "vacuous consequence" of the sort that characterizes the familiar classical notion. But the general bent is familiar: A sentence is a consequence of some other sentences if no systematic substitition of ideas for terms makes all the premises true and the conclusion false (§155).

It is natural to ask how these two notions relate to one another, and Bolzano did not miss the chance. What is surprising is that although we see in his analytic proof system an anticipation of formal deductive systems, and although we see in his thin concept of consequence as a precursor of the modern model-theoretic definition, Bolzano had things the other way around. He labeled the thin notion the "derivability" relation and asked about its adequacey to the substantial relation of objective grounding. So whereas we ask, "are our systems 'complete' in the sense that every logical consequence can be represented as such with a formal proof?" in §200, Bolzano asked instead whether to every objective consequence relation, i.e., everything for which there is a purely analytic proof, there corresponds a "derivation," i.e., a thin consequence relation:

> If truths are supposed to be related to each other as ground and consequence, they must always, one might believe, be derivable from one another as well. The relation of ground and consequence would then be such as to be considered a particular species of the relation of derivability; the first concept would be subordinate to the second.

Bolzano's "completeness" question was about the adequacy of his thin definition of logical consequence to his metaphysically substantial concept of following analytically. But he found nothing in his proof transformation techniques or in the details of the specifications of either notion of consequence that indicated a method for answering it: "Probable as this seems to me," he concluded, "I know no proof that would justify me in looking upon it as settled."

Thus inspired by Aristotle's account of the role of analytic thinking, Bolzano embarked on a logical program that led to two impasses. By identifying synthetic reasoning with the syllogism, Bolzano was inspired to articulate principles of non-syllogistic inference that should comprise all properly scientific expositions. This provided the framework for a theory of proof transformation, with the goal of systematically eliminating syllogisms from proofs—a goal that he himself concluded was likely unattainable. Meanwhile, with this concept of a purely analytical proof Bolzano isolated a concept of logical consequence distinct from the relation he called derivability. This exposed a question at the heart of his program—*How are these two concepts related?*—a question that Bolzano realized he did not have the resources to answer. In section 5 we will see how with the elimination theorems the Göttingen school surmounted both of these impasses. But first we consider how another Aristotelean thread recurs in their work.

Structural reasoning in Göttingen

In Gerhard Gentzen's first paper, he designed a formal calculus of structural reasoning.[12] Gentzen's main results show that his system is an adequate formalization of an informal notion of logical consequence. This formalization of logical consequence differs conceptually from semantic formalizations by Tarski and others, because the proof system is the locus of analysis. The full meaning of these results, we will see in the following section, is that they provide an interpretation of Gentzen's later work on cut-elimination: the eliminabilty of the CUT rule in a logical calculus means that the calculus's logical rules are complete in the sense that they suffice to derive all the logical consequences of a set of assumptions. But the results themselves and the proof Gentzen presented of them, recounted here, should first be compared with Aristotle's completeness proof as reconstructed in section 2.

Gentzen called his system a "formal definition of provability." It consists of "sentences" of the form $M \longrightarrow v$, where v is an "element" and M is a "complex" (a non-empty set of finitely many elements.) Gentzen represented set-theoretical union by concatenation, so complexes can be denoted with a single uppercase letter, with a string if lowercase letters, or even with a string that mixes the two. Gentzen referred to the complex at the left of a sentence's arrow symbol as its antecedent and to the lone element on the right of the arrow symbol as the succedent. "Tautologies" are sentences whose antecedent is the singleton set containing the same element that appears in the sentence's succedent, and more generally sentences whose antecedent contains the element in the succedent are "trivial."

Among the obstacles in the way of a full analysis of the concept of logical consequence is a convincing demarcation between the logical and non-logical parts of language. For example, determining whether propositions stand in Bolzano's "derivability" relation requires first knowing how finely one can carve them up into "ideas" and also how the parts of the propositions that are not "ideas" do their binding. "To be sure," Bolzano wrote, "this distinction has its ambiguity, because the domain of concepts belonging to logic is not so sharply demarcated that no dispute could ever arise over it" (1937: §148). Similar observations famously led Alfred Tarksi (1901–83) to the conclusion that the distinction was inevitably conventional.[13]

One of Gentzen's brilliancies was his decision simply to drop the whole question about which parts of a sentence are properly "logical" from the analysis of logical consequence. A fully general account of logical consequence, he reasoned, should not bore so far into the sub-sentential structure as to capture all possible nuance. It should remain as shallow as possible, picking up just the general features of sentences that figure into an all-purpose intuitive concept:

> We say that a complex of elements *satisfies* a given sentence if it either does not contain all antecedent elements of the sentence, or alternatively, contains all of them and also the succedent of that sentence. [...] We now look at the complex K of all (finitely many) elements of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ and $\mathfrak{q}$ and call $\mathfrak{q}$ a *consequence* of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ if (and only if) every subcomplex of K which satisfies the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ also satisfies $\mathfrak{q}$. (33)

Gentzen remarked: "[o]ur considerations do not depend on any particular kind of informal interpretation of the 'sentences,' since we are concerned only with their formal structure," and the modicum of formal structure his "sentences" display suffices (33). Gentzen wanted to clarify the intuitive concept of logical consequence just enough that it could be formalized.

He specified two inference rules for the system he designed to carry out this formalization, which he called "THINNING" and "CUT":

$$\frac{L \longrightarrow v}{ML \longrightarrow v}\ \textit{thinning} \qquad\qquad \frac{L \longrightarrow u \qquad Mu \longrightarrow v}{LM \longrightarrow v}\ \textit{cut}$$

Then he defined a "proof" of a sentence $\mathfrak{q}$ from the sentences $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ to be "an ordered succession of inferences (i.e., THINNINGs and CUTs) arranged in such a way that the conclusion of the last inference is $\mathfrak{q}$ and that its premises are either premises of the $\mathfrak{p}$'s or tautologies" (31). Like Bolzano, Gentzen wrote his proofs in tree-form. Here is an example proof of $Kf \longrightarrow d$ from $c \longrightarrow e$, $ef \longrightarrow a$, and $ac \longrightarrow d$, where $K = bc$:

$$\cfrac{\cfrac{\cfrac{c \longrightarrow e}{bc \longrightarrow e}\ \textit{thinning} \qquad ef \longrightarrow a}{Kf \longrightarrow a}\ \textit{cut} \qquad ac \longrightarrow d}{Kf \longrightarrow d}\ \textit{cut}$$

Gentzen's sentence system is a refinement of that of Hertz. Hertz had referred to the central rule, not as "cut," but as "syllogism."[14] The reason to do so is obvious: These rules share with the classical syllogistic forms an intermediate term that appears in each premise but is missing from the conclusion. In the terminology of both Aristotle and Bolzano, such inference is "synthetic" in that it passes from premises to a conclusion in which a new connection is made (between the antecedent of one premise and the succedent of another). The earlier connections to the middle term are lost. This contrasts with Bolzano's "analytic" rules in which all the terms that appear in the premises appear again, at least as sub-terms, in the conclusion.

In section 4, Gentzen wrote:

> Our formal definition of provability, and, more generally, our choice of the forms of inference will seem appropriate only if it is certain that a sentence $\mathfrak{q}$ is "provable" from the sentences $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ if and only if it represents informally a consequence of the $\mathfrak{p}$'s. (33)

The sense in which Gentzen meant for this proof system to be an adequate formalization of the intuitive concept of logical completeness is then captured by his two central theorems. The first states that the proof system is "correct."[15]

Theorem 1. *If a sentence $\mathfrak{q}$ is provable from the sentences $\mathfrak{p}_1, \dots, \mathfrak{p}_\mathfrak{v}$ then it is a consequence of them.*

Proof. Observe first that the conclusion of a thinning of a sentence is a consequence of that sentence: Suppose the complex K satisfies $L \longrightarrow v$. Either it does not contain every element in L or else it contains v. Either way, K also satisfies $ML \longrightarrow v$. Similarly, the conclusion of a CUT of two sentences is a consequence of those sentences: Any K that does not satisfy

$LM \longrightarrow v$ must contain all the elements in L and all the elements in M but not v. If K furthermore contains u, then it fails to satisfy $Mu \longrightarrow v$; otherwise it fails to satisfy $L \longrightarrow u$. Observe, now, that every tautology is a consequence of every sentence, and every sentence is a consequence of itself. From this last observation, it follows that every initial sentence in a proof from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ is a consequence of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$. By the first observation and the evident transitivity of consequence, if the premises of a THINNING or a CUT are consequences of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$, then so too is their conclusion. Therefore, every sentence in a proof from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ of $\mathfrak{q}$ is a consequence of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$, and in particular $\mathfrak{q}$ is. □

Gentzen then proved the converse: "If a sentence $\mathfrak{q}$ is a 'consequence' of the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$, then it is also 'provable' from them." In fact he showed that proofs of a specific "normal form" suffice. Normal proofs are chains of applications of CUT followed by a single, terminal application of THINNING.

Theorem 2. *If a sentence* $\mathfrak{q}$ *is a consequence of the sentences* $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$*, then there exists a normal proof of* $\mathfrak{q}$ *from* $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$.

Proof. Observe first that every trivial sentence obviously has such a normal proof (a proof with zero CUTs is normal). Suppose, therefore, that $\mathfrak{q}$ is non-trivial and of the form $L \longrightarrow v$. Let $\mathfrak{S}$ be the set of all non-trivial sentences with succedent v for which there is a normal proof from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ without THINNING. Notice that, because the conclusion of a CUT never contains an element that did not appear in its premises, $\mathfrak{S}$ is finite. Clearly if the antecedent of some $\mathfrak{s}$ in $\mathfrak{S}$ is entirely contained in L, there would be a normal proof of $\mathfrak{q}$ from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ (to construct it, just add one THINNING at the bottom of the normal proof of $\mathfrak{s}$ from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$). We will show that this must be the case.

Suppose that no $\mathfrak{s}$ in $\mathfrak{S}$ has an antecedent that is contained in L. Define a sequence of complexes $L = M_1, M_2, \ldots, M_k = N$ recursively: If every $\mathfrak{p}$ in $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ is satisfied in M_i, let M_i be N. Otherwise, choose one $\mathfrak{p}$ ($O \longrightarrow u$) that is not satisfied in M_i. Its succedent u does not belong to M_i, and we define $M_{i+1} = uM_i$. Notice that this sequence is necessarily finite, because the complex consisting of all the finitely many elements among $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ necessarily satisfies them all. We will show that N is a counter-example to the claim that $\mathfrak{q}$ is a consequence of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$. N satisfies $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$, and it contains L, so we need only show that N does not contain v.

An inductive argument shows that in fact each M_i satisfies all the sentences in $\mathfrak{S}$ and does not contain v.

Base: $M_1 = L$ does not contain v, because of the assumption that $\mathfrak{q}$ is not trivial. Furthermore, if $M_1 = L$ did not satisfy some $\mathfrak{s}$ in $\mathfrak{S}$, then the antecedent of $\mathfrak{s}$ would be contained in L, contrary to our current assumption.

Induction: Assume that M_i satisfies all the sentences in $\mathfrak{S}$ and does not contain v, and consider $M_{i+1} = uM_i$ where $O \longrightarrow u$ is a sentence from among $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ that M_i does not satisfy. O belongs to M_i, because M_i does not satisfy $O \longrightarrow u$. By hypothesis v does not occur in M_i, and therefore v does not occur in O. Now, were M_{i+1} to contain v, then v would be u. But in that case, $O \longrightarrow u$ would be in $\mathfrak{S}$ despite not being satisfied by M_i, contrary to the hypothesis. Therefore M_{i+1} does not contain v.

Suppose now that there is a sentence in $\mathfrak{S}$ that M_{i+1} does not satisfy. Any such sentence must have the form $Pu \longrightarrow v$, with P contained in M_i (again, u is the succedent of the sentence $O \longrightarrow u$ that M_i does not satisfy, the element appended toM_i to arrive at M_{i+1}.) Consider the CUT:

$$\frac{O \longrightarrow u \qquad Pu \longrightarrow v}{OP \longrightarrow v}\ \textit{cut}$$

The sentence $OP \longrightarrow v$ belongs to $\mathfrak{S}$, because (1) it has the succedent v, (2) both $Pu \longrightarrow v$ (as a memeber of $\mathfrak{S}$) and $O \longrightarrow u$ (as a sentence from among $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$) have normal proofs from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ without THINNING, (3) v belongs neither to O nor to P. But O and P both belong to M_i, although v does not, so M_i does not satisfy $OP \longrightarrow v$, contrary to the hypothesis. □

The parallel with Aristotle's completeness proof in *Prior Analytics* I23 is vivid. Like Aristotle, Gentzen was not working with distinct syntactic and semantic realms. The elements and complexes that satisfy sentences are not different sorts of things than those sentences themselves. This fact plays out dramatically in the proof itself, where sentences constructed from elements failing to satisfy certain other sentences are used as premises of a CUT rule to generate another sentence whose satisfiability by those same elements is considered.

And just as Aristotle had done, Gentzen observed that any sentence that follows, in the general intuitive sense, from a set of premises can be shown to do so, and that a perspicuously simple set of formal inference patterns suffices for the demonstration. Following Bolzano, who announced that a single syllogism could stand in place of them all, Gentzen observed further that with just one synthetic inference rule, all logical consequence relations can be formally demonstrated.

These remarkable parallels between Gentzen's formalization of the consequence relation and Aristotle's syllogistic completeness proof point to further significance of Gentzen's framework. Because he succeeded in formalizing the intuititive concept of logical consequence in a single, synthetic inference rule, Gentzen was able to rephrase questions about the adequacy of other inferential procedures in terms of the relationships between them and the syllogism rather than in terms of their correspondence to abstract notions of another sort. We now turn to how with the elimination theorems he and Herbrand did just this.

The elimination theorems in context

The way forward from Bolzano's impasse urged by the prevailing conception of logical completeness is to drop the metaphysical ideology that led him to consider his proof system in absolutist terms and to see his definition of logical consequence as procedural. One readily sees the thin consequence definition as abstract and the analytic proof system as concrete. The question of logical completeness is then, as it was for Gödel and Henkin, about the adequacy of the latter to the former.

Equipped with Gentzen's analysis of logical consequence in terms of structural inference, a different way forward from Bolzano is available, one that preserves the centrality of proof transformation program. In a phrase, Gentzen showed that Bolzano's two dualities—the one between the analytic and synthetic portions of his framework for logical inference and the other between logical inference itself and the modal concept of logical consequence—are actually the same. For Gentzen's (1932) results project the modal notion, what he called the intuitive concept of logical consequence, into the inferential framework itself. By formalizing logical consequence, Gentzen recast it as an immanent feature of inference rather than as an abstract relation to which inference could be hoped to correspond. In fact, consequence is realized precisely as the syllogism rule that Bolzano had tried to eliminate.

Gentzen leaves little evidence of his direct influences. We do not know if he was familiar with Bolzano's work from the previous century, although there are interesting textual similarities. Both writers (Bolzano 1810: Preface) referred to the presence of "intermediate concepts" in rules of syllogism or cut as introducing "detours" [*Umwege*] in proofs and described the merit of purely analytic proofs in terms of their avoidance of such detours.[16] In a similar vein, describing the structure of cut-free proofs, Gentzen wrote, "The final result is, as it were, gradually built up from its constituent elements. The proof represented by the derivation is not roundabout in that it contains only concepts which recur in the final result," and beyond the obvious echoes of Bolzano's analyticity condition, Gentzen's image of a theorem being "gradually built up from its constituent elements" is suggestive of Bolzano's ontological program (88).

We do know that Gentzen was directly influenced by Hertz, who was explicit about his indebtedness to Aristotle in (1929). As well, we know that thinking about logic in Göttingen both in his student years and his post-doctoral years was dominated by the influence of David Hilbert (1862–1943). Among Hilbert's own ideological commitments was his eliminitivist attitude towards abstract semantics. He did not, for example, much appreciate Bernays's contributions to the question of logical completeness of classical propositional logic, because of his abiding suspicions about the concepts of truth and content involved. Hilbert thought of interpretations as technical tools and avoided deliberation about which are correct, and so he and his school customarily referred to the purely syntactic formulation of completeness (so-called "Post-completeness"), which in the context of classical propositional logic is equivalent to Bernays's interpretation-theoretic one, as a "stricter" and more "mathematically precise" than the other.[17]

This eliminitivist attitude fed into Gentzen's work in two important ways. Most obviously, it supplied the right sort of environment to orient his conception of logic away from the substantive metaphysical concerns of about grounding that dominated Bolzano's thought. This reorientation proved crucial in Gentzen's reaction to the role of synthetic reasoning in axiomatic settings, as described in section 3. It also can be seen reflected in the work of Herbrand, whose work in Göttingen of 1930 has seemed to many commentators as supplying all the necessary ingredients to establish the logical completeness for classical quantification theory except for the proper conceptual terrain.[18] In place of models, Herbrand worked with the idiosyncratic notion of *champs finis*, which simultaneously blocked the crucial inference needed to establish completeness and opened up the way to observing the transformations needed to establish his fundamental theory and corollary "syllogism elimination."

But beyond this, Hilbert's suspicions about abstract conceptions of truth and interpretations led him to seek meaning elsewhere, and he found what he was after in rules of logical inference. In his lectures from early 1920, one finds the following possibly earliest statement of what today is known as inferentialist semantics: "*Diese Regel kann als die Definition des Seinszeichens aufgefasst werden*."[19] So whereas in his more famous lectures of 1917–18[20] Hilbert had been explicitly, if reluctantly, pursuing questions about formal systems and their interpretations, in just a few years he had so modified his perspective that "questions of completeness" and "of the relationship between the formalism and its semantics [...] receded into the background." In their place, seeking "a more direct representation of mathematical thought," Hilbert redesigned his logical calculus for quantification theory so that the rules governing the calculus, rather than a semantic theory external to that calculus, could be thought of as directly "defining" or "giving the meaning" of the propositional connectives and quantifiers (Ewald and Sieg 2013: 298).

Gentzen's work is illuminated if understood as the realization of this Göttingen attitude. Pursuing just such a "direct representation of mathematical thought," Gentzen turned to an empirical study of the written records of mathematical proofs to isolate and organize the distinctive inferential moves afforded by various sentential connectives and determiners. These patterns of inference, he felt, exhibit all that mathematicians mean by the particles they govern. Further organization and systematization of this empirical data uncovered surprising properties, such as some inferences being more general than and therefore inclusive of others, a sort of "interderivability" relation between introduction and elimination rules associated with each particle, and crucially, in the most mature organizational scheme of the *sequent calculus*, the observation that the definitive rules of inference are "analytic" in the same sense that the rules of Bolzano's theory of grounding are: Their premises contain no ideas that their conclusion does not contain. In Gentzen's modern terminology, the formulas in a premise of such a rule are all sub-formulas of the formulas in the conclusion.

The sequent calculus just mentioned is the culmination of these several threads in Gentzen's thought. The basic framework is just the elementary sentence system of 1932. Because Gentzen had left the sub-sentential structure of that system's elements unanalyzed in order to capture nothing more than the fully general concept of logical consequence, its elements can stand for any of a variety of complex sentences. So Gentzen mapped the sentences of quantification theory onto these elements and supplemented the resulting calculus with the empirically discovered, meaning-constitutive, analytical rules of inference governing the sentential connectives and quantifiers. Thus like Bolzano before him, Gentzen devised a logical system of a hybrid nature. For each logical particle of quantification theory there were associated analytic inference rules, and these operated alongside the synthetic, structural inference rule CUT.

The sequent calculus, though, is a conceptual advance beyond Bolzano's hybrid system. It supplies an extra layer of significance to both its analytic and its synthetic components. Gentzen's analytic rules are no longer overburdened with the task of uncovering a truth's objective grounds, but for that very reason they are now understood as the site of logical meaning. One need not look to something external to the logical system for its semantics. Meanwhile, the CUT rule is distinguished not only by being synthetic but by being the formalization of the intuitive concept of logical consequence. One need not look to anything external to the logical system for that relation either.

From Gentzen's orientation, then, Bolzano's question about the relationship between analytical consequence and derivability is not, as it seems to most contemporary readers, a confused reversal of the question of logical completeness. It is a natural question preliminary to a proper formalization. By providing a system whose analytic rules comprise its semantics and whose synthetic rule captures the independent notion of logical consequence, the question of completeness—"Do the rules governing the meanings of the particles of quantification theory suffice to derive everything that logically follows?"—can be precisely stated as: "Can everything derivable in fact be derived without CUT, or is the CUT rule an essential ingredient in some logical derivations?" Gentzen's "elimination" of syllogism is a demonstration of the analytical rules' logical completeness.[21]

The cut-elimination theorem therefore provides a clear way forward from Bolzano's conundrum about the relationship between (what he called) consequence and derivability. But as with Aristotle's own completeness proof from the *Prior Analytics* it is hard to see the theorem for what it is if you expect a completeness theorem to establish a correspondence between syntax and semantics. According to Gentzen, following Hilbert, semantics is not something independent of logical syntax. It is something provided by a properly designed syntax. Its adequacy is established not in terms of its relationship with an abstract consequence relation, but in terms of whether its inferential scope is at all strengthened by the addition of a syllogism rule that makes that relation concrete.

At the same time, the theorem sheds light on the frustration Bolzano encountered in developing his proto-proof theory. In section 3 we saw Bolzano observing that the syllogism rule disrupts the analyticity of proofs but worried that it is for all we know unavoidable. His analytical rules are not applicable to multiple "simple sentences," and it is possible that such sentences are found both among the groundless, basic truths and also among the grounded, consequent truths. If there are simple truths that are not basic, then there are truths that do not have purely analytical proofs. This must have been a devastating realization for Bolzano, whose entire motivation was to provide a proof system that would disclose every truth's objective grounds. Reflecting on this situation at the very end of (1837), he confessed misgivings about his entire ontological scheme. Maybe there are no ultimate grounds. Maybe axiom choice is a mere convention for efficiently organizing facts:

> This seems to be the most suitable place to admit that I occasionally doubt whether the concept of ground and consequence [...] is not complex after all; it may turn out to be none other than the concept of an ordering of truths which allows us to deduce from the smallest number of simple premises the largest possible number of the remaining truths as conclusions. (§221)

The elimination theorems corroborate Bolzano's observation, by showing that syllogism is inessential only when reasoning purely logically, establishing logical validities. When one is reasoning from arbitrary non-logical hypotheses, it is essential. For Gentzen and Herbrand, however, this situation in no way diminishes the theorem's significance. Instead, it enhances it. The Göttingen logicians never expected all truths, even all mathematical truths, to rest on ultimate simple grounds, much less that an analytical proof system would disclose such grounds. What an analytical proof system *can* do, provided that it is complete in the way that the elimination theorems establish, is highlight exactly what the role of logical analysis is in a scientific exposition, and where it yields to synthesis. Herbrand's observation about the the essential use of syllogistic inference in the presence of non-logical hypotheses is readily recast in terms of "admissibility": If one removes the rule *modus ponens* from Herbrand's formulation of classical logic, the result is a system in which modus ponensis admissible but not derivable. The distinction between admissibility and derivability appears to have not occurred to such contemporaries of Herbrand as Gödel, who worked successfully with a rough and ambiguous notion of deductive validity, and did not receive wide attention until 1955, but the study of syllogism in Göttingen led to it naturally.[22] An extension of Gentzen's work by Gasai Takeuti (1926–2017) shows further that in sequent calculus proofs with non-logical hypotheses at the leaf nodes, only "anchored" CUTs (instances of the CUT rule, at least one of whose premises is a descendent, in the proof-tree, of an axiom) are essential; "free" CUTs are still eliminable.[23]

An Aristotelean legacy

The story of the elimination of syllogism in the Göttingen school should make plain the fact that the fate of Aristotelean logic was not, as many have suggested, to fade away with the arrival of the predicate calculus. As indicated in section 4, those elimination results themselves rest atop a proof idea Aristotle introduced. And although it is the syllogism, or something essentially like it, that is “eliminated” in the work of Gentzen and Herbrand, this is hardly to say that it is no longer present in modern logic. After all, the Göttingen logicians taught us that syllogism-free reasoning is significant *because* with it one can replicate the syllogism’s inferential scope. The syllogism is revealed to be, not deadwood or a redundancy of logistic theory, but the gold-standard of inference against which meaning-constitutive rules are measured.

This is the reading of the elimination theorems that one is left with when one is able to shed certain preconceptions about the meaning of logical completeness and the ingredients that make up its formal verification. Those preconceptions have a rightful place in our modern conception of logic. They are recommended Gödel’s theorems and nearly a century of research conducted in their wake. But they are not the sole devices for understanding logic’s central notions. As much as they inform us, they obscure the fact that Aristotle was able to pose and settle perfectly legitimate questions about logical completeness phrased in completely different terms.

Following Smiley’s lead, we saw that the conceptual terrain that Aristotle navigated in the *Prior Analytics* recurs in key moments in history and especially so in the work of Göttingen logicians. Reading Gentzen and Herbrand as navigating that same conceptual terrain uncovers the nearly forgotten place of syllogism in 20th century logic. But more than that, it points us to Gentzen’s observation that the syllogism is able to capture, in concrete form, the whole abstract concept of logical consequence, so that its elimination confers on a logic the label of completeness. And this, we saw, is just the role that Aristotle saw it playing from the beginning.

Notes

[1]Page 276 of Herbrand, J. (1931). “Unsigned note on (Herbrand 1930),” Annales de l’Université de Paris 6, 186-9. Translated in W. Goldfarb (1971), *Jacques Herbrand: Logical Writings*. Cambridge: Harvard University Press: 272-6.

[2]Page 175 of Herbrand, J. (1930). *Recherches sur la théorie de la démonstration*. Doctoral thesis, University of Paris. Translated in (Goldfarb 1971): 44-202.

[3]All quotations of Aristotle are from J. Barnes (1984). *The Complete Works of Aritstotle*. Princeton: Princeton Uniersity Press.

[4]Page 26 of Smiley, T. (1994) “Aristotle’s completeness proof” *Ancient Philosophy* **14**: 25–38. Smiley’s quotation is from Corcoran, J. (1972) “Completeness of an ancient logic” *Journal of Symbolic Logic* **37**: 696–702.

[5]Smiley’s quotation is from Lear, J. (1980). *Aristotle and Logical Theory*. Cambridge: Cambridge University Press.

[6]The proof presented is a paraphrase of (Smiley 1994: 33–34).

[7]Bolzano (1810). *Beyträge zu einer begründeteren Dartellung der Mathematik*. Translated in Ewald (1996) *From Kant to Hilbert* vol. I Oxford: Oxford Univeristy Press, 174–224, Bolzano (1837). *Wissenschaftslehre*. Translated in J. Berg (ed.) *Theory of Science* 1973. Boston: D. Reidel Publishing Company. For a full elaboration, one can consult Mark Siebel’s contribution to this volume.

[8]See for example (1810: §2) and Franks (2014) "Logical completeness, form, and content: an archaeology" in J. Kennedy (ed.), *Interpreting Gödel: Critical Essays*. Cambridge University Press. 2014 for elaboration.

[9]in *Posterior Analytics* I7: "It follows that we cannot in demonstrating pass from one genus to another. We cannot, for instance, prove geometrical truths by arithmetic."

[10]"crossing over to another kind"

[11]cf. page 251 of Schroeder-Heister (2002) "Resolution and the origins of structural reasoning: early proof-theoretic ideas of Hertz and Gentzen," *Bulletin of Symbolic Logic* **8**: 246–265 and page 314 of van Plato, J. (2012) "Gentzen's proof systems: byproducts in a work of genius" *Bulletin of Symbolic Logic* **18**(3): 313–67.

[12]Genten (1932) "Über die Existenz unabhängiger Axiomensysteme zu unendlichenSatzsystemen," *Mathematische Annalen* **107**, 329-50. Translated in Szabo (1969). *The Collected Papers of Gerhard Gentzen*. London: North Holland: 29–52.

[13]Tarksi (1936) "On the concept of logical consequence," trans. in J. H. Woodger (ed.) (1956) *Logic, Semantics, Metamathematics: Papers from 1923 to 1938 by Alfred Tarski*. Oxford: Oxford University Press. 1956

[14]Hertz, P. (1929) "Über Axiomensysteme für beliebige Satzsysteme," *Mathematische Annalen* **101**: 457-514

[15]Although the proofs in this section are reworded and rhetorically restructured, the reasoning and constructions in them are Gentzen's original ones. The notation and lables have been preserved from Gentzen's proofs to assist with cross-reference.

[16]Gentzen, G. (1935) "Untersuchungen über das logische Schliessen," Gentzen's doctoral thesis at the University of Göttingen. Translated in (Szabo 1969): 68–131.

[17]See Franks, C. (2017) "Hilbert's logic," in A. P. Malpass and M. Antonutti-Marfori (eds.) *The History of Philosophical and Formal Logic*. Bloomsbury for elaboration of this theme and William Ewald's contribution to this volume for other aspects of Hilbert's engagement with syllogism.

[18]notably in Derben, B. and J. van Heijenjoort (1986) "Introductory note to (Gödel 1929) and (Gödel 1930)" in Feferman, S., J. W. Dawson, Jr., S. C. Kleene, G. H. Moore, R. M. Solovay, and J. van Heijenoort (eds.) (1986). *Kurt Gödel: Collected Works vol. I* Oxford: Oxford University Press: 44–59

[19]Ewald, W. and W. Sieg (eds) (2013) *David Hilbert's Lectures on the Foundations of Arithmetic and Logic 1917–1933*, New York: Springer: 323.

[20]These lectures eventually supplied the bulk of the material in the influential Hilbert, D. and W. Ackermann (1928). *Grundzüge der theoretischen Logik*, Berlin: Springer.

[21]This reading of the cut-elimination theorem was first offered in Franks (2010) "Cut as consequence" *History and Philosophy of Logic* **31**(4). Not indicated there is the strong parallel with Aristotle's own approach to completeness, as it is reflected in Gentzen's 1932 proofs and as it provides the conceptual categories for understanding the elimination theorem.

[22]Franks, C. (2018) "The context of inference" *History and Philosophy of Logic* **39**(4): 365–395 contains a conceptual history of the derivability/admissibility distinction, and in section 8 of Franks, C. (2021) "The deduction theorem (before and ater Herbrand)" *History and Philosophy of Logic* **42**(2): 219–59 Herbrand is identified as probably the earliest writer to articulate the distinction. The wide attention attribution is to Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik*. Berlin: Springer.

[23]Takeuti, G. (1987). *Proof Theory*. Amsterdam: North Holland. See pages 44–47 of Buss (1998). *Handbook of Proof Theory*. Amsterdam: Elsevier for a perspicuous demonstration and further references.