AME 50531
Intermediate Thermodynamics
Examination 2: Solution
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4 April 2007

1. (40) Helium and argon exist in a chamber with total pressure $P = 100\ kPa$, temperature $T = 300\ K$, $V = 10\ m^3$. These noble gases are well described as an ideal mixture of inert calorically perfect ideal gases. The mass fraction of helium is 0.5.

   (a) Find the mole fractions of helium and argon.

   (b) The mixture is heated isochorically until $T = 1000\ K$. Find the heat transfer to the mixture.

Solution

The mole fraction of species i, y_i can be related to the mass fraction of species i, c_i via the equation

$$y_i = \frac{c_i/M_i}{\sum_{j=1}^{N} c_j/M_j}$$

We have

$$M_{He} = 4\ kg/kmole,$$

$$M_{Ar} = 40\ kg/kmole.$$

Now the mass fraction of each species is $c_{He} = 0.5$, $c_{Ar} = 0.5$. So the mole fractions are

$$y_{He} = \frac{c_{He}/M_{He}}{c_{He}/M_{He} + c_{Ar}/M_{Ar},}$$

$$y_{He} = \frac{0.5/4}{0.5/4 + 0.5/40,}$$

$$y_{He} = 0.909091.$$

So

$$y_{Ar} = 1 - y_{He} = 1 - 0.909091 = 0.0909091.$$

The first law holds that

$$U_2 - U_1 = Q_{12} - W_{12}.$$

Since the process is isochoric, $W_{12} = 0$, so

$$\begin{aligned} Q_{12} &= U_2 - U_1, \\ &= n_{He}\overline{u}_{He2} + n_{Ar}\overline{u}_{Ar2} - n_{He}\overline{u}_{He1} - n_{Ar}\overline{u}_{Ar1}, \\ &= n_{He}\overline{c}_{vHe}(T_f - T_i) + n_{Ar}\overline{c}_{vAr}(T_2 - T_1), \\ &= (n_{He}\overline{c}_{vHe} + n_{Ar}\overline{c}_{vAr})(T_2 - T_1) \end{aligned}$$

Now

$$P_{He1} = y_{He}P_1 = .909091(100\ kPa) = 90.9091\ kPa.$$

$$P_{Ar1} = y_{Ar} P_1 = 0.0909091(100\ kPa) = 9.09091\ kPa.$$

From the ideal gas law and Dalton's law, we have

$$n_{He} = \frac{P_{He1} V}{\overline{R} T_1} = (90.9091)(10)/(8.314)(300) = 0.364\ kmole.$$

$$n_{Ar} = \frac{P_{Ar1} V}{\overline{R} T_1} = (9.09091)(10)/(8.314)(300) = 0.0364\ kmole.$$

Now for the specific heats, one finds

$$\overline{c}_{vHe} = c_{vHe} M_{He} = \left(3.116\ \frac{kJ}{kg\ K}\right)\left(4\ \frac{kg}{kmole}\right) = 12.464\ \frac{kJ}{kmole\ K}.$$

$$\overline{c}_{vAr} = c_{vAr} M_{Ar} = \left(0.312\ \frac{kJ}{kg\ K}\right)\left(40\ \frac{kg}{kmole}\right) = 12.48\ \frac{kJ}{kmole\ K}.$$

Note the remarkable fact that on a molar basis, the specific heats of helium and argon are essentially identical! This is not a coincidence, but a fundamental property of noble gases.

So the heat transfer is

$$Q_{12} = ((0.364)(12.464) + (0.0364)(12.48))(1000 - 300) = 3498\ kJ.$$

2. (40) When a material which is solid at standard temperature and pressure is subjected to a strong shock, all of the elastic bonds are broken, and it behaves as a dense gas. A common model for such a material is the so-called Tait equation. In canonical form, the specific Helmholtz free energy of such a material is sometimes modelled by

$$a(v, T) = c_v T \left(1 - \ln\left(\frac{T}{T_o}\right) + (k-1)\ln\left(\frac{v_o}{v}\right)\right) + \frac{\varepsilon}{k}\frac{v}{v_o}.$$

Here constant parameters are specific heat c_v, reference temperature and specific volume v_o and T_o, ratio of specific heats k, and configuration energy ε. Find the entropy $s(v, T)$, the thermal state equation $P(v, T)$ and caloric state equation $u(v, T)$.

Solution

Start with the Gibbs equation

$$du = T ds - P dv.$$

Recall the definition of a:

$$a \equiv u - Ts.$$

So

$$da = du - T ds - s dT.$$

Eliminate du in the Gibbs equation to get

$$da + Tds + sdT = Tds - Pdv.$$

Simplify to get

$$da = -sdT - Pdv.$$

So the canonical variables for a are $a(T, v)$. Moreover

$$\left.\frac{\partial a}{\partial v}\right|_T = -P, \qquad \left.\frac{\partial a}{\partial T}\right|_v = -s.$$

Taking $\left.\frac{\partial a}{\partial v}\right|_T$, we find

$$-\frac{c_v(k-1)T}{v} + \frac{\varepsilon}{kv_o} = -P.$$

So the thermal state equation is

$$P(v, T) = c_v(k-1)\frac{T}{v} - \frac{\varepsilon}{kv_o}.$$

Taking $\left.\frac{\partial a}{\partial T}\right|_v$, we find

$$-c_v + c_v\left(1 - \ln\left(\frac{T}{T_o}\right) + (k-1)\ln\left(\frac{v_o}{v}\right)\right) = -s.$$

So that

$$s(v, T) = c_v\left(\ln\left(\frac{T}{T_o}\right) - (k-1)\ln\left(\frac{v_o}{v}\right)\right)$$

Now from the definition of a, we have

$$u = a + Ts.$$

Combining and simplifying, we get

$$u(v, T) = c_v T + \frac{\varepsilon}{k}\frac{v}{v_o}.$$

3. (20) Consider 100 m^3 of atmospheric air, which is an air-water vapor mixture at 100 kPa, 15 C, and 40% relative humidity. Find the mass of water in the mixture.

Solution

First get the humidity ratio

$$\omega = 0.622\frac{\phi P_g}{P - \phi P_g}.$$

At 15 C, the saturated steam tables give

$$P_g = 1.705\ kPa.$$

So

$$\omega = 0.622\frac{0.40(1.705)}{100 - 0.40(1.705)} = 0.00427.$$

Now the pressure of the air is

$$P_a = P - P_v = P - \phi P_g = 100 - 0.4(1.705) = 99.318\ kPa.$$

The mass of the air is then

$$m_a = \frac{P_a V}{R_a T} = \frac{99.318(100)}{0.287(15 + 273)} = 120.158\ kg$$

So the mass of the vapor is

$$m_v = \omega m_a = 0.00427(120.158) = 0.513\ kg.$$