VASSILIEV INVARIANTS VIA MANIFOLD CALCULUS

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Contents

Introduction

One of the main questions in knot theory is to find computable knot invariants which can classify knots up to isotopy. Motivated by this, varies knot invariants have been constructed and studied. For example, the Alexander polynomial [Ale27] classifies many knots, but it can not distinguish mirror images of knots. The Jones polynomial [Jon85] can distinguish mirror images of knots, but it does not distinguish all knots, cf. [Kan86]. Khovanov Homology [Kho00], a categorification of the Jones polynomial, detects the unknot cf. [KM11], but there are still infinitely many families of knots that have identical Khovanov homology, cf. [Wat07]. These invariants already provide us with new knowledge about knots, but it seems that each of them individually can not give a satisfactory answer to the classification problem.

In this thesis we will first introduce finite type invariants (Section 2), which are a collection of knot invariants discovered by Vassiliev [Vas90; Bar95]. Instead of focusing on a single knot invariant, Vassiliev's idea was to study the structure of all knot invariants with values in an abelian group A, i.e. $\mathrm{H}^0(\mathrm{Emb}(S^1, S^3); A)$, where $\mathrm{Emb}(S^1, S^3)$ is the space of knots with Whitney C^∞-topology. The collection of Vassiliev invariants is conjectured to distinguish knots.

Our approach to study Vassiliev invariants, inspired by [BCKS17] and [GKW01], is to use manifold calculus (Section 3). For this, we consider the space $\mathcal{K} = \mathrm{Emb}_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c)$ of long knots (Section 1) with boundary condition c. See Figure 1 for an example of a long knots. The space of long knots has the property that $\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3) \simeq \mathcal{K} \times_{\mathrm{SO}(2)} \mathrm{SO}(3)$, cf. [Bud08, Theorem 2.1]. As a consequence we have $\mathrm{H}^0(\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3); A) \cong \mathrm{H}^0(\mathcal{K}; A)$.

More precisely, we consider the embedding functor $\mathrm{Emb}(-)$ associated to the space $\mathcal{K}$ (Section 3.2), that is

$$\begin{aligned} \mathrm{Emb}(-)\colon \mathbf{Open}_\partial(\mathrm{I}) &\to \mathbf{Top} \\ V &\mapsto \mathrm{Emb}_\partial(V, \mathbb{R}^2 \times \mathrm{D}^1, c). \end{aligned}$$

which maps open subset $\partial\mathrm{I} \subseteq V$ of I to the space $\mathrm{Emb}_\partial(V, \mathbb{R}^2 \times \mathrm{D}^1, c)$ of embeddings.

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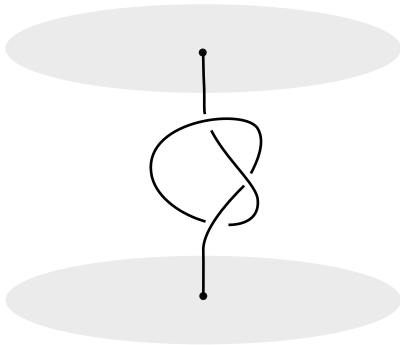

FIGURE 1. A visualisation of a trefoil long knot

Manifold calculus associates to $\mathrm{Emb}(-)$ a sequence of polynomial functors $\mathrm{T}_n \mathrm{Emb}(-)$ "approximating" the functor $\mathrm{Emb}(-)$,

$$\begin{tikzcd}
& \mathrm{Emb}(-) \arrow[dl] \arrow[d, "\eta_n"] \arrow[dr, "\eta_{n-1}"] \arrow[drrr, "\eta_0"] & & & \\
\cdots \arrow[r, "r_{n+1}"'] & \mathrm{T}_n \mathrm{Emb}(-) \arrow[r, "r_n"'] & \mathrm{T}_{n-1} \mathrm{Emb}(-) \arrow[r, "r_{n-1}"'] & \cdots \arrow[r, "r_1"'] & \mathrm{T}_0 \mathrm{Emb}(-).
\end{tikzcd}$$

The natural transformation η_n evaluated at the unit interval I induces a knot invariant $\eta_n(\mathrm{I})_*\colon \pi_0(\mathcal{K}) \to \pi_0(\mathrm{T}_n \mathrm{Emb}(\mathrm{I}))$. Our first result is to give a geometric proof of the following theorem, stated in [BCKS17], which implies that $\eta_n(\mathrm{I})_*$ is an additive Vassiliev invariant of degree at most n.

Theorem 1 (Theorem 3.2.6)**.** *Let K_1 and K_2 be knots such that*[1] *$[K_1]$ and $[K_2]$ have the same values for any Vassiliev invariants of degree at most $n-1$. Then we have $\eta_n(\mathrm{I})_*([K_1]) = \eta_n(\mathrm{I})_*([K_2])$.*

The two main ingredients of the proof are clasper surgery[2] and grope cobordism of knots. Clasper surgery of a long knot K in a manifold M is a special case of Dehn surgery in M performed along certain framed links, which does not change the diffeomorphism type of M, but can change the isotopy class of K. If two long knots can be transformed into each other by clasper surgeries of degree k, then they have the same values for any Vassiliev invariant of degree at most $k-1$, cf. Theorem 2.2.20. A grope is an embedded (CW)-complex in a 3-manifold, whose boundary are knots. Two knots have the same values for any Vassiliev invariant of degree at most $k-1$ if and only if they cobound a grope of degree k. Actually, claspers and gropes are two equivalent characterisation of the universal additive Vassiliev invariants. The relation between claspers, gropes and Vassiliev invariants is explained in detail in [CT04a] and [CT04b]. In order to prove the above theorem, we prove the following technical lemma about gropes, stated less precisely[3].

Lemma 2 (Lemma 2.3.19)**.** *Let $(G_n^c, \{C_i\}_{i=1}^n)$ be a capped grope of degree $n \geq 2$. Then there exists a continuous map*

$$h_n\colon \mathrm{D}^2 \times \Delta^{n-1} \to \mathbb{R}^3,$$

which is a Δ^{n-1}-family of embeddings of disks, with fixed boundary, in a neighbourhood of G_n^c.

[1] The square bracket indicates the isotopy equivalence class of knots.

[2] The original proof in [BCKS17] also used this characterisation

[3] We will explain precisely the properties that h_n has to satisfy in Section 2.3.

Note the sequence of polynomial functors $\mathrm{T}_n \operatorname{Emb}(-)$, evaluated at I, gives us a tower of fibrations

$$\cdots \xrightarrow{r_{n+1}(\mathrm{I})} \mathrm{T}_n \operatorname{Emb}(\mathrm{I}) \xrightarrow{r_n(\mathrm{I})} \mathrm{T}_{n-1} \operatorname{Emb}(\mathrm{I}) \xrightarrow{r_{n-1}(\mathrm{I})} \cdots \xrightarrow{r_1(\mathrm{I})} \mathrm{T}_0 \operatorname{Emb}(\mathrm{I}). \tag{0.0.1}$$

Goodwillie constructed a cosimplicial space associated to this tower of fibrations, which we explain in detail in Construction 4.1.20. With help of this cosimplicial space, we are able to make some computation with the Bousfield–Kan homotopy spectral sequence $\{E^r_{p,q}\}_{p,q\geq 0}$ with integral coefficients associated to this tower of fibrations (Section 4.2.3). Inspired by work of Conant [Con08], we are able to give a combinatorial interpretation for the groups $E^1_{p-1,p}$ and $E^1_{p,p}$, and the differential $d^1\colon E^1_{p-1,p} \to E^1_{p,p}$.

Proposition 3 (Proposition 4.2.31)**.** *Denote by $\mathcal{T}_{p-1}$ the abelian group generated by labelled unitrivalent trees (Definition 2.2.11) of degree $p-1$ with a total ordering on its leaves, modulo AS- and IHX-relations (Definition 4.2.28). Then $E^1_{p,p} \cong \mathcal{T}_{p-1}$.*

In Figure 2 we draw an example of a labelled unitrivalent tree of degree 4.

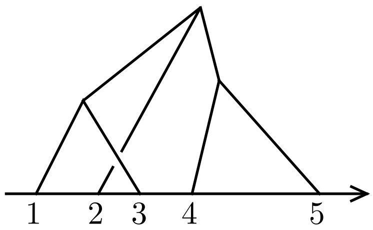

FIGURE 2. A labelled unitrivalent tree of degree 4.

Proposition 4 (Proposition 4.2.39)**.** *Denote by $\mathcal{D}_{p-1}$ the abelian group generated by $(i, p-1)$-marked unitrivalent graphs (Definition 4.2.34) modulo AS- and IHX^{sep}-relations. Then we have $E^1_{p-1,p}/torsion \cong \mathcal{D}_{p-1}$.*

In Figure 3 we draw an example of a labelled unitrivalent tree of degree 4.

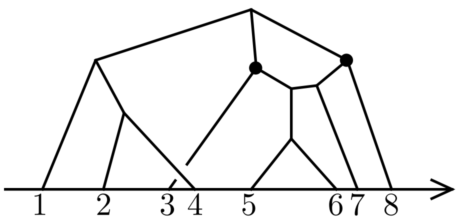

FIGURE 3. A $(3, 8)$-marked unitrivalent graph. The blacks dots indicate the marked nodes.

It is enough to consider d^1 modulo torsion, because $E^1_{p,p}$ is torsion free.

Proposition 5 (Proposition 4.2.44)**.** *A tree $\tau \in \mathcal{T}_{p-1}$, τ is STU^2-equivalent to 0 (Definition 4.2.41) if and only if $\tau \in \operatorname{im}(d^1)$ under the isomorphism $E^1_{p,p} \cong \mathcal{T}_{p-1}$ from Proposition 4.2.31.*

We use an example in Figure 4 to show the combinatorial interpretation the differential d^1. Observe that in Figure 4 that we can interpret $\Gamma_1 - \Gamma_2$ as performing an STU-relation in $\Gamma_{1,5}$ at the trivalent node connected by an edge to the leaf 5, and $\Gamma_3 - \Gamma_4$ as performing an STU-relation in $\Gamma_{1,5}$ at the trivalent node connected by an edge to the leaf 1. The name STU^2 is inspired by this observation that we perform a pair of STU-relation in a $(i, p-1)$-marked labelled unitrivalent graph.

As a corollary we obtain a combinatorial interpretation of $E^2_{p,p}$ for $p \geq 1$.

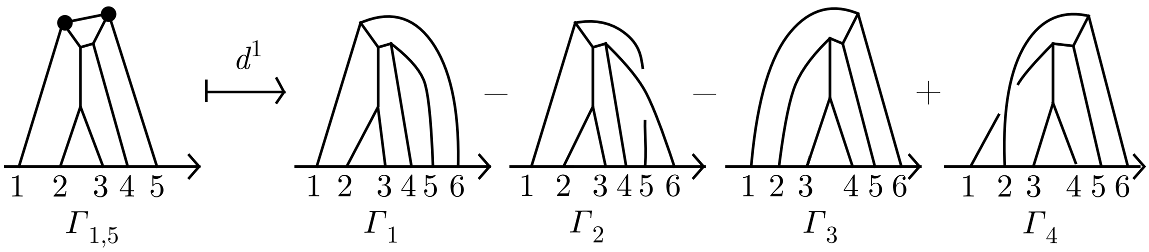

FIGURE 4. An example of d^1 applied to a $(1,5)$-marked unitrivalent graphs.

Corollary 6 (Corollary 4.2.46)**.**
i) *For $p \geq 4$, $E^2_{p,p}$ is isomorphic to the abelian group generated by unitrivalent tree of degree $p-1$, modulo AS-, IHX-, and STU^2-relations.*
ii) *For $p = 3$, $E^2_{3,3} \cong E^1_{3,3} \cong \mathcal{T}_2 \cong \mathbb{Z}$, because $d^1 \colon E^1_{2,3} \to E^1_{3,3}$ is trivial.*
iii) *For $p = 0, 1, 2$, we have $E^2_{p,p} = 0$.*

In [Con08], Conant computed similar results of the above corollary for the rational homotopy spectral sequence, with help of Sinha's cosimplicial model for the tower of fibration 0.0.1.

We conclude in Section 5 by highlighting some related work and problems, e.g. whether we can give a new (geometric) interpretation to the sequence of polynomial functors $\mathrm{T}_n \operatorname{Emb}(-)$ using higher genus grope cobordism of knots.

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Notation. Throughout the text, we will use the following notation and the embeddings given here to parametrise D^n, S^n and Δ^n.
i) Let $\mathrm{I} := [0,1]$ denote the unit interval.
ii) Let $\mathrm{D}^n := \{p \in \mathbb{R}^n \mid \|p\| \leq 1\}$ denote the unit n-ball in $\mathbb{R}^n$.
iii) Let $\mathrm{S}^n := \{p \in \mathbb{R}^{n+1} \mid \|p\| = 1\}$ denote the unit n-sphere in $\mathbb{R}^{n+1}$.
iv) Let $\Delta^n := \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=0}^n t_i = 1\}$ denote the standard n-simplex. For $\emptyset \neq S \subseteq \{0, 1, \dots, n\}$, denote by $\Delta_S := \{t \in \Delta^n \mid t_s = 0, \text{ for } s \notin S\}$ the face of Δ^n, labelled by S.

1. Long knots

Classically, knot theory studies smooth embeddings from S^1 to S^3 up to isotopy. For technical reasons, we shall consider long knots instead of knots in this thesis. A long knot is an embedding from I to $\mathbb{R}^2 \times \mathrm{D}^1$ coinciding with a fixed linear embedding near the boundary. The one point compactification of each long knot induces an isomorphism between $\pi_0(\mathcal{K})$ and $\pi_0(\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3))$. Thus for the study of knot invariants with values in an abelian group A, i.e. elements of $\mathrm{H}^0(\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3); A)$, it does no harm to use long knots instead of knots. However, note that $\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3)$ and the space of long knots $\mathcal{K}$ have different higher homotopy groups, cf. Remark 1.0.8.

Definition 1.0.1. Let M and N be smooth manifolds.

i) We define $\mathrm{Emb}(M, N)$ to be the space of smooth embeddings of M into N.
ii) When M and N are smooth manifolds with boundary, we define $\mathrm{Emb}_\partial(M, N)$ to be the space of smooth embeddings $F\colon M \to N$ which preserve the boundary, i.e. $F(\partial N) \subseteq \partial M$.
iii) We define $\mathrm{Emb}_\partial(M, N, f)$ to be the space of smooth embeddings that coincide with a given smooth embedding $f\colon M \hookrightarrow N$ near the boundary that are transverse to ∂N, i.e.

$$\mathrm{Emb}_\partial(M, N, f) := \{F \in \mathrm{Emb}_\partial(M, N) \mid F \pitchfork \partial N, F \text{ and } f \text{ are germ equivalent at } \partial M\}.$$

We topologise $\mathrm{Emb}(M, N)$ and $\mathrm{Emb}_\partial(M, N, f)$ with the Whitney C^∞-topology. For a detailed introduction of Whitney C^∞-topology, see [GG73].

Definition 1.0.2. Fix the embedding $c\colon \mathrm{I} \to \mathbb{R}^2 \times \mathrm{D}^1$, $t \mapsto (0, 0, -1 + 2t)$ and define the space of long knots $\mathcal{K}$ as $\mathrm{Emb}_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c)$. Elements of $\mathcal{K}$ are called *long knots*.

For the joy of the reader, see Figure 5 for an example of long knots.

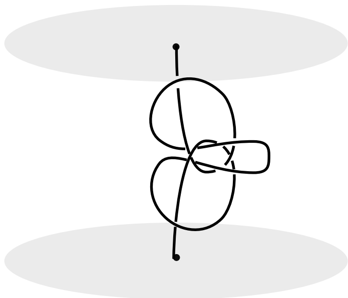


FIGURE 5. An example of a long knot.

Definition 1.0.3. Two long knots $K_0, K_1 \in \mathcal{K}$ are called *isotopic* if there is a smooth map $F\colon \mathrm{I} \times \mathrm{I} \to \mathbb{R}^2 \times \mathrm{D}^1$ such that $F|_{\mathrm{I}\times\{0\}} = f_0$ and $F|_{\mathrm{I}\times\{1\}} = f_1$, and $F|_{\mathrm{I}\times\{t\}} \in \mathcal{K}$ for every $t \in \mathrm{I}$. We call F an isotopy between K_0 and K_1, write $K_0 \sim K_1$

Remark 1.0.4. By [Hir76, Theorem 2.3.3] and [Fox45, Theorems 1–4], an isotopy F between $K_0, K_1 \in \mathcal{K}$ is the same as a path from K_0 to K_1 in $\mathcal{K}$.

Definition 1.0.5. Denote by $\mathrm{Diff}_\partial(\mathbb{R}^2 \times \mathrm{D}^1)$ the space (with the Whitney C^∞-topology) of self-diffeomorphisms of $\mathbb{R}^2 \times \mathrm{D}^1$ which restrict to the identity map on the boundary. Two long knots $K_0, K_1 \in \mathcal{K}$ are called *ambient isotopic* if there is a path $F\colon \mathrm{I} \to \mathrm{Diff}_\partial(\mathbb{R}^2 \times \mathrm{D}^1)$ such that $F(0) = \mathrm{id}_{\mathbb{R}^2\times\mathrm{D}^1}$ and $F(1) \circ K_0 = K_1$.

Remark 1.0.6. Isotopy and ambient isotopy are equivalence relations on $\mathcal{K}$. The isotopy extension theorem (cf. [Hir76, Theorem 8.1.3]) implies that K_0 and K_1 are isotopic if and only if they are ambient isotopic.

Remark 1.0.7. The smoothness of the embedding in the definition of a long knot guarantees the tameness of the long knot, i.e. the image of the embedding of I in $\mathbb{R}^2 \times \mathrm{D}^1$ admits a tubular neighbourhood $\mathrm{N}(I)$. In this way, we exclude wild long knots, which are topological embeddings $w\colon \mathrm{I} \hookrightarrow \mathbb{R}^1 \times \mathrm{D}^1$ that do not extend to an embedding $\mathrm{N}(I) \hookrightarrow \mathbb{R}^2 \times \mathrm{D}^1$.

Remark 1.0.8. Classically, a knot is defined as an element of $\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3)$. The relation between the space of long knots $\mathcal{K}$ and the space of knots $\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3)$ is given by

$$\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3) \simeq \mathcal{K} \times_{\mathrm{SO}(2)} \mathrm{SO}(3),$$

cf. [Bud08, Theorem 2.1]. In particular, this induces an isomorphism between $\pi_0(\mathcal{K})$ and $\pi_0(\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3))$.

Convention 1.0.9. From now on, we abbreviate long knots as knots and we call an element in $\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3)$ an ordinary knot.

Definition 1.0.10. Let K and K' be two knots. The *connected sum* $K\#K'$ is the concatenation of K and K'. More explicitly, $K\#K' \colon \mathrm{I} \to \mathbb{R}^2 \times \mathrm{D}^1$ is defined by

$$t \mapsto \begin{cases} \frac{1}{2}K(2t) - (0,0,\frac{1}{2}) & \text{if } t \in [0, \frac{1}{2}] \\ \frac{1}{2}K'(2t-1) + (0,0,\frac{1}{2}) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Remark 1.0.11. The connected sum of two knots K and K' is an element of $\mathcal{K}$, because K and K' are germ equivalent to c near the boundary $\partial\mathrm{I}$ of I.

Proposition 1.0.12.

i) *If $K_1 \sim K_1'$ and $K_2 \sim K_2'$, then $K_1\#K_2 \sim K_1'\#K_2'$. In particular, the connected sum operation $\#$ is well-defined on $\pi_0(\mathcal{K})$.*

ii) *The connected sum operation is commutative, i.e. $K\#K' \sim K'\#K$.*

Sketch. i) For $i = 1, 2$, let F_i be an isotopy between K_i and K_i'. Then the concatenation of F_1 and F_2 gives an isotopy of $K_1\#K_2$ and $K_1'\#K_2'$.

ii) See [Sch49] for details. The desired isotopy is given by sliding K_1 through K_2, as visualised in Figure 6. □

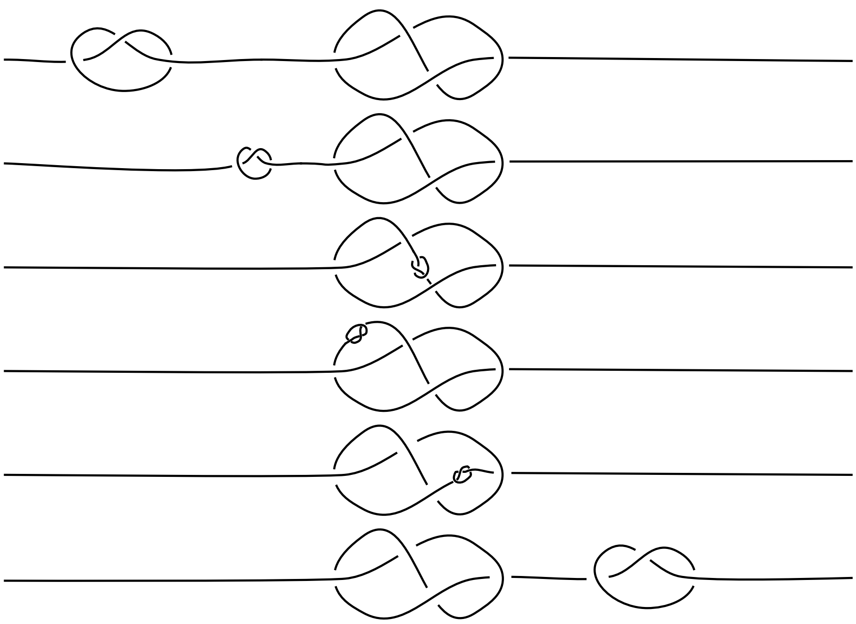


FIGURE 6. The connected sum of two knots is commutative up to isotopy.

Corollary 1.0.13. *The connected sum operation $\#$ endows $\pi_0(\mathcal{K})$ with the structure of a commutative monoid, where the identity is given by the unknot.*

Sketch. Recall that the operation of a monoid needs to be associative. The associativity of $\#$ follows from a standard argument involving a reparameterisation of the interval I. Note that concatenation of a knot K with an unknot does not change the isotopy class of the knot K. □

1.1. **Knot invariants.** Knot invariants are functions that assign to every knot an element in a set, group, ring etc., such that equivalent knots have the same value. It has been an important problem to find practically computable knot invariants ν that distinguish two knots up to isotopy. This section aims at giving a basic introduction to knot invariants.

Definition 1.1.1. A *knot invariant* with values in a set R is a map[4] $f\colon \pi_0(\mathcal{K}) \to R$.

A continuous map between two topological spaces induces a map between their sets of connected components, which yields the following proposition.

Proposition 1.1.2. *A continuous map $f\colon \mathcal{K} \to X$ of topological spaces induces a knot invariant $f_*\colon \pi_0(\mathcal{K}) \to \pi_0(X)$.* □

Remark 1.1.3. Using the bijection $\pi_0(\mathcal{K}) \cong \pi_0(\mathrm{Emb}(\mathrm{S}^1, \mathrm{S}^3))$, the notions of knot invariant and invariant of ordinary knots coincide. □

Convention 1.1.4. From now on, we only consider knot invariants with values in sets underlying abelian groups. However, we do not require that knot invariants are monoid homomorphisms.

Proposition 1.1.5. *Let A be an abelian group, then $\mathrm{H}^0(\mathcal{K}; A) = \mathrm{Mor}_{\mathbf{Set}}(\pi_0(\mathcal{K}), A)$ is the group of all knot invariants with values in A.* □

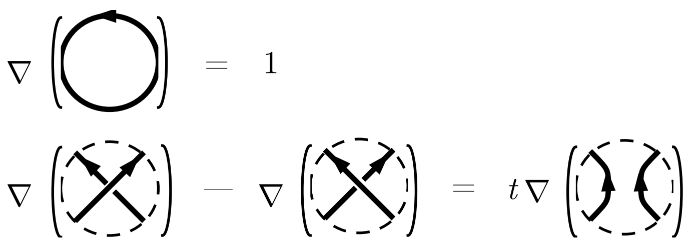
Let us discuss some examples of knot invariants.

Example 1.1.6. Consider the free abelian group $\mathbb{Z}[\pi_0(\mathcal{K})]$ generated by elements of $\pi_0(\mathcal{K})$. The *canonical knot invariant* is $i_{\mathcal{K}}\colon \pi_0(\mathcal{K}) \hookrightarrow \mathbb{Z}[\pi_0(\mathcal{K})], [K] \mapsto [K]$. Note that any knot invariant factors through $i_{\mathcal{K}}$.

Example 1.1.7. A *polynomial knot invariant* is a knot invariant with values in a polynomial ring. For ordinary knots, there are well-known polynomial invariants like the Alexander polynomial $\Delta(t)$ (cf. [Ale27]), the Conway polynomial $\nabla(t)$ (cf. [Con70]) and the Jones polynomial $V(t)$ (cf. [Jon85]).

We recall briefly the definition of the Conway polynomial for ordinary knots.

Definition 1.1.8. The *Conway polynomial* is a knot invariant taking values in $\mathbb{Z}[t]$. It is defined by the following "skein relations":



Definition 1.1.9. A knot invariant $f\colon \pi_0(\mathcal{K}) \to A$ is *additive* if it is a monoid homomorphism, i.e. $f(K_1 \# K_2) = f(K_1) + f(K_2)$ for any $K_1, K_2 \in \pi_0(\mathcal{K})$.

Proposition 1.1.10. *The second coefficient c_2 of the Conway polynomial is an additive knot invariant.*

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[4]Sometimes we abuse notation and call the induced map $\mathcal{K} \to \pi_0(\mathcal{K}) \to R$ a knot invariant.

Proof. Let K_1, K_2 be two knots and let U be the unknot. By [CDM12, Exrcise 2.6] we have that $\nabla(K_1 \# K_2) = \nabla(K_1)\nabla(K_2)$. By [Kau81], we can write for any knot K

$$\nabla(K) = 1 + c_2(K)t^2 + c_4(K)t^4 + \cdots + c_{2n}(K)t^{2n}$$

Thus we have $c_2(K_1 \# K_2) = c_0(K_1)c_2(K_2) + c_2(K_1)c_0(K_2) = c_2(K_1) + c_2(K_2)$ as desired. $\square$

2. Vassiliev invariants

In this section we present the definition of Vassiliev's knot invariants and discuss their properties and the related topics of clasper surgery and grope cobordism of knots.

2.1. **Definition of Vassiliev invariants.**

We will use a combinatorial description for the definition of Vassiliev invariants, which is easy to compute but lacks a geometric interpretation. To remedy this, we want to sketch Vassiliev's original approach as a motivation, cf. [Vas90].

Instead of focusing on one specific knot invariant, Vassiliev considered the whole set[5] $\mathrm{H}^0(\mathcal{K}; A)$ of all knot invariants with values in a given abelian group A. The main steps of his computation are the following:

i) Embed $\mathcal{K}$ in the space $\mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c)$ of all smooth maps from I into $\mathbb{R}^2 \times \mathrm{D}^1$ which are germ equivalent with c on the boundary (cf. Definition 1.0.2).
ii) Compute the homology of the complement of $\mathcal{K}$ in $\mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c)$.
iii) Use Alexander duality to obtain $\mathrm{H}^\bullet(\mathcal{K}; A)$, and in particular $\mathrm{H}^0(\mathcal{K}; A)$.

In order to perform step ii) and iii), Vassiliev finds a filtration by finite dimensional vector spaces $\{\Gamma_i\}_{i \in \mathbb{N}}$, which approximate the space $\mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c)$. Intersecting this sequence with $\mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c) \setminus \mathcal{K}$ yields a filtration

$$\sigma_1 \subseteq \sigma_2 \subseteq \cdots \subseteq \sigma_n \subseteq \sigma_{n+1} \subseteq \cdots \subseteq \mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c) \setminus \mathcal{K}.$$

Now, Vassiliev computes $\mathrm{H}_\bullet(\mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c) \setminus \mathcal{K}; A)$ via the homology spectral sequence associated to this filtration. Furthermore, this filtration gives a filtration of $\mathrm{H}_\bullet(\mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c) \setminus \mathcal{K}; A)$. In each of the finite dimensional vector spaces we can apply Alexander duality to obtain a filtration

$$V_1^A \subseteq V_2^A \subseteq \cdots \subseteq V_n^A \subseteq V_{n+1}^A \subseteq \cdots \subseteq \mathrm{H}^0(\mathcal{K}; A).$$

Finally, a Vassiliev invariant of degree n with values in A is defined to be an element of V_n^A.

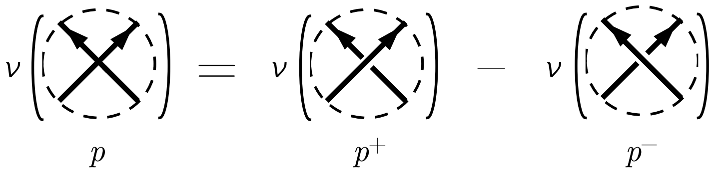
Remark 2.1.1. Let $K \in \mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c)$ be a smooth map. We call a point $p \in \mathrm{im}(K)$ a singularity of K if $K^{-1}(p)$ contains more than one element. The filtration $(\sigma_i)_{i \geq 1}$ of $\mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c) \setminus \mathcal{K}$ arises by distinguishing K by the type and the number of its singularities. Thus it is natural to conjecture that the system of Vassiliev invariants classify knots. On the other hand, it is still open whether Vassiliev invariants detect the unknot.

We now give the combinatorial definition of Vassiliev invariants.

Definition 2.1.2. Let $K \colon \mathrm{I} \to \mathbb{R}^2 \times \mathrm{D}^1$ be a smooth map such that K and the fixed c (cf. Definition 1.0.2) are germ equivalent at $\partial \mathrm{I}$. A *double point* of K is a point $p \in \mathrm{im}(K)$ such that $K^{-1}(p)$ consists of exactly two points t_1 and t_2 with linearly independent $T_{t_1} f$ and $T_{t_2} f$. Define by $\Sigma \subseteq \mathrm{C}^\infty_\partial(\mathrm{I}, \mathbb{R}^2 \times \mathrm{D}^1, c) \setminus \mathcal{K}$ the subspace of maps, which have only finitely many double points as singularities.

[5] Vassiliev used another mapping space instead of $\mathcal{K}$, which is homotopy equivalent to $\mathcal{K}$. For simplicity, we just write $\mathcal{K}$ instead.

Construction 2.1.3. Let $\nu\colon \pi_0(\mathcal{K}) \to A$ be a knot invariant. Then ν induces a function $\widetilde{\nu}$ on $\Sigma \cup \mathcal{K}$ by resolving any double point p into an overcrossing p^+ and an undercrossing p^-, and setting $\nu(p) := \nu(p^+) - \nu(p^-)$.



Proposition 2.1.4. *The extension $\widetilde{\nu}$ of ν in Construction 2.1.3 does not depend on the order in which we resolve the double points.*

Proof. Let $K^\times \in \Sigma$ and enumerate the double points of $K^\times$ by $p_1, p_2, \dots, p_m$. We can resolve each p_i into an overcrossing p^+ and an undercrossing p^-. By resolving the singularities in all possible orders, we obtain a set $\{K_\varepsilon\}_{\varepsilon\in\{0,1\}^m}$ of 2^m knots (up to isotopy). Here the knot K_ε with $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)$ is obtained from $\mathrm{im}(f)$ by resolving the double point p_i into p_i^+ for $\varepsilon_i = 0$ and p_i^- for $\varepsilon_i = 1$. Set $\mid \varepsilon \mid := \sum_{i=1}^m \varepsilon_i$, then one obtains inductively the formula

$$\widetilde{\nu}(K^\times) = \sum_{\varepsilon\in\{0,1\}^m} (-1)^{|\varepsilon|}\nu(K_\varepsilon),$$

which witnesses that $\widetilde{\nu}$ does not depend on the order in which we resolve double points. $\square$

Definition 2.1.5. A *Vassiliev invariant of degree at most n* is a knot invariant ν such that the induced function $\widetilde{\nu}$ on $\Sigma \cup \mathcal{K}$ vanishes for all $K^\times$ in Σ with more than n double points.

Example 2.1.6 ([Bar95, Section 1.4])**.**

i) The n-th coefficient c_n of the Conway polynomial is a Vassiliev invariant of degree at most n.
ii) After a suitable change of variables, each coefficient in the Taylor expansion of the Jones, HOMFLY, and Kauffman polynomials is a Vassiliev invariant.
iii) After a suitable change of variables, each coefficient in the Taylor expansion of the Reshetikhin-Turaev 'quantum-group' invariant is a Vassiliev invariant.

We can compute the set of Vassiliev invariants in degree 0, 1 and 2.

Proposition 2.1.7.

i) *Every Vassiliev invariant of degree at most 1 is constant.*
ii) *The group of Vassiliev invariants of degree at most 2 is generated by c_2.*

Proof. See [CDM12, Proposition 3.3.1–3.3.3] for a proof. $\square$

The number of double points of elements in Σ induces us a filtration on $\mathbb{Z}[\pi_0(\mathcal{K})]$ as follows.

Definition 2.1.8. Recall the free abelian group $\mathbb{Z}[\pi_0(\mathcal{K})]$ from Example 1.1.6. Denote by $\mathcal{K}_n$ the subgroup of $\mathbb{Z}[\pi_0(\mathcal{K})]$ generated by the linear combinations of equivalence classes of knots which are obtained by resolution of singular knot $K \in \Sigma$ with n double points. Note that $\mathcal{K}_{n+1} \subseteq \mathcal{K}_n$ by resolving just one double point. This yields the *Vassiliev filtration*

$$\mathcal{K}_0 := \mathbb{Z}[\pi_0(\mathcal{K})] \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \cdots \supseteq \mathcal{K}_n \supseteq \mathcal{K}_{n+1} \supseteq \dots$$

We can reformulate the definition of Vassiliev invariants as follows.

Proposition 2.1.9. *A knot invariant ν is a Vassiliev invariant of degree at most n if and only if its extension $\bar{\nu}\colon \mathbb{Z}[\pi_0(\mathcal{K})] \to A$ factors through $\mathbb{Z}[\pi_0(\mathcal{K})]/\mathcal{K}_{n+1}$.* $\square$

In order to check whether Vassiliev invariants classify knots up to isotopy, one needs to see whether for any two non-equivalent knots K_1 and K_2, there exists a Vassiliev invariant v such that $\nu(K_1) \neq \nu(K_2)$. This motivates the definition of n-equivalence of knots.

Definition 2.1.10. Let $n \in \mathbb{N}$. We say two knots K_1 and K_2 are *n-equivalent* if $[K_1] - [K_2] \in \mathcal{K}_n$. In other words, we have that $\nu([K_1]) = \nu([K_2])$ for every Vassiliev invariant ν of degree at most n.

Remark 2.1.11 ([CDM12, Section 3.2.4]).

i) The collection of Vassiliev invariants classifies knots if and only if $\cap_{n\geq 0}\mathcal{K}_n = 0$.
ii) (Gourssarov filtration) Let $\Gamma_n\mathcal{K}$ be the set of knots which are $(n-1)$-equivalent to the unknot. We get the filtration

$$\pi_0(\mathcal{K}) = \Gamma_1\mathcal{K} \supseteq \cdots \supseteq \Gamma_n\mathcal{K} \supseteq \Gamma_{n+1}\mathcal{K} \supseteq \cdots ,$$

and see that $\cap_{n\geq 0}\Gamma_n\mathcal{K} = \{\text{unknot}\}$ if and only if Vassiliev invariants detect the unknot.

2.2. **Clasper surgery.** Clasper surgery is a combinatorial description of the notion of n-equivalence of knots. This section briefly recalls the special case of simple tree clasper surgery, which will be sufficient for our applications. The reference for this section is [Hab00].

Convention 2.2.1. From now on, we abbreviate simple tree clasper surgery by clasper surgery.

Definition 2.2.2. A *clasper* $C = N \cup B \cup E$ in $\mathbb{R}^2 \times \mathrm{D}^1$ for a knot $K \in \mathcal{K}$ is a connected, oriented, compact surface embedded in the interior of $\mathbb{R}^2 \times \mathrm{D}^1$ together with a decomposition into three (unconnected) subsurfaces N, B, E that satisfy the conditions below. We call the connected components of N, B, E nodes, leaves[6] and edges respectively.

i) Nodes are disks that are disjoint from K.
ii) Edges are disks, say parametrised by $[0,1] \times [0,1]$, that are disjoint from K.
iii) Leaves are disks that intersect K transversally in the interior and exactly once.
iv) For each edge, the two arcs $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ are attached to the boundaries of leaves or nodes. These two arcs are not allowed to be attached to the same leaves or nodes.
v) For each node, there are exactly three edges attached to it, where the attaching regions are pairwise disjoint.
vi) For each leaf, there is exactly one edge attached to it.

Convention 2.2.3. Since we do not change our ambient 3-manifold $\mathbb{R}^2 \times \mathrm{D}^1$, we will not mention the ambient manifold where a clasper is embedded.

Example 2.2.4. Figure 7 depicts two claspers for knots K_1 and K_2 respectively.

[6]"B" stands for "Blatt", which means leaf in German.

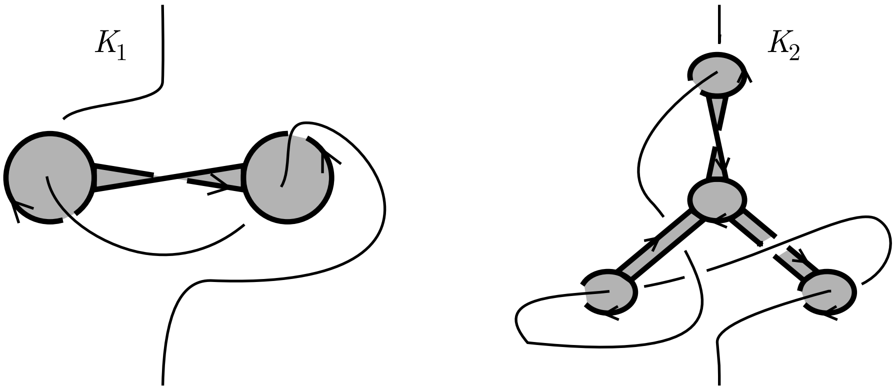
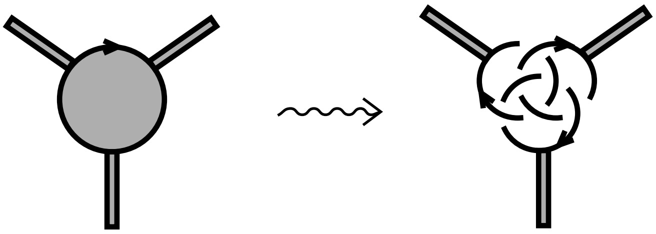


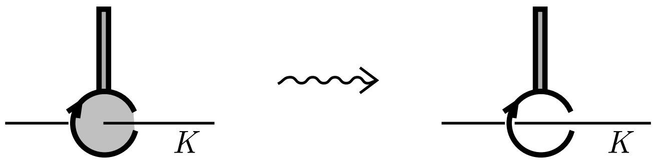
FIGURE 7. Visualisation of two claspers

Definition 2.2.5. For any clasper $C = N \cup B \cup E$ in $\mathbb{R}^2 \times \mathrm{D}^1$, we can associate to it a link L_C (defined up to isotopy) in a regular neighbourhood of C by the following steps.

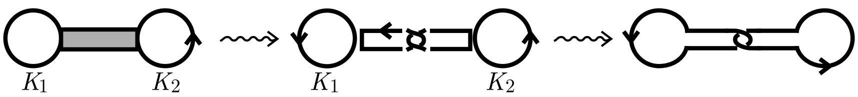
i) Replace each node by a Borromean ring, such that each of the three edges meeting the node becomes attached to different link component of the Borromean ring. Each link component inherits the orientation of the boundary of the node. The over- and undercrossings of the Borromean ring should be exactly as depicted in the figure.



ii) Remove the interior of each leaf.



iii) After performing step i) and ii), each edge connects two (images of) unknots K_1 and K_2. We take the boundary of the edge, cut it into two connected components, and add a full twist to connect these two components as shown in the figure below. Thus we obtain a Hopf link. Finally we take connect sum of the Hopf link with K_1 and K_2.



Definition 2.2.6. *Clasper surgery* along a clasper C is just 0-framed Dehn surgery (cf. [Sav12, Chapter 2])of $\mathbb{R}^2 \times \mathrm{D}^1$ along the associated link L_C. We denote the resulting 3-manifold $(\mathbb{R}^2 \times \mathrm{D}^1)^C$.

Remark 2.2.7. More generally, we can perform clasper surgery on an arbitrary 3-manifold M along a clasper $C \subseteq M^\circ$ with framing coming from the disks N and B. We denote the resulting manifold by M^C.

Proposition 2.2.8 ([Hab00, Proposition 3.3]). *Let C be a clasper for a knot $K \in \mathcal{K}$, and denote by $\mathrm{N}(C)$ a regular neighbourhood of C Then there is an orientation-preserving diffeomorphism*

$$\Phi'_C \colon \mathrm{N}(C) \to \mathrm{N}(C)^C$$

that restrits to the identity on $\partial \mathrm{N}(C)$. In particular, Φ'_C extends to a diffeomorphism

$$\Phi_C \colon \mathbb{R}^2 \times \mathrm{D}^1 \to (\mathbb{R}^2 \times \mathrm{D}^1)^C$$

restricting to the identity outside the interior of $\mathrm{N}(C)$.

Remark 2.2.9. The map Φ_C is unique up to isotopy relative to $\mathbb{R}^2 \times \mathrm{D}^1 \setminus \mathrm{N}(C)^\circ$, cf. [GS99, Page 154].

Notation 2.2.10. Let C be a clasper for a knot $K \in \mathcal{K}$, and denote by $K^C \in \mathcal{K}$ the result[7] of clasper surgery along C.

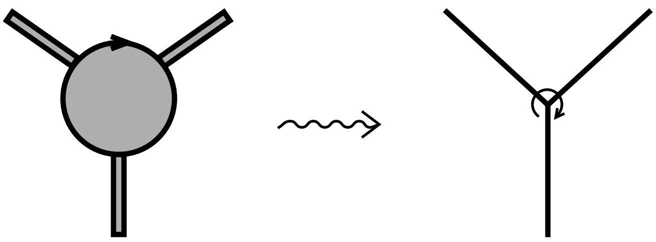
We will use the following combinatorial notation for claspers.

Definition 2.2.11. A *unitrivalent tree* is a (unrooted) binary tree with oriented nodes, i.e. a cyclic ordering of the edges at each node. The *degree*[8] of a unitrivalent tree is the number of nodes divided by 2.

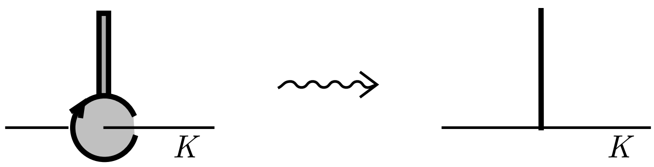
Remark 2.2.12. Let Γ be a unitrivalent tree with n leaves. Then the degree of Γ is $n - 1$.

Notation 2.2.13. We can represent a clasper in a 3-manifold using a unitrivalent tree in the following way:

i) To each node of the clasper we associate an inner node of the tree. The cyclic ordering at an inner node is induced by the orientation of the boundary of the corresponding node of the clasper.



ii) To each leaf we associate a leaf of the tree, which intersects the knot at the point where the leaf of the clasper and the knot once intersected.



iii) To each edge of the clasper we associate an edge of the tree, attached to nodes and leaves in the same (combinatorial) way as the corresponding edge in the clasper.

We define the *degree* of a clasper C as the degree of the associated unitrivalent tree.

Surgery in a 3-manidold M along two framed links L_1 and L_2 gives diffeomorphic results if and only if L_1 can be transformed to L_2 by a sequence of Kirby moves, cf. [Kir78].

Translating Kirby moves into the language of clasper surgery, we obtain (among others) the following move.

[7]The knot K^C depends on the choice of Φ_C, but it is independent up to isotopy by Remark 2.2.9.

[8]Warning: The degree of a unitrivalent is different from the notion of the degree of the underlying tree.

Proposition 2.2.14 ([Hab00, Proposition 2.7])**.** *Let C and C' be claspers for knots K and K' respectively. We have $K^C \sim K'^{C'}$ when the pairs (K, C) and (K', C') are related by "Habiro's move 1", which is indicated in Figure 8.*

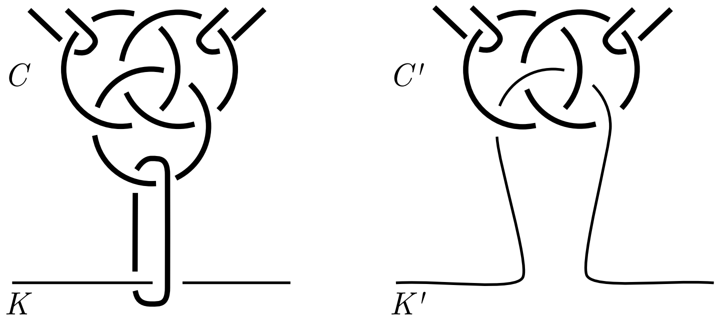


FIGURE 8. Habiro's move 1. The only difference of the pair (K, C) and (K', C') are all drawn in the figure.

We will see in a moment that a clasper surgery along a clasper of degree $n+1$ gives an equivalence relation on $\pi_0(\mathcal{K})$, which coincides with n-equivalence of knots (cf. Definition 2.1.10).

Definition 2.2.15. Two knots K_1 and K_2 are called *C_k-equivalent*, $k \geq 1$, if there exists a finite sequence of clasper surgeries along claspers of degree k that transforms K_1 into K_2 (up to isotopy). We denote this relation by $\sim_{C_k}$.

Remark 2.2.16. C_k-equivalence is an equivalence relation. Indeed, reflexivity and transitivity are clear. For symmetry, see [Hab00, Proposition 3.23].

Remark 2.2.17. Since Dehn surgery is invariant up to ambient isotopy of the framed link, cf. [Sav12, Chapter 2], C_k-equivalence makes sense on $\pi_0(\mathcal{K})$.

Proposition 2.2.18 ([Hab00, Proposition 3.7])**.** *Let $1 \leq k \leq l$, then C_l-equivalence is stronger than C_k-equivalence.* ☐

The following theorem is an important criterium to detect additive Vassiliev invariants. We will use this in the proof of Theorem 3.2.6.

Theorem 2.2.19 ([Hab00, Theorem 6.17])**.**

i) *The set $\pi_0(\mathcal{K})/\sim_{C_k}$ of C_k-equivalence classes becomes an abelian group, using connected sum (cf. Definition 1.0.10) as addition and unknot as identity.*
ii) *The quotient map $\psi_k\colon \pi_0(\mathcal{K}) \to \pi_0(\mathcal{K})/\sim_{C_k}$ is a universal additive Vassiliev invariant of degree at most $k-1$. More precisely, for every abelian group A and additive Vassiliev invariant $\nu\colon \pi_0(\mathcal{K}) \to A$ of degree at most $k-1$, there is a unique group homomorphism $\overline{\nu}\colon \pi_0(\mathcal{K})/\sim_{C_k} \to A$ such that the following diagram commutes.*

$$\begin{tikzcd} \pi_0(\mathcal{K}) \arrow[r, "\nu"] \arrow[d, "\psi_k"'] & A \\ \pi_0(\mathcal{K})/\sim_{C_k} \arrow[ur, dashed, "\exists!\,\overline{\nu}"'] & \end{tikzcd}$$

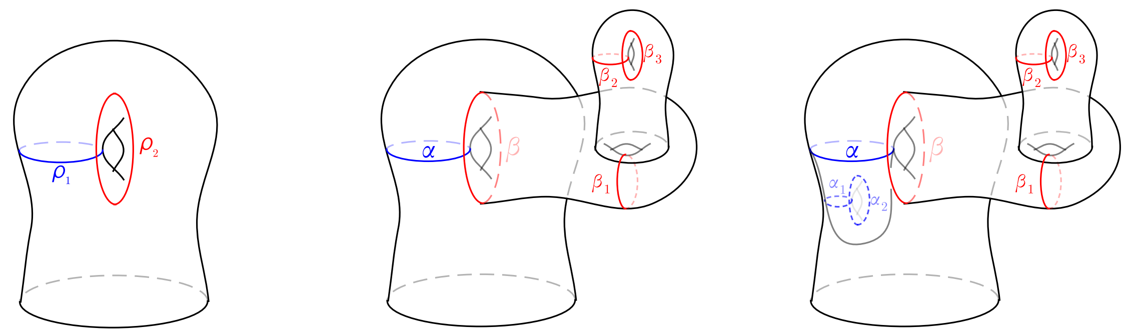
Theorem 2.2.20 ([Hab00, Theorem 6.18])**.** *Two knots K_1 and K_2 are C_{n+1}-equivalent if and only if they are n-equivalent.*

2.3. **Gropes.** Another characterisation of n-equivalence of knots can be given via grope cobordism of knots. A grope is an embedded (CW)-complex in $\mathbb{R}^3$, whose boundary components are knots. Conant and Teichner [CT04a] explain the connection between gropes and clasper surgery and concluded that two knots K_1 and K_2 are C_n-equivalent, if and only if K_1 and K_2 (or more precisely their images) cobound a grope of degree n, called a grope cobordism between K_1 and K_2. We will encounter a reformulated version of this connection in Theorem 3.2.7.

The aim of this section is to prove that a genus one capped grope of degree n gives a Δ^{n-1}-family of embedded disks in $\mathbb{R}^3$ with a common boundary, cf. Lemma 2.3.19. This technical statement will be an important ingredient in the proof of Theorem 3.2.6 in Section 3. We will use this theorem in the next chapter to prove Theorem 3.2.6.

Definition 2.3.1. We define *(genus one) grope* recursively. A grope (G_1, ρ) of degree 1 is a circle ρ embedded in $\mathbb{R}^3$ which is ambient isotopic to an unknot. A grope $(G_n, \{\rho_i\}_{i=1}^n)$ of degree $n \geq 2$ is a 2-complex embedded in $\mathbb{R}^3$ with n marked embedded simple closed curves ρ_i, which is constructed by the following steps:

i) A grope $(G_2, \{\rho_1, \rho_2\})$ of degree 2 is a punctured torus with two embedded simple closed curves ρ_1 and ρ_2, which is ambient isotopic in $\mathbb{R}^3$ to the one depicted in Figure 9A.



(A) $(G_2, \{\alpha, \beta\})$ (B) $(G_4, \alpha \cup \{\beta_j\}_{j=1}^3)$ (C) $(G_5, \{\alpha_i\}_{i=1}^2 \cup \{\beta_j\}_{j=1}^3)$

FIGURE 9. Visualisations of gropes of degree 2, 4 and 5.

ii) For $n > 2$, a grope of degree n is a union
$$\left(G_n \coloneqq G_k \cup_\alpha \cup G_2 \cup_\beta G_l, \{\rho_i\}_{i=1}^n = \{\alpha_i\}_{i=1}^k \cup \{\beta\}_{j=1}^l\right)$$
of gropes $(G_2, \{\alpha, \beta\})$, $\left(G_k, \{\alpha_i\}_{i=1}^k\right)$ and $\left(G_l, \{\beta_j\}_{j=1}^l\right)$ of degree 2, k and l respectively, that satisfy
  a) $1 \leq k \leq n-1$, and $1 \leq l \leq n-1$, and $k + l = n$,
  b) $\partial G_k = \alpha$ and $G_2 \cap G_k = \alpha$; $\partial G_l = \beta$ and $G_2 \cap G_l = \beta$, and
  c) $G_k \cap G_l = \alpha \cap \beta$.

Notation 2.3.2. We allow to abbreviate $(G_n, \{\rho_i\}_{i=1}^n)$ as G_n.

Remark 2.3.3. Note that the definition of gropes we give here is not as general as the one given in [CT04a].

Definition 2.3.4. For a degree n grope $(G_n, \{\rho_i\}_{i=1}^n)$, we defined the *bottom stage* T_{G_n} as the degree 2 grope $(G_2, \{\alpha, \beta\})$ to which $(G_k, \{\alpha_i\}_{i=1}^k)$ and $(G_l, \{\beta_j\}_{j=1}^l)$ are attached in Definition 2.3.1.

Similar to the case of claspers, we can also give a combinatorial description for gropes using trees. To do this, let us first introduce a decomposition of a grope into tori.

Proposition 2.3.5. *For $n \geq 2$, a grope $(G_n, \{\rho_i\}_{i=1}^n)$ can be decomposed canonically into $n-1$ punctured tori. This is examplified in Figure 10.*

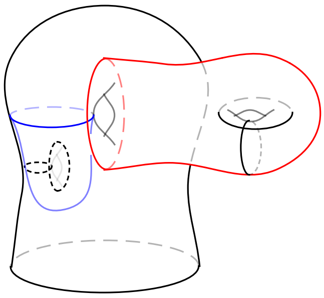


FIGURE 10. The decomposition of a degree 4 grope into 3 punctured tori, which are marked by different colours.

Proof. We prove the proposition by induction.

The base case is clear, since a grope G_2 of degree 2 is by definition a punctured torus.

Now let $n \geq 2$ and assume that the proposition is true for gropes of degree at most n. For a grope G_{n+1} of degree $n+1$, we can by definition decompose it into gropes G_2, G_k and G_l of degree 2, k and l respectively such that $k + l = n + 1$. By assumption, G_k and G_l can be decomposed into $k-1$ and $l-1$ punctured tori respectively. Thus G_{n+1} can be decomposed into $1 + (k-1) + (l-1) = n$ punctured tori. □

Making use of the above decomposition, we associate to each grope G_n a rooted labelled unitrivalent tree of degree n.

Definition 2.3.6. A *rooted labelled unitrivalent tree* of degree n is a unitrivalent tree of degree n, which has a chosen leaf as root and a bijection of the other leaves with the set $\{1, 2, \ldots, n\}$. See Figure 11 for two examples.

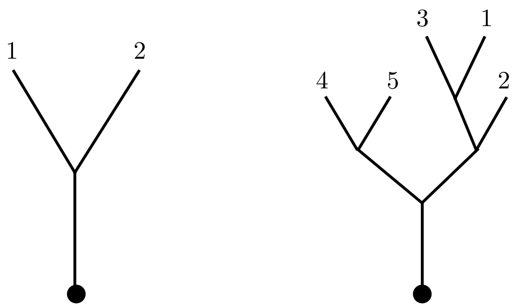


FIGURE 11. Two labelled rooted trees. The black dots designate the roots.

Recall that in a rooted tree Γ, the parent (when it exists) of a node v is a node adjacent to v which lies on the cycle-free path from v to the root. The other adjacent nodes to v are called children. Call a node w a descendant of v if v lies on the cycle-free path from v to the root. A unitrivalent subtree of $\widetilde{\Gamma}$ is a unitrivalent subtree of Γ which contains the root.

Definition 2.3.7. Let $(G_n, \{\rho_i\}_{i=1}^n)$ be a grope of degree $n \geq 2$. Define its *associated rooted labelled unitrivalent tree* Γ_{G_n} using the procedures. Recall from Proposition 2.3.5 the decomposition of G_n into $n-1$ tori.

i) To each punctured torus in the decomposition of G_n, we assign an inner node[9].

[9]We can also assign cyclic ordering to each inner node, depending on the orientation of the tori from the decomposition, cf. [CT04a]

ii) To each ρ_i we assign a leaf, labelled by i.
iii) To the boundary of the bottom stage, we assign the root.
iv) Connect the inner node representing the bottom stage to the root by an edge.
v) Connect two inner nodes by an edge if the two tori T_1, T_2 they represent have non-empty intersection (i.e. $T_1 \cap T_2 = \partial T_1$ or $T_1 \cap T_2 = \partial T_2$).
vi) Connect a leaf with an inner node by an edge if the curve ρ_i that is represented by the leaf lies in the torus represented by the inner node.

Remark 2.3.8. Let G_n be a grope of degree n and let Γ_{G_n} be its associated tree. Then every rooted labelled subtree $\widetilde{\Gamma}$ of Γ_{G_n} represents a subgrope of G_n which is given as the union of the punctured tori that are represented by the inner nodes of $\widetilde{\Gamma}$. That is

$$\{\text{unitrivalent subtree of } \Gamma_{G_n}\} \longleftrightarrow \{\text{subgropes of } G_n\}.$$

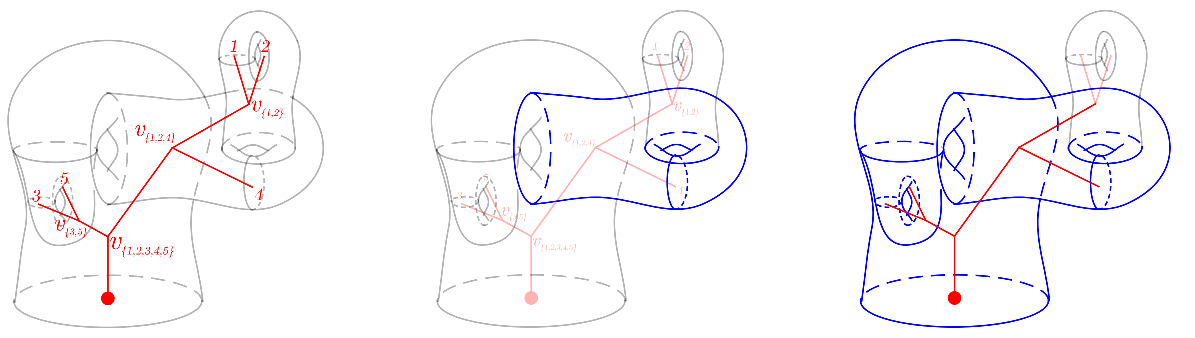
Using this combinatorial description, let us fix some notation for gropes.

Notation 2.3.9. Let $(G_n, \{\rho_i\}_{i=1}^n)$ be a grope of degree n and and let Γ_{G_n} be its associated rooted labelled unitrivalent tree. Let $\emptyset \neq S \subseteq \{1, \ldots, n\}$.

i) Let v_S denote the inner node in Γ_{G_n} such that the set of leaves that are descendants of v_S is exactly $\{\rho_i\}_{i \in S}$. Note that this is not defined for every S.
ii) Denote by T_S the punctured torus corresponding to v_s. Note that $T_S = T_{G_n}$ for $S = \{1, 2, \ldots, n\}$.
iii) Let G_S be the subgrope of G_n corresponding to the smallest rooted unitrivalent subtree Γ_S of Γ_{G_n}, which contains all leaves $\{\rho_i\}_{i \in S}$.
iv) Denote by $\mathrm{N}(T_S)$ be a tubular neighbourhood of T_S in $\mathbb{R}^3$, when T_S is well-defined, otherwise take $\mathrm{N}(T_S) \coloneqq \emptyset$. Define the "tubular neighbourhood" of G_S as

$$\mathrm{N}(G_S) \coloneqq \bigcup_{S' \subseteq S} \mathrm{N}(T_{S'}).$$

Example 2.3.10. We use the grope G_5 from Figure 9C to illustrate some of the notations.



(A) G_5 and Γ_{G_5}(red) (B) Punctured torus $T_{\{1,2,4\}}$ (C) Subgrope $G_{\{3,4,5\}}$(*blue*)

FIGURE 12. A grope of degree 5 and its corresponding rooted labelled unitrivalent tree.

Given a grope $(G_n, \{\rho_i\}_{i=1}^n)$, we can attach n disks to $\{\rho_i\}_{i=1}^n$ respectively. This way, we obtain a 'capped grope' of degree n.

Definition 2.3.11. A *capped grope* $(G_n^c, \{C_i\}_{i=1}^n)$ of degree $n \geq 2$ is a 2-complex embedded in $\mathbb{R}^3$ as the union of a grope $(G_n, \{\rho_i\}_{i=1}^n)$ of degree n and n disks $\{C_i\}_{i=1}^n$, called caps, that satisfy

i) $\partial C_i = \rho_i$ and $C_i \cap G_n = \rho_i$ for $1 \leq i \leq n$, and

ii) $C_i^\circ \cap C_j^\circ = \emptyset$ for $1 \leq i \neq j \leq n$.

For the joy of the readers, see Figure 13 for an example of a capped grope of degree 5.

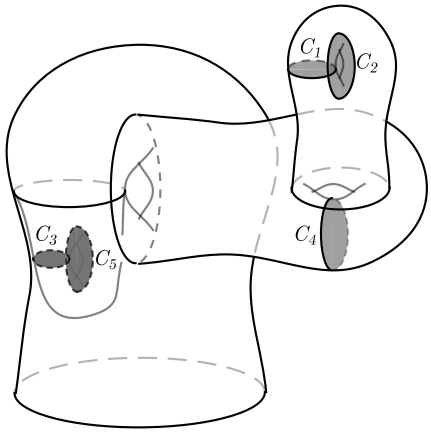


FIGURE 13. A visualisation of a capped grope $(G_5^c, \{C_i\}_{i=1}^5)$ whose associated uncapped grope is homeomorphic to Figure 9C.

Notation 2.3.12. We allow to abbreviate the notation $(G_n^c, \{C_i\}_{i=1}^n)$ by G_n^c.

Notation 2.3.13. Given a capped grope $(G_n^c, \{C_i\}_{i=1}^n)$ of degree $n \geq 2$, we denote by $(G_n, \{\rho_i\}_{i=1}^n) \coloneqq (G_n^c \setminus \cup_{i=1}^n C_i^\circ, \{\partial C_i\}_{i=1}^n)$ the uncapped grope associated to G_n^c. Also let $\mathrm{N}(C_i)$ be a tubular neighbourhood of C_i in $\mathbb{R}^3$.

Now we introduce cap surgery of a capped grope $(G_n^c, \{C_i\}_{i=1}^n)$ along one of its caps C_i, which is the main ingredient of the proof of Theorem 2.3.19.

Construction 2.3.14. Let $\psi\colon (\mathrm{D}^k \times \mathrm{D}^{m-k}, \mathrm{S}^{k-1} \times \mathrm{D}^{m-k}) \hookrightarrow (M, N)$ be a smooth embedding of pairs of smooth manifolds, where M is a m-dimensional manifold and N is a codimension 1 submanifold of M. Furthermore assume that $\mathrm{im}(\psi) \cap N = \psi(\mathrm{S}^{k-1} \times \mathrm{D}^{m-k})$. An *ambient surgery* along ψ is the procedure to obtain a new codimension one submanifold N^ψ of M by the following

$$N^\psi \coloneqq N \setminus \psi(\mathrm{S}^{k-1} \times \mathring{\mathrm{D}}^{m-k}) \bigcup_{\psi(\mathrm{S}^{k-1} \times \mathrm{S}^{m-k-1})} \psi(\mathrm{D}^k \times \mathrm{S}^{m-k-1}).$$

Lemma 2.3.15 ([GS99, Page 154])**.** *The submanifold N^ψ of M that is obtained by ambient surgery along ψ is uniquely determined up to diffeomorphism by the isotopy class of ψ.* □

Definition 2.3.16. A *surgery of $(G_n^c, \{C_i\}_{i=1}^n)$ along one of its caps C_i* is an ambient surgery along an embedding

$$\psi_i\colon (\mathrm{D}^2 \times \mathrm{D}^1, \mathrm{S}^1 \times \mathrm{D}^1) \to (\mathbb{R}^3, T_i)$$

where T_i is the punctured torus in the decomposition of G_n (cf. Proposition 2.3.5) in which $\rho_i = \partial C_i$ is embedded, and where ψ_i satisfies the following properties:

i) $\psi_i(\mathrm{D}^2 \times \{\frac{1}{2}\}) = C_i$ and $\psi_i(\mathrm{S}^1 \times \{\frac{1}{2}\}) = \partial C_i$,
ii) $\psi_i(\mathrm{S}^1 \times \mathring{\mathrm{D}}^1)$ is a tubular neighbourhood of ∂C_i in T_i,
iii) $\psi_i(\mathrm{D}^2 \times \mathring{\mathrm{D}}^1)$ is a tubular neighbourhood of C_i in $\mathbb{R}^3$, and
iv) $\psi_i(\mathrm{D}^2 \times \mathrm{D}^1) \cap T_i = \psi_i(\mathrm{S}^1 \times \mathrm{D}^1)$.

Remark 2.3.17. The existence of a suitable ψ_i follows from the existence of tubular neighbourhoods of submanifolds, cf. [Hir76, Theorem 4.6.3, Theorem 4.6.4]. Since any two tubular neighbourhoods of a submanifold are isotopic (cf. [Hir76, Theorem 4.6.5]), the map ψ_i is determined up to isotopy by the conditions i), ii), iii) and iv) in Construction 2.3.16. Thus, the new submanifold $T_i^{\psi_i}$ is determined up to diffeomorphism.

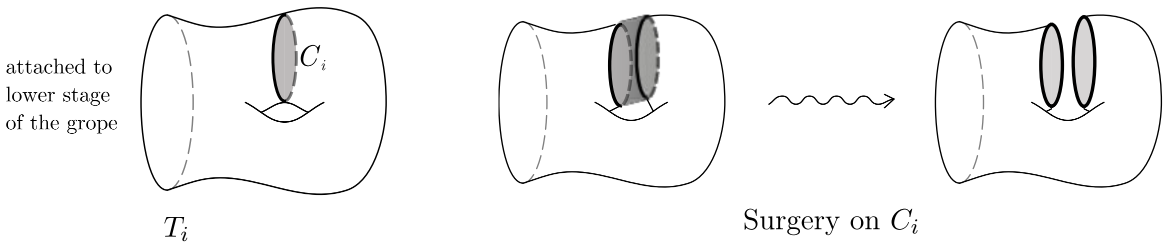


FIGURE 14. Surgery of a capped grope along its cap C_i.

Proposition 2.3.18. *In the situation of Definition 2.3.16, the new submanifold $T_i^{\psi_i}$ obtained from the ambient surgery is homeomorphic to a disk with boundary ∂T_i.*

Proof. For a visualisation of this, see Figure 14. First we attach a disk along the boundary of T_i, making it a torus $\widehat{T}_i$, which has Euler characteristic -2. The procedure of doing the surgery is to remove an annulus from T_i and glue back two disks. Thus the difference of the Euler characteristic of $T_i^{\psi_i} \cup_{\partial T_i} \mathrm{D}^2$ and $\widehat{T}_i$ is 2. In other words, the Euler characteristic of $T_i^{\psi_i} \cup_{\partial T_i} \mathrm{D}^2$ is 0. By the classification of surfaces we have $T_i^{\psi_i} \cup_{\partial T_i} \mathrm{D}^2 \approx \mathrm{S}^2$ and thus $T_i^{\psi_i} \approx \mathrm{D}^2$ such that $\partial T_i^{\psi_i} = \partial T_i$. $\square$

Finally, we can state precisely and prove the theorem at which we worked towards the whole section.

Theorem 2.3.19. *Let $(G_n^c, \{C_i\}_{i=1}^n)$ be a capped grope of degree $n \geq 2$. Then there exists a continuous map*

$$h_n \colon \mathrm{D}^2 \times \Delta^{n-1} \to \mathbb{R}^3,$$

which is a family of embeddings with fixed boundary in a neighbourhood of G_n^c, i.e. it has the following properties:

i) *For every $v \in \Delta^{n-1}$, the restriction $h_n(- \times \{v\}) \colon \mathrm{D}^2 \to \mathbb{R}^3$ is an embedding,*
ii) *$h_n(\mathrm{S}^1 \times \{v\}) = \partial G_n$, for every $v \in \Delta^{n-1}$. Furthermore, $h_n(p, v) = h_n(p, w)$ for every $v, w \in \Delta^{n-1}$, $p \in \mathrm{S}^1$.*
iii) *For every $v = (t_1, t_2, \dots, t_n) \in \Delta^{n-1}$ in barycentric coordinates, we have*

$$h_n(\mathrm{D}^2 \times \{v\}) \subset \mathrm{N}(G_{S_v}) \cup \bigcup_{s \in S_v} \mathrm{N}(C_s),$$

where $S_v \coloneqq \{1 \leq i \leq n \mid t_i \neq 0\}$.

Proof. We prove the theorem by induction.

For the base case $n = 2$, we want a map $h_2 \colon \mathrm{D}^2 \times \Delta^1 \to \mathbb{R}^3$ satisfying i), ii) and iii). First we construct two disks D_1 and D_2 that will in a moment turn out to be the images $h_2(\mathrm{D}^2 \times \{(0,1)\})$ and $h_2(\mathrm{D}^2 \times \{(1,0)\})$. For $i = 1, 2$, let us do surgery of G_2^c along C_i via the embedding $\psi_i \colon (\mathrm{D}^2, \mathrm{S}^1) \times \mathrm{D}^1 \to (\mathbb{R}^3, G_2)$ as in Definition 2.3.16. By Proposition 2.3.18, we obtain two disks (See Figure 15 for a visualisation of this two surgeries.)

$$D_i \coloneqq G_2^{\psi_i} = G_2 \setminus \psi_i(\mathrm{S}^1 \times \mathring{\mathrm{D}}^1) \cup_{\psi_i(\mathrm{S}^1 \times \mathrm{S}^0)} \alpha_i(\mathrm{D}^2 \times \mathrm{S}^0) \subset \mathbb{R}^2 \times \mathrm{D}^1,$$

such that $D_i = \partial G_2$. We see from the construction that $D_i \subset G_{\{i\}} \cup \mathrm{N}(C_i)$.

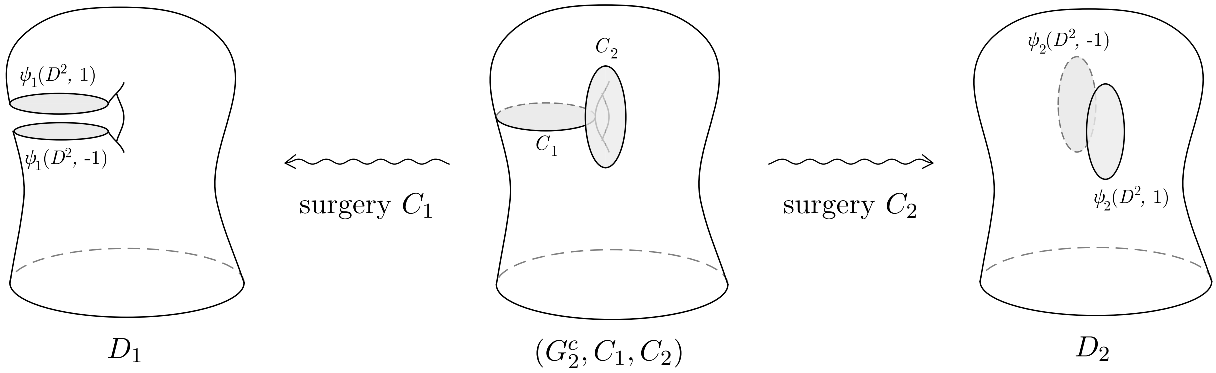


FIGURE 15. Surgery of G_2^c along the caps C_1 and C_2 yields disks D_1 and D_2 respectively.

Now, let us define h_2, which will be an isotopy from D_1 to D_2 in $\mathbb{R}^3$. First, let us denote the space $Q \coloneqq \psi_1(\mathrm{S}^1 \times \mathrm{D}^1) \cap \psi_2(\mathrm{S}^1 \times \mathrm{D}^1) = \psi_1(\mathrm{D}^2 \times \mathrm{D}^1) \cap \psi_2(\mathrm{D}^2 \times \mathrm{D}^1)$. Consider the space $D \coloneqq \psi_1(\mathrm{D}^2 \times \mathrm{D}^1) \cup_Q \psi_2(\mathrm{D}^2 \times \mathrm{D}^1)$. Using for example Seifert–van Kampen theorem, we can easily see that D is homeomorphic to the standard unit 3-ball D^3 in $\mathbb{R}^3$. The intersection $T \coloneqq D \cap G_2 = \psi_1(\mathrm{S}^1 \times \mathrm{D}^1) \cup_Q \psi_2(\mathrm{S}^1 \times \mathrm{D}^1)$ is a punctured torus, by an argument via Euler characteristic. See Figure 16 for a visualisation of D and T.

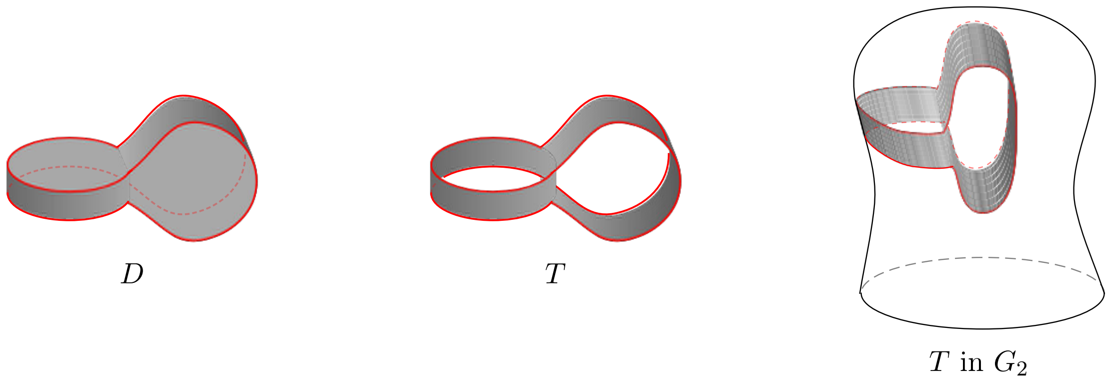


FIGURE 16. A visualisation of D, T and T embedded in G_2.

In particular, we have $\partial D = \widetilde{D}_1 \cup_{\partial T} \widetilde{D}_2$ where,

$$\widetilde{D}_1 \coloneqq \psi_1(\mathrm{D}^2 \times \mathrm{S}^0) \cup (\psi_2(\mathrm{S}^1 \times \mathrm{D}^1) \setminus Q^\circ),$$

$$\widetilde{D}_2 \coloneqq \psi_1(\mathrm{S}^1 \times \mathrm{D}^1) \cup (\psi_2(\mathrm{D}^2 \times \mathrm{S}^0) \setminus Q^\circ).$$

See Figure 17 for a visualisation of $\widetilde{D}_1$ and $\widetilde{D}_2$. Furthermore, we have

$$D_1 = (G_2 \setminus T^\circ) \cup_{\partial T} \widetilde{D}_1,$$

$$D_2 = (G_2 \setminus T^\circ) \cup_{\partial T} \widetilde{D}_2.$$

Thus we can fix an isotopy[10] $g_{12} \colon \widetilde{D}_1 \times \mathrm{I} \to D$ in D from $\widetilde{D}_1$ to $\widetilde{D}_2$ restricting to the identity on ∂T. In other words, $g_{1,2}|_{\widetilde{D}_1 \times \{t\}}$ is an embedding for $t \in \mathrm{I}$, $g_{1,2}|_{\widetilde{D}_1 \times \{0\}} = \mathrm{id}_{\widetilde{D}_1}$ and $g_{1,2}(\widetilde{D}_1 \times \{1\}) = \widetilde{D}_2$.

[10]We can choose a homeomorphism from D to D^3 which maps $\widetilde{D}_1$ and $\widetilde{D}_2$ to the upper- and lower hemisphere of $\partial D^3 = \mathrm{S}^2$ respectively. Pulling back an isotopy in D^3 from the upper hemisphere to the lower hemisphere gives such an isotopy g_{12}

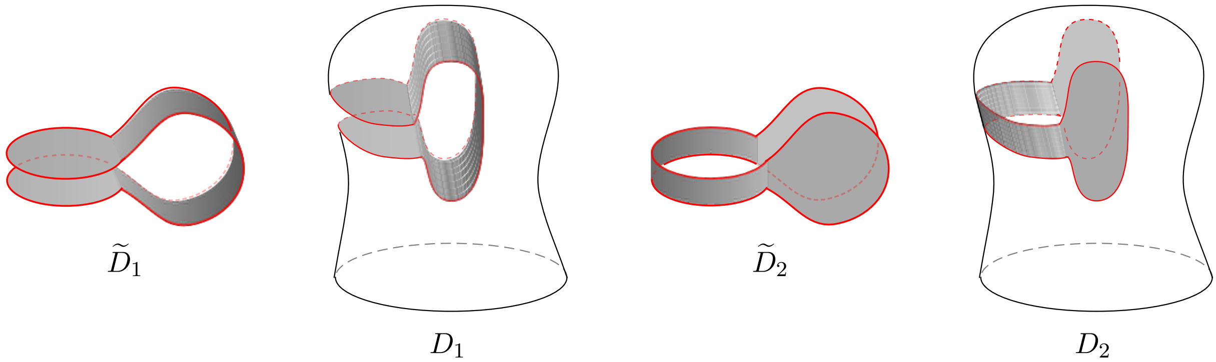


FIGURE 17. $\widetilde{D}_1 \subseteq D_1$ and $\widetilde{D}_2 \subseteq D_2$.

This leads us to define

$$\hat{h}_2 \colon D_1 \times \Delta^1 \to D \cup G_2 \subseteq \mathbb{R}^3$$
$$(p,(1-t,t)) \mapsto \begin{cases} g_{12}(p,t) & \text{if } p \in \widetilde{D}_1 \\ p & \text{otherwise} \end{cases}$$

As for h_2, we choose a parametrisation $q \colon \mathrm{D}^2 \to D_1$ of D_1 and define $h_2 \coloneqq \hat{h}_2 \circ (q \times \mathrm{id}_{\Delta^1})$. From the surgery construction of D_1 and D_2 and definition of $\hat{h}_2$, we have that h_2 satisfies conditions i), ii), iii) and iv) as desired.

Induction step: For $n \geq 3$, assume that the theorem is true for capped gropes G_k^c of degree $k \leq n-1$. We prove the theorem for capped gropes $(G_n^c, \{C_i\}_{i=1}^n)$ of degree n.

We consider the associated uncapped grope G_n as a union of $(T_{G_n}, \{\alpha, \beta\})$, $(G_k, \{\rho_i\}_{i \in S_\alpha})$ and $(G_l, \{\rho_j\}_{j \in S_\beta})$ with $\#S_\alpha = k$, $\#S_\beta = l$ and $k + l = n$ (cf. Definition 2.3.1).

Let $r = \alpha$ or $r = \beta$. First we construct two continuous families of embedded disks $\{\mathrm{D}_{v^r}\}_{v^r \in \Delta^{\#S_r - 1}}$ in $\mathbb{R}^3$, which will turn out to be the images $h_n(\mathrm{D}^2 \times \{v^r \in \Delta_{S_r}^{\#S_r - 1} \subseteq \Delta^n\})$

By assumption, $G_{\#S_r - 1}$ gives continuous maps

$$h^r \colon \mathrm{D}^2 \times \Delta^{\#S_r - 1} \to \mathbb{R}^3,$$

such that for every $v^r = (t_k)_{k \in S_r}, w^r \in \Delta^{\#S_r - 1}$ we have the following.

i) The map $h^r|_{\mathrm{D}^2 \times \{v^r\}}$ is a smooth embedding of a disk in $\mathbb{R}^3$.
ii) We have $h^r(\mathrm{S}^1 \times \{v^r\}) = r$. Furthermore, $h^r(p, v^r) = h^r(p, w^r)$ for every $p \in \mathrm{S}^1$.
iii) We have

$$h^r(\mathrm{D}^2 \times \{v^r\}) \subset \mathrm{N}(G_S) \cup \bigcup_{s \in S} \mathrm{N}(C_s).$$

where $S_{v^r} \coloneqq \{k \in S_r \mid t_k \neq 0\}$, and G_{S_v} with is the subgrope of $G_{\#S_r}$ defined via Notation 2.3.9.iii).

Before we continue, let us define the abbreviation

$$C_{v^r} \coloneqq h^r(\mathrm{D}^2 \times \{v^r\}),$$

and for $v^\alpha \in \Delta^{|S_\alpha| - 1}$ and $v^\beta \in \Delta^{|S_\beta| - 1}$, denote

$$G^c_{v^\alpha, v^\beta} \coloneqq C_{v^\alpha} \cup_\alpha T_{G_n} \cup_\beta C_{v^\beta}.$$

Note that $(G^c_{v^\alpha, v^\beta}, \{C_{v^\alpha}, C_{v^\beta}\})$ is a degree 2 capped grope whose associated uncapped grope is T_{G_n}.

By of [Pal60, Theorem C][11], we obtain two continuous families of embeddings ψ^r

$$\psi^r \colon (\mathrm{D}^2 \times \mathrm{D}^1 \times \Delta^{\#S_r - 1}, \mathrm{S}^1 \times \mathrm{D}^1 \times \Delta^{\#S_r - 1}) \to (\mathbb{R}^3, T_{G_n})$$

[11]This theorem states that under certain conditions, an n-isotopy (isotopy parametrized by Δ^n) of a submanifold can be extended to an n-isotopy of its tubular neighbourhood. A more general theorem

extending $h^r, r \in \{\alpha, \beta\}$ such that for every $v^r, w^r \in \Delta^{\#S_r - 1}$ we have the following:

i) The restriction $\psi_{v^r} := \psi^r|_{(\mathrm{D}^2\times\mathrm{D}^1)\times\{v^r\}}$ is an embedding and $\psi_{v^r}|_{\mathrm{D}^2\times\{0\}} = h^r|_{\mathrm{D}^2\times\{v^r\}}$. Furthermore $\mathrm{im}(\psi^r) \cap T_{G_n} = \mathrm{im}(\psi^r|_{\mathrm{S}^1\times\mathrm{D}^1\times\Delta^{\#S_r-1}})$.

ii) We have $\psi_{v^r}(\mathrm{D}^2 \times \{\frac{1}{2}\}) = C_{v^r}$ and $\psi_{v^r}(\mathrm{S}^1 \times \{\frac{1}{2}\}) = r$.

iii) We have $\psi_{v^r}|_{\mathrm{S}^1\times\mathrm{D}^1} = \psi_{w^r}|_{\mathrm{S}^1\times\mathrm{D}^1}$ and $\psi_{v^r}(\mathrm{S}^1 \times \mathrm{D}^1)$ is a tubular neighbourhood of r in T_{G_n}.

iv) We have that $\mathrm{im}(\psi_{v^r})$ is a tubular neighbourhood of C_{v^r} in $\mathbb{R}^3$.

The above properties i)–iv) determine $\psi_{v^r} \colon (\mathrm{D}^2 \times \mathrm{D}^1, \mathrm{S}^1 \times \mathrm{D}^1) \to (\mathbb{R}^3, T_{G_n})$ up to isotopy. Thus, for $r \in \{\alpha, \beta\}$, we can perform surgery on $G^c_{v^\alpha,v^\beta}$ along C_{v^r} via the embedding ψ_{v^r} and obtain a disk

$$D_{v^r} := (T_{G_n} \setminus \psi_{v^r}(\mathrm{S}^1 \times \mathrm{D}^1)) \cup_{\psi_{v^r}(\mathrm{S}^1\times\mathrm{S}^1)} \psi_{v^r}(\mathrm{D}^2 \times \mathrm{S}^0).$$

As a result, we obtain two continuous families of embedded disks $\{D_{v^r}\}_{v^r\in\Delta^{\#S_r-1}}$ in $\mathbb{R}^3$ with boundary ∂G_n. By construction we have $D_{v^r} \subseteq \mathrm{N}(G_S) \cup \bigcup_{s\in S} \mathrm{N}(C_s)$ for every element $v^r = (t_k)_{k\in S_r} \in \Delta^{\#S^r-1}$, and $S_{v^r} := \{k \in S_r \mid t_k \neq 0\}$ and G_S the corresponding subgrope of $G_{\#S_r\#-1}$.

Now we are going to define h_n. In other words, we need to find an embedded disk in $\mathbb{R}^3$, satisfying ii) and iii) of the theorem for every $w \in \Delta^{n-1}$. We are going to use that for $k + l = n$, the join $\Delta^{k-1} \star \Delta^{l-1}$ is homeomorphic to Δ^{n-1}. In other words, for every $w \in \Delta^{n-1}$, there exists a unique $v^\alpha \in \Delta^{k-1}_{S_\alpha}$, $v^\beta \in \Delta^{l-1}_{S_\beta}$ and $t \in [0,1]$ such that $w = (1-t)v^\alpha + tv^\beta$. Recall that $\#S_\alpha = k$ and $\#S_\beta = l$ and we intend to make $D_{v^r} = h_n(\mathrm{D}^2 \times \{v^r\})$ for $v^r \in \Delta^{\#S_r-1} \subseteq \Delta^{n-1}$ and $r \in \{\alpha, \beta\}$. Therefore, for every pair of v^α and v^β, we will construct an isotopy h_{v^α,v^β} from D_{v^α} to D_{v^β} such that $h_{v^\alpha,v^\beta}(D_{v^\alpha} \times \{t\})$ will turn out to be the image $h_n(\mathrm{D}^2 \times \{(1-t)v^\alpha + tv^\beta\})$.

Let us define, similar to the base case, the isotopy h_{e^α,e^β} for $e^\alpha = (1, 0, \ldots, 0) \in \Delta^{k-1}$ and $e^\beta = (1, 0, \ldots, 0) \in \Delta^{l-1}$. We consider the degree 2 capped grope $G^c_{e^\alpha,e^\beta}$ with caps C_{e^α} and C_{e^β}. Let us define the space $Q := \psi_{e^\alpha}(\mathrm{S}^1 \times \mathrm{D}^1) \cap \psi_{e^\beta}(\mathrm{S}^1 \times \mathrm{D}^1)$, and consider the space $D := \psi_{e^\alpha}(\mathrm{D}^2 \times \mathrm{D}^1) \cup_Q \psi_{e^\beta}(\mathrm{D}^2 \times \mathrm{D}^1)$, which is homeomorphic to D^3. The intersection $T := D_{e^\alpha,e^\beta} \cap T_{G_n} = \psi_{e^\alpha}(\mathrm{S}^1 \times \mathrm{D}^1) \cup_Q \psi_{e^\beta}(\mathrm{S}^1 \times \mathrm{D}^1)$ is a punctured torus. Similar as in the base case, we have

$$\partial D = \widetilde{D}_{e^\alpha} \cup_{\partial T_{e^\alpha,e^\beta}} \widetilde{D}_{e^\beta}$$

where

$$\widetilde{D}_{e^\alpha} := \psi_{e^\alpha}(\mathrm{D}^2 \times \mathrm{S}^0) \cup (\psi_{e^\beta}(\mathrm{S}^1 \times \mathrm{D}^1) \setminus Q),$$
$$\widetilde{D}_{e^\beta} := \psi_{e^\alpha}(\mathrm{S}^1 \times \mathrm{D}^1) \cup (\psi_{e^\beta}(\mathrm{D}^2 \times \mathrm{S}^0) \setminus Q),$$

and

$$D_{e^\alpha} = (T_{G_n} \setminus T^\circ_{e^\alpha,e^\beta}) \cup_{\partial T_{e^\alpha,e^\beta}} \widetilde{D}_{e^\alpha},$$
$$D_{e^\beta} = (T_{G_n} \setminus T^\circ_{e^\alpha,e^\beta}) \cup_{\partial T_{e^\alpha,e^\beta}} \widetilde{D}_{e^\beta}.$$

Thus we fix an isotopy $g_{e^\alpha,e^\beta} \colon \widetilde{D}_{e^\alpha} \times \mathrm{I} \to D$ from $\widetilde{D}_{e^\alpha}$ to $\widetilde{D}_{e^\beta}$ which restricts to identity on $\partial T_{e^\alpha,e^\beta}$. Then we define

$$h_{e^\alpha,e^\beta} \colon D_{e^\alpha} \times \mathrm{I} \to D_{e^\alpha,e^\beta} \cup T_{G_n} \subseteq \mathbb{R}^3$$
$$(p,t) \mapsto \begin{cases} g_{e^\alpha,e^\beta}(p,t) & \text{if } p \in \widetilde{D}_{e^\alpha} \\ p & \text{otherwise} \end{cases}$$

says that for two manifold M, N (without boundary), the map $\mathrm{Emb}(M,N) \to \mathrm{Diff}(M,N)$ is a Hurewicz fibration, cf. [Theorem C][Pal60].

For any other pair v^α, v^β, define $D_v \coloneqq \psi_{v^\alpha}(\mathrm{D}^2 \times \mathrm{D}^1) \cup_Q \psi_{v^\beta}(\mathrm{D}^2 \times \mathrm{D}^1)$. Then we define g_{v^α,v^β} by mapping $(\widetilde{D}_{v^\alpha}, \widetilde{D}_{v^\beta}, D_v)$ homemorphically to $(\widetilde{D}_{e^\alpha}, \widetilde{D}_{e^\beta}, D)$ via ψ^r for $r \in \{\alpha, \beta\}$, and pull back the isotopy g_{e^α,e^β}:

$$\begin{tikzcd} \widetilde{D}_{v^\alpha} \times \mathrm{I} \arrow[r, "g_{v^\alpha,v^\beta}"] \arrow[d, "\psi_{v^\alpha} \circ \psi_{e^\alpha}^{-1} \times \mathrm{id}_I"'] & D_v \\ \widetilde{D}_{e^\alpha} \times \mathrm{I} \arrow[r, "g_{e^\alpha,e^\beta}"'] & D \arrow[u, "\psi_{e^r} \circ \psi_{v^r}^{-1} \times \mathrm{id}_I"'] \end{tikzcd}$$

Now we define $\widehat{h}_n \colon D_{e^\alpha} \times \Delta^{n-1} \to \mathbb{R}^3$ by mapping

$$(p, (1-t)v^\alpha + tv^\beta) \mapsto (\psi_{v^r} \circ \psi_{e^r}^{-1}) \left(g_{e^\alpha,e^\beta}(\psi_{e^\alpha} \circ \psi_{v^\alpha}^{-1}(p), t) \right)$$

if $g_{e^\alpha,e^\beta}(\psi_{e^\alpha} \circ \psi_{v^\alpha}^{-1}(p), t) \in \psi_{e^r}(\mathrm{D}^2 \times \mathrm{D}^1)$ with $r \in \{\alpha, \beta\}$, and $(p, w) \mapsto p$ otherwise.

Finally, we choose a parametrisation $q_{e^\alpha} \colon \mathrm{D}^2 \to D_{e^\alpha}$ of D_{e^α} and define the sought-after map $h_n \coloneqq \widehat{h}_n \circ (q_{e^\alpha} \times \mathrm{id}_{\Delta^{n-1}})$. By construction h_n satisfies i) to iii). □

3. Manifold calculus on knots

Manifold calculus is a method introduced by Goodwillie and Weiss [GKW01; Wei99], which produces a sequence of functors “approximating” a given "good" functor on the category of open sets of a manifold. Given two smooth manifolds M and N, we can apply manifold calculus to the embedding functor $\mathrm{Emb}(-, N) \colon \mathbf{Open}\,(M)^{\mathrm{op}} \to \mathbf{CGH}$. In this way we can obtain information about the space $\mathrm{Emb}(M, N)$ by studying the embedding functor, and analysing its sequence of approximations. In this section we are going to briefly introduce the basic building blocks of manifold calculus, apply this technique to the space of knots and see how it relates to Vassiliev invariants.

3.1. Manifold Calculus.

The main reference for this section is [BW13] and [Wei99], which also contains further motivation for manifold calculus.

Definition 3.1.1.

i) Denote by **CGH** the topological category of compactly generated weak Hausdorff spaces.

ii) For a manifold M, denote by $\mathbf{Open}_\partial\,(M)$ the category of open subsets of M which contain ∂M. Objects of $\mathbf{Open}_\partial\,(M)$ are the open subsets V of M such that $\partial M \subseteq V$, and morphisms of $\mathbf{Open}_\partial\,(M)$ are the inclusions of these open subsets.

For a manifold M' without boundary, we will simplify the notation as $\mathbf{Open}\,(M')$.

Definition 3.1.2 ([Wei99])**.** A smooth codimension zero embedding $i_v \colon (V, \partial V) \to (W, \partial W)$ between smooth manifolds V and W is an *isotopy equivalence* if there exists a smooth embedding $i_w \colon (W, \partial W) \to (V, \partial V)$ such that $i_v \circ i_w$ and $i_w \circ i_v$ are isotopic to $\mathrm{id}_{(W,\partial W)}$ and $\mathrm{id}_{(V,\partial V)}$ respectively.

Definition 3.1.3. Let M be a smooth manifold dimension m. A *good functor* on $\mathbf{Open}_\partial\,(M)$ is a functor $F \colon \mathbf{Open}_\partial\,(M)^{\mathrm{op}} \to \mathbf{CGH}$ of topological categories, which satisfies the following conditions:

i) (isotopy invariant) If $i \in \mathrm{Mor}_{\mathbf{Open}(M)}(V, W)$ is an isotopy equivalence, then $F(i)$ is a weak homotopy equivalence;

ii) For any filtration $\dots V_i \subseteq V_{i+1} \dots$ of open subsets of M, the canonical map (coming from the universal property of holim[12])

$$F(\cup_{i\in\mathbb{N}} V_i) \to \operatorname*{holim}_{i\in\mathbb{N}} F(V_i)$$

is a weak homotopy equivalence.

Theorem 3.1.4 ([Wei99, Proposition 1.4])**.**

i) *Let M and N be smooth manifolds without boundary with $\dim M \leq \dim N$. Then the embedding functor*

$$\begin{aligned} \mathrm{Emb}(-,N)\colon \mathbf{Open}\,(M)^{\mathrm{op}} &\to \mathbf{CGH} \\ V &\mapsto \mathrm{Emb}(V,N) \end{aligned}$$

is a good functor.

ii) *Let M and N be smooth manifolds with boundary with $\dim M \leq \dim N$, and let $f\colon M \hookrightarrow N$ be a fixed smooth embedding. Then the embedding functor*

$$\begin{aligned} \mathrm{Emb}_\partial(-,N,f)\colon \mathbf{Open}_\partial\,(M)^{\mathrm{op}} &\to \mathbf{CGH} \\ V &\mapsto \mathrm{Emb}_\partial(V,N,f) \end{aligned}$$

is a good functor.

iii)

Now we are going to construct the approximation sequence for good functors.

Definition 3.1.5. Denote by $[n]$ the set $\{0,1,\dots,n\}$.

i) Define the category $\mathbf{Pow}([n])$ as the category whose objects are the subsets of $[n]$ and the morphisms are inclusions of subsets.

ii) Define the full subcategory $\mathbf{Pow}([n])_{\neq\emptyset}$ of $\mathbf{Pow}([n])$, whose objects are the non-empty subsets of $[n]$.

Definition 3.1.6. For a manifold M without boundary, a good functor F is a *polynomial functor of degree at most n* if for every open subset $U \in \mathbf{Open}\,(M)$ and $A_0, A_1, \dots, A_n$ pairwise disjoint closed subsets of M which lie in U, the $(n+1)$-cube

$$\begin{aligned} \chi\colon \mathbf{Pow}([n]) &\to \mathbf{CGH} \\ S &\mapsto F(U \setminus \cup_{i\in S} A_i) \end{aligned}$$

is homotopy cartesian, i.e. we have that $\chi(\emptyset) \to \operatorname{holim}_{S\neq\emptyset} \chi(S)$ is a weak homotopy equivalence. In other words, $F(U) \to \operatorname{holim}_{S\neq\emptyset} F(U \setminus \cup_{i\in S} A_i)$ is a weak homotopy equivalence.

Remark 3.1.7. One obtains the definition of polynomial functors of degree at most n for manifolds M with boundary by replacing $\mathbf{Open}\,(M)$ with $\mathbf{Open}_\partial\,(M)$ and requiring that each A_i has empty intersection with ∂U so that $F(U \setminus \cup_{i\in S} A_i)$ is well-defined.

The name 'polynomial functor' may come from the following criterium for polynomial functions.

Lemma 3.1.8. *A smooth function $p\colon \mathbb{R} \to \mathbb{R}$ satisfies*

$$\sum_{S\subseteq[n]} (-1)^{\#S} p\left(\sum_{i\in S} x_i\right) = 0 \qquad (3.1.1)$$

for any real number $x_0, x_1, \dots, x_n$ if and only if p is a polynomial of degree $\leq n$. □

[12]For the definition and properties of the homotopy limit, see [BK72, Chapter XI.3.2].

Definition 3.1.9. Let M be a smooth manifold and let $n \in \mathbb{N}$. Define $\mathbf{Open}^n_\partial(M)$ to be the full subcategory of $\mathbf{Open}_\partial(M)$ whose objects are the open subsets W of M that are diffeomorphic to $\mathrm{N}(\partial M) \sqcup (\bigsqcup_{i=1}^k \mathbb{R}^m)$ with $1 \leq k \leq n$. Here $\mathrm{N}(\partial M)$ denotes a (non-fixed) tubular neighbourhood of ∂M in M.

For a manifold M' without boundary, we will simplify the notation as $\mathbf{Open}^n(M)$.

Definition 3.1.10. Let M be a manifold and let F be a good functor, the *n-th Taylor approximation* $\mathrm{T}_n F$ of F is the homotopy right Kan extension

$$\begin{tikzcd}
\mathbf{Open}^n_\partial(M)^{\mathrm{op}} \arrow[r, hook, "i_n"] \arrow[d, "F"'] & \mathbf{Open}_\partial(M)^{\mathrm{op}} \arrow[dl, dashed, "\mathrm{T}_n F"] \arrow[dl, phantom, shift right=4ex, "\Longrightarrow"] \\
\mathbf{CGH} &
\end{tikzcd}$$

of $F|_{\mathbf{Open}^n_\partial(M)^{\mathrm{op}}}$ along the inclusion $i_n\colon \mathbf{Open}^n_\partial(M)^{\mathrm{op}} \hookrightarrow \mathbf{Open}_\partial(M)^{\mathrm{op}}$, together with the natural transformation $\eta_n\colon F \to \mathrm{T}_n F$ coming from the universal property of the homotopy right Kan extension. Written as a homotopy limit, the functor $\mathrm{T}_n F$ is

$$\mathrm{T}_n F(V) \coloneqq \operatorname*{holim}_{\substack{W \subseteq V \\ W \in \mathbf{Open}^n_\partial(M)}} F(W).$$

Example 3.1.11 ([Wei99, Section 0])**.** Let M, N be smooth manifolds (without boundary). The first Taylor approximation $\mathrm{T}_1 \operatorname{Emb}(V)$ of the embedding functor $\operatorname{Emb}(-, N)$ is weakly homotopy equivalent to the immersion functor $\operatorname{Imm}(-, N)$, which associates to an open subset $V \subseteq M$ the space of immersions $\operatorname{Imm}(V, N)$.

Proposition 3.1.12 ([Wei99, Theorem 3.9, Theorem 6.1])**.** *Using the notation from Definition 3.1.10, the pair $(\mathrm{T}_n F, \eta_n)$ has the following properties:*

i) *The functor $\mathrm{T}_n F$ is a polynomial functor of degree at most n,*
ii) *For any $V \in \mathbf{Open}^n_\partial(M)$, $\eta_n(V)$ is a weak homotopy equivalence.*
iii) *If F is a polynomial functor of degree at most n, then η_n is a weak equivalence, i.e. $\eta_n(V)$ is a weak homotopy equivalence for every $V \in \mathbf{Open}_\partial(M)$.*
iv) *If $\mu\colon F \to G$ is a natural transformation where G is a polynomial functor of degree at most n, then (up to weak equivalence) the natural transformation μ factors through $\mathrm{T}_n F$.*

Remark 3.1.13. In other words, the natural transformation $\eta_n\colon F \to \mathrm{T}_n F$ is the best approximation of F by a polynomial functors of degree at most n and $\mathrm{T}_n F$ is unique up to weak homotopy equivalence.

Definition 3.1.14. Let F be a good functor. The *Taylor tower* of F is the sequence of natural transformations $r_i\colon \mathrm{T}_i F \to \mathrm{T}_{i-1} F$ with $i \geq 1$, that are induced by the inclusion $\mathbf{Open}^{i-1}_\partial(M) \hookrightarrow \mathbf{Open}^i_\partial(M)$.

$$\begin{tikzcd}
 & & & F \arrow[dlll] \arrow[dll, "\eta_{n+1}"'] \arrow[d, "\eta_n"] \arrow[dr, "\eta_{n-1}"] \arrow[drrr, "\eta_0"] & & & \\
\dots \arrow[r, "r_{n+2}"'] & \mathrm{T}_{n+1} F \arrow[r, "r_{n+1}"'] & & \mathrm{T}_n F \arrow[r, "r_n"'] & \mathrm{T}_{n-1} F \arrow[r, "r_{n-1}"'] & \dots \arrow[r, "r_1"'] & \mathrm{T}_0 F.
\end{tikzcd}$$

Definition 3.1.15. Let M be a smooth manifold and let F be a good functor. The Taylor tower of F *converges* if for every $V \in \mathbf{Open}_\partial(M)$, the map $\eta(V)\colon F(V) \to \operatorname{holim}_{n \neq 0} \mathrm{T}_n F(V)$ is a weak homotopy equivalence.

The Taylor tower of a good functor does not converge in general. However, for some embedding functors, we have the following convergence criterium.

Theorem 3.1.16 ([GKW01, Corollary/Summary 4.2.4])**.** *Let M and N be two smooth manifolds. If $\dim N - \dim M \geq 3$, then the Taylor tower of the embedding functor $\mathrm{Emb}(-, N)$ on M converges.*

The theorem is not applicable to the embedding functor

$$\mathrm{Emb}_\partial(-, \mathbb{R}^2 \times \mathrm{D}^1, c)\colon \mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}} \to \mathbf{CGH}$$
$$V \mapsto \mathrm{Emb}_\partial(V, \mathbb{R}^2 \times \mathrm{D}^1, c)$$

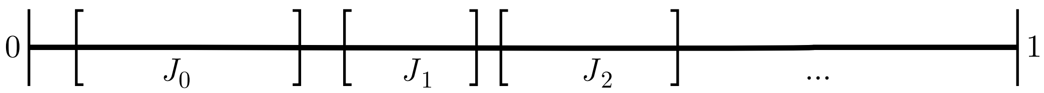
corresponding to the space $\mathcal{K}$. Indeed, the convergence fails, because the set $\pi_0(\mathcal{K})$ is countable, but the homotopy limit of the corresponding Taylor tower can be shown to be uncountable. However, we will see that the Taylor tower of this functor is related closely to Vassiliev invariants. This way, it still gives interesting information about $\mathcal{K}$.

3.2. **Vassiliev invariants via Taylor tower of $\mathcal{K}$.** In this section we will prove that the n-th Taylor approximation of the embedding functor corresponding to $\mathcal{K}$ induces a Vassiliev invariant of degree at most $n-1$, cf. Theorem 3.2.6.

Notation 3.2.1. For the rest of this section, we will only consider the embedding space $\mathcal{K}$ and the corresponding embedding functor $\mathrm{Emb}_\partial(-, \mathbb{R}^2 \times \mathrm{D}^1, c)$, denoted by $\mathrm{Emb}(-)$.

For the proof of Theorem 3.2.6, we need the following a non-functorial description of the spaces $\mathrm{T}_n \mathrm{Emb}(\mathrm{I})$ for $n \geq 1$.

Situation 3.2.2. Let $J_0 = [a_0, b_0], J_1 = [a_1, b_1], \ldots, J_n = [a_n, b_n], \ldots$ be pairwise disjoint closed subintervals of I such that $b_i < a_{i+1}$ for $i \geq 0$ (see the picture below). For $S \subseteq \mathbb{N}$, let $J_S := \cup_{i \in S} J_i$ and let $\mathrm{Emb}_S(\mathrm{I}) := \mathrm{Emb}(\mathrm{I} \setminus J_S)$. For an element $K \in \mathcal{K} = \mathrm{Emb}(\mathrm{I})$, let $K_S := K|_{\mathrm{I} \setminus J_S}$ be the restriction of K to $\mathrm{I} \setminus J_S$.



Definition 3.2.3. Recall the category $\mathbf{Pow}([n])_{\neq\emptyset}$ of subsets of $[n] = \{0, 1, \cdots, n\}$ that are non-empty, cf. Definition 3.1.5.

i) Define the functor $\Delta_{-\subseteq[n]}\colon \mathbf{Pow}([n])_{\neq\emptyset} \to \mathbf{CGH}$ via $S \mapsto \Delta_S$, where Δ_S is the standard $(\#S - 1)$-simplex labelled by S.
ii) Define the functor $\mathrm{Emb}_{-\subseteq[n]}\colon \mathbf{Pow}([n])_{\neq\emptyset} \to \mathbf{CGH}$ via $S \mapsto \mathrm{Emb}_S(\mathrm{I})$.

Proposition 3.2.4. *Let $n \geq 1$.*

i) *The space $\mathrm{T}_n \mathrm{Emb}(\mathrm{I})$ is weakly homotopy equivalent to the space*
$$\operatorname*{holim}_{\substack{\emptyset \neq S \\ S \subseteq [n]}} \mathrm{Emb}_S(\mathrm{I}) := \mathrm{Nat}(\Delta^{-\leq n}, \mathrm{Emb}_{-\subseteq[n]}(\mathrm{I})),$$
where Nat *denotes the space of natural transformations.*
ii) *The natural map $\eta_n(\mathrm{I})\colon \mathrm{Emb}(\mathrm{I}) \to \mathrm{T}_n \mathrm{Emb}(\mathrm{I})$ sends an embedding $f \in \mathrm{Emb}(\mathrm{I})$ to the "constant" natural transformation Const_f, given by*
$$\mathrm{Const}_f\colon \Delta_{-\subseteq[n]} \to \mathrm{Emb}_{-\subseteq[n]}$$
$$\Delta_S \mapsto f_S.$$

Proof. i) Since $\mathrm{T}_n\,\mathrm{Emb}$ is a polynomial functor of degree at most n, the map
$$\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}) \to \operatorname*{holim}_{\emptyset\neq S\subseteq[n]} \mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}\setminus J_S)$$
is a weak homotopy equivalence, by Definition 3.1.6. Since we have $\mathrm{I}\setminus J_S \in \mathbf{Open}^n_\partial(\mathrm{I})$, we know that $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}\setminus J_S) \simeq \mathrm{Emb}(\mathrm{I}\setminus J_S)$, by Proposition 3.1.12.ii). Therefore we see that $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}) \simeq \operatorname{holim}_{\emptyset\neq S\subset[n]} \mathrm{Emb}_S(\mathrm{I})$.

ii) For any $S\subseteq\mathbb{N}$, the composition of maps
$$\mathrm{Emb}(\mathrm{I}) \xrightarrow{\eta_n(\mathrm{I})} \mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}) \xrightarrow{\text{i)}} \operatorname*{holim}_{\substack{\emptyset\neq S\\ S\subset[n]}} \mathrm{Emb}_S(\mathrm{I}) \xrightarrow{\mathrm{pr}} \mathrm{Emb}_S(\mathrm{I})$$
sends an embedding $f\in\mathrm{Emb}(\mathrm{I})$ to the restriction $f|_{\mathrm{I}\setminus J_S} \in \mathrm{Emb}_S(\mathrm{I})$. Thus $\eta_n(\mathrm{I})(f)$ maps every Δ_S to f_S. □

The map $\eta_n(\mathrm{I})$ induces a knot invariant $\eta_n(\mathrm{I})_*\colon \pi_0(\mathcal{K}) \to \pi_0(\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}))$, cf. Proposition 1.1.2. The rest of this section is concerned with the proof that $\eta_n(\mathrm{I})_*$ is an additive Vassiliev invariant of degree at most $n-1$.

Theorem 3.2.5 ([BCKS17, Section 4, Section 5])**.** *Let $n\geq 1$. The set $\pi_0(\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}))$ of path components of $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$ admits an abelian group structure, such that the map $\eta_n(\mathrm{I})_*\colon \pi_0(\mathcal{K}) \to \pi_0(\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}))$ becomes an additive knot invariant.*

Thus in order to prove that $\eta_n(\mathrm{I})_*$ is a Vassiliev invariant of degree at most $n-1$, we only need to prove that $\eta_n(\mathrm{I})_*$ factor through $\pi_0(\mathcal{K}) \to \pi_0(\mathcal{K})/\!\sim_{C_n}$, cf. Theorem 2.2.19.

Theorem 3.2.6 ([BCKS17, Theorem 6.5])**.** *Let K_1 and K_2 be two knots with $K_1 \sim_{C_n} K_2$, then $\eta_n(\mathrm{I})_*([K_1]) = \eta_n(\mathrm{I})_*([K_2])$.*

Now we give a new proof of Theorem 3.2.6 using the description of $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$ from Proposition 3.2.4, clasper surgery and gropes (Lemma 2.3.19). Compared with the original proof in [BCKS17], our proof is more gemeotric and does not need the construction of a further model for $\mathrm{T}_n\,\mathrm{Emb}(-)$.

Recall from Definition 2.2.15 that $[K_1] \sim_{C_n} [K_2]$ if K_1 and K_2 are related by a finite sequence of (simple tree) clasper surgeries of degree n (and isotopies). Thus it is sufficient to prove that for any knot $K\in\mathcal{K}$ and any clasper C of degree n for K, there is a path between $\eta_n(K)$ and $\eta_n(K^C)$ in $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$.

Recall from Situation 3.2.2 that $J_0, J_1, \dots$ are defined as pairwise disjoint closed subintervals of I°. We assume that $(J_i)_{i\geq 0}$ is chosen such that for a given knot K and clasper C, each $J_0, J_1, \dots, J_n$ contains exactly one preimage of an intersection point of K with a leaf of the clasper C.

In [CT04b], Conant and Teichner discovered how to relate a capped grope of degree n with a clasper surgery of degree n, which plays an important role in the proof.

Theorem 3.2.7 ([CT04a])**.** *Let $n\geq 2$. There exists a knot $\widetilde{K}\in\mathcal{K}$ which is isotopic to K^C, and such that*

i) *the images of K and $\widetilde{K}$ are equal outside J_0. Furthermore, $K|_{J_0}$ and $\widetilde{K}|_{J_0}$ intersect only at their endpoints, denoted by A and B;*

ii) *the knot*[13] *$K(J_0)\cup_{A,B}\widetilde{K}(J_0)$ is an unknot, and bounds a capped grope $(G^c_n, \{D_i\}^n_{i=1})$, such that for $1\leq i\leq n$, the cap D_i intersects $K(J_i) = \widetilde{K}(J_i)$ transversally and exactly once;*

iii) *the intersection of G^c_n with $\mathrm{im}(K)\cup\mathrm{im}(\widetilde{K})$ is*
$$\left(K(J_0)\cup_{A,B}\widetilde{K}(J_0)\right) \cup \bigcup_{1\leq i\leq n} \left(D_i\cap K(J_i)\right);$$

[13] We abuse the word "knot" here for the image of a knot

iv) *the associated rooted unitrivalent tree Γ_{G_n} of G_n^c is isomorphic to the associated unitrivalent tree of the clasper Γ_C. The root of Γ_{G_n} is the leaf corresponding to the leaf of Γ_C intersecting $K(J_0)$.*

In the case where $n = 2$, see Figure 18 for an illustration of Theorem 3.2.7.

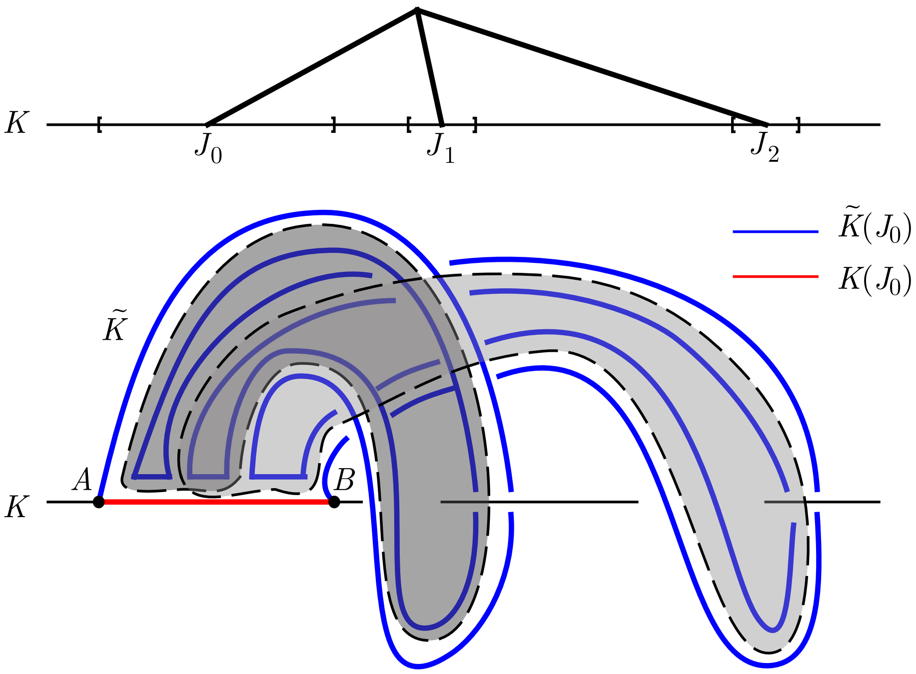


FIGURE 18. A clasper surgery of degree 2 on a knot K. The curves $K(J_0)$ and $\widetilde{K}(J_0)$ cobound a capped grope of degree 2. The discs in grey are the caps D_1 and D_2.

Therefore it is sufficient for proving Theorem 3.2.6 to show that there is a path in $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$ connecting $\eta_n(\mathrm{I})(K)$ and $\eta_n(\mathrm{I})(\widetilde{K})$.

Remark 3.2.8. Before we prove the theorem, let us see how a path in $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$ looks like. A path in $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$ is a continuous map $P\colon \mathrm{I} \to \mathrm{T}_n\,\mathrm{Emb}(\mathrm{I}) = \mathrm{Nat}(\Delta_{-\subseteq[n]}, \mathrm{Emb}_{-\subseteq[n]})$ by definition. By [Fox45, Theorem 1], we can rewrite P as a collection of continuous maps $P_S\colon \Delta_S \times \mathrm{I} \to \mathrm{Emb}_S(\mathrm{I})$ for $\emptyset \neq S \subseteq [n]$, such that for $\emptyset \neq S' \subseteq S \subseteq [n]$, the following diagram commutes:

$$\begin{tikzcd}
\Delta_S \times \mathrm{I} \arrow[r, "P_S"] & \mathrm{Emb}_S(\mathrm{I}) \\
\Delta_{S'} \times \mathrm{I} \arrow[u, "S'\subseteq S"] \arrow[r, "P_{S'}"'] & \mathrm{Emb}_{S'}(\mathrm{I}). \arrow[u, "S'\subset S"']
\end{tikzcd}$$

Thus in the proof of Theorem 3.2.6, we want to find such a collection of maps

$$H_n = (H_S\colon \Delta_S \times \mathrm{I} \to \mathrm{Emb}_S(\mathrm{I}))_{\emptyset\neq S\subseteq[n]}$$

such that for every $\emptyset \neq S \subseteq [n]$ and every $v \in \Delta_S$, we have that $H_S|_{\{v\}\times\mathrm{I}}$ is a path between K_S and $\widetilde{K}_S$ in $\mathrm{Emb}_S(\mathrm{I})$.

Proof of Theorem 3.2.6. We prove the cases $n = 1$ and $n \geq 2$ independently. For $n = 1$, the clasper C of degree 1 consists of two leaves connected by an edge. By Proposition 2.2.14, we can find a knot $\widetilde{K} \in \mathcal{K}$ which is equivalent to K^C, such that $\widetilde{K}$ and K are equal outside J_0 (See Figure 19 for a visualisation.). Note that $K(J_0) \cup_{A,B} \widetilde{K}(J_0)$ bounds an embedded disk D in $\mathbb{R}^2 \times \mathrm{D}^1$, which intersects $K(J_1)$ transversally once and is disjoint from $(K(\mathrm{I}) \cup \widetilde{K}(\mathrm{I})) \setminus (K(J_0) \cup \widetilde{K}(J_0) \cup K(J_1))$. Thus we can define an isotopy $\widehat{H} \colon J_0 \times \mathrm{I} \to D$ between $K|_{J_0}$ and $\widetilde{K}|_{J_0}$, relative endpoints.

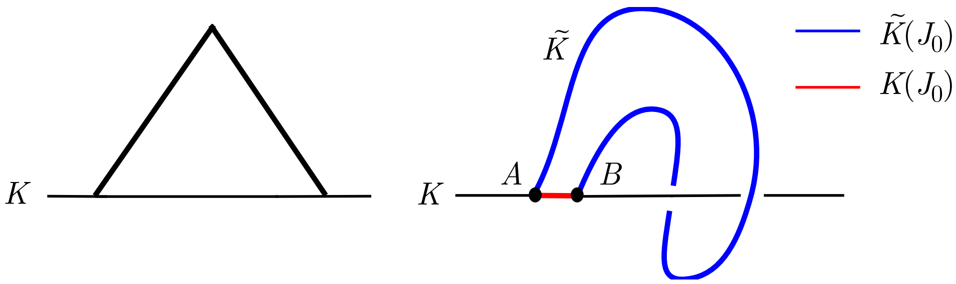


FIGURE 19. Clasper surgery of along C.

Thus we define a path H_1 between $\eta_1(\mathrm{I})(K)$ and $\eta_1(\mathrm{I})(\widetilde{K})$ via the following natural transformation, cf. Remark 3.2.8.

$$\begin{aligned}
H_1 \colon \Delta_{-\subseteq[1]} \times \mathrm{I} &\to \mathrm{Emb}_{-\subseteq[1]}(\mathrm{I}) \\
H_1|_{\{0,1\}} \colon \quad t &\mapsto K_{\{0,1\}} \\
H_1|_{\{0\}} \colon \quad t &\mapsto K_{\{0\}} \\
H_1|_{\{1\}} \colon \quad (\mathrm{pt}, t) &\mapsto i_{\widehat{H}}(-, t)
\end{aligned}$$

where $i_{\widehat{H}}$ is defined as

$$\begin{aligned}
i_{\widehat{H}} \colon (\mathrm{I} \setminus J_1) \times \mathrm{I} &\to \mathbb{R}^2 \times \mathrm{D}^1 \\
(p, t) &\mapsto \begin{cases} K(p) & \text{if } p \notin J_0 \\ \widehat{H}(p, t) & \text{otherwise} \end{cases}
\end{aligned}$$

One can easily check that the $H_1 \in \mathrm{Nat}(\Delta_{-\subseteq[1]}, \mathrm{Emb}_{-\subseteq[1]})$.

Now we prove the case $n \geq 2$. So $K(J_0) \cup_{A,B} \widetilde{K}(J_0)$ bounds a capped grope $(G_n^c, \{D_i\}_{i=1}^n)$ of degree n, as in Theorem 3.2.7. We apply Lemma 2.3.19 to this capped grope G_n^c and thus obtain a Δ^{n-1}-family of embedded disks

$$h_n \colon \mathrm{D}^2 \times \Delta^{n-1} \to \mathbb{R}^2 \times \mathrm{D}^1,$$

satisfying the following conditions.

i) The map $h_n(- \times \{v\})$ is an embedding, for every $v \in \Delta^{n-1}$.

ii) We have $h_n(\mathrm{S}^1 \times \{v\}) = K(J_0) \cup_{A,B} \widetilde{K}(J_0)$ for every $v \in \Delta^{n-1}$. Furthermore for $v, w \in \Delta^{n-1}$ and $p \in \mathrm{S}^1$, we have $h_n((p, v)) = h_n((p, w))$.

iii) Let $v = (t_1, t_2, \ldots, t_n) \in \Delta^{n-1}$ and set $S_v := \{1 \leq i \leq n \mid t_i \neq 0\}$. We have

$$h_n(\mathrm{D}^2 \times \{v\}) \subset \mathrm{N}(G_{S_v}) \cup \bigcup_{s \in S_v} \mathrm{N}(D_s).$$

where G_{S_v} is the subgrope of G_n corresponding to the subtree Γ_{S_v} of Γ_{G_n}, cf. Notation 2.3.9.

Since we have by construction

$$G_n \cap \left(K(\mathrm{I}) \cup \widetilde{K}(\mathrm{I})\right) = \left(K(J_0) \cup_{A,B} \widetilde{K}(J_0)\right) \cup \bigcup_{1 \leq i \leq n} \left(D_i \cap K(J_i)\right),$$

we can refine[14] $\mathrm{N}(G_S)$ and $\mathrm{N}(C_i)$ so that for every $S \subseteq \{1, 2, \ldots, n\}$ and $1 \leq i \leq n$ we see

$$\mathrm{N}(G_S) \cap (K(\mathrm{I}) \cup \widetilde{\gamma}(\mathrm{I})) = K(J_0) \cup \widetilde{K}(J_0),$$
$$\mathrm{N}(C_i) \cap (K(\mathrm{I}) \cup \widetilde{\gamma}(\mathrm{I})) \subset K(J_i).$$

This way, $h_n(\mathrm{D}^2 \times \{v\})$ is an embedded disc in D^3, bounded by $K(J_0) \cup_{A,B} \widetilde{K}(J_0)$. In particular, $h_n(\mathrm{D}^2 \times \{v\}) \cap \Big(K(\mathrm{I}) \cup \widetilde{K}(\mathrm{I})\Big) \subset \Big(K(J_0) \cup_{A,B} \widetilde{K}(J_0)\Big) \cup \bigcup_{i \in S_v} \gamma(J_i)$. Each $h_n(\mathrm{D}^2 \times \{v\})$ (essentially) is an isotopy relative endpoints between $K(J_0)$ and $\widetilde{K}(J_0)$. However, to define H_n, a path between $\eta_n(\mathrm{I})(K)$ and $\eta_n(\mathrm{I})(\widetilde{K})$, we need the isotopies to vary continuously as v varies in Δ^{n-1}.

For $e = (1, 0, \ldots, 0) \in \Delta^{n-1}$, we fix an isotopy $H_e \colon J_0 \times \mathrm{I} \to h_n(\mathrm{D}^2 \times \{e\})$ relative endpoints between $K(J_0)$ and $\widetilde{K}(J_0)$. For any other $v \in \Delta^{n-1}$, we can define an isotopy H_v by mapping $h_n(\mathrm{D}^2 \times \{v\})$ homeomorphically to $h_n(\mathrm{D}^2 \times \{e\})$ via h_n and pull back the isotopy H_e. More concretely, the following diagram commutes.

$$\begin{tikzcd}
J_0 \times \mathrm{I} \arrow[r, "H_v"] \arrow[d, "H_e"'] & h_n(\mathrm{D}^2 \times \{v\}) \\
h_n(\mathrm{D}^2 \times \{e\}) \arrow[ur, "h_n(-,v) \circ h_n^{-1}(-,e)"'] &
\end{tikzcd}$$

Therefore we can define H_n as the natural transformation

$$\begin{aligned}
H_n \colon \Delta_{-\subseteq[n]} \times \mathrm{I} &\to \mathrm{Emb}_{-\subseteq[n]}(\mathrm{I}) \\
H_n|_{0 \in S} \colon \quad (v, t) &\to K_S \\
H_n|_{0 \notin S} \colon \quad (v, t) &\to i_{H_v}(-, t),
\end{aligned}$$

where $i_{H_v} \colon \ \big(\mathrm{I} \setminus \bigcup_{i \in S_v} J_i\big) \times \mathrm{I} \to \mathbb{R}^2 \times \mathrm{D}^1 \subseteq \mathbb{R}^3$ is defined by

$$(p, t) \mapsto \begin{cases} K(p) & \text{if } p \notin J_0 \\ H_v(p, t) & \text{otherwise} \end{cases}$$

□

4. Homotopy spectral sequence for space of long knots

Goodwillie observed that the collection of good functors on $\mathbf{Open}_\partial(\mathrm{I})$ is in one to one correspondence with cosimplicial objects in the category of compactly generated weakly Hausdorff spaces, cf. [GKW01, Section 5]. We will give details for this correspondence in Section 4.1. To the cosimplicial space $\mathfrak{Emb}_\bullet$ (Construction 4.1.11) corresponding to the functor $\mathrm{Emb}(-)$, we can associate the Bousfield-Kan homotopy spectral sequence $\{E_{p,q}\} q \geq p \geq 0$ with integral coefficients. In Section 4.2, we will briefly introduce the construction of this spectral sequence, show concrete computation of the d^1-differentials that maps in to the diagonal and give a combinatorial interpretation to these differentials. From the combinatorial interpretation we will see that this spectral sequence relates closely to the theory of Vassiliev invariants.

4.1. A cosimplicial model for the Taylor tower of $\mathrm{Emb}(-)$.

In this section we will give details for the correspondence (Construction 4.1.20) between good functors on $\mathbf{Open}_\partial(\mathrm{I})$ and cosimplicial objects in $\mathbf{CGH}$. This correspondence facilitates further computations in Section 4.2.3.

The part of this section after Construction 4.1.11 is a digression, and not necessary for our discussion in Section 4.2.

[14]For example by scaling the length of the normal vectors, cf. Notation 2.3.9

Definition 4.1.1. We define the following two categories.

i) The *simplicial index category* $\mathbf{\Delta}$ consists of the objects $[n] = \{0, 1, \dots, n\} \subseteq \mathbb{N}$ with $n \geq 0$, and the morphisms are the order-preserving maps.
ii) Denote by $\mathbf{\Delta}_+$ the category of finite, totally ordered sets. The morphisms are order-preserving maps.

Remark 4.1.2. The simplicial index category $\mathbf{\Delta}$ is equivalent to the category of non-empty finite totally ordered sets, which we also denote, by abuse of notation, by $\mathbf{\Delta}$

Definition 4.1.3.

i) A *cosimplicial space* is a functor $X_\bullet \colon \mathbf{\Delta} \to \mathbf{CGH}$.
ii) An *augmented cosimplicial space* $Y_\bullet$ is a functor $Y_\bullet \colon \mathbf{\Delta}_+ \to \mathbf{CGH}$.

Notation 4.1.4. Let $X_\bullet$ be a cosimplicial space. Denote by X_k the value $X_\bullet([k])$, for $k \geq 0$.

Notation 4.1.5. By restricting an augmented cosimplicial space $Y_\bullet$ to the subcategory $\mathbf{\Delta} \subseteq \mathbf{\Delta}_+$, we obtain the *associated cosimplicial space*, which we also write, by abuse of notation, as $Y_\bullet$.

Convention 4.1.6. By *totalisation* $\operatorname{Tot} X_\bullet$ of a cosimplicial space $X_\bullet$ we always mean the totalisation $\operatorname{Tot} \widetilde{X_\bullet}$ of a fibrant replacement $\widetilde{X_\bullet}$ of $X_\bullet$, for the model structure, see [BK72, Section X.4.6]. Similarly by *n-th partial totalisation* $\operatorname{Tot}^n X_\bullet$ of $X_\bullet$ we always mean $\operatorname{Tot}^n \widetilde{X_\bullet}$.

Remark 4.1.7 ([BK72, Chapter XI.4.4]). We have

$$\operatorname{Tot} X_\bullet \simeq \operatorname{holim} \widetilde{X_\bullet}$$
$$\operatorname{Tot}^n X_\bullet \simeq \operatorname*{holim}_{k \leq n} \widetilde{X_k}$$

Definition 4.1.8. Let $\mathbf{Open}_\partial(\mathrm{I})_{\mathrm{fin}}$ be the full subcategory of $\mathbf{Open}_\partial(\mathrm{I})$ whose objects are the open subsets of I that contain $\partial\mathrm{I}$ and have only finitely many path connected components, i.e.

$$\operatorname{Ob}\left(\mathbf{Open}_\partial(\mathrm{I})_{\mathrm{fin}}\right) \coloneqq \{\mathrm{I}\} \cup \bigcup_{n \geq 0} \operatorname{Ob}\left(\mathbf{Open}^n_\partial(\mathrm{I})\right)$$

Proposition 4.1.9 ([GKW01, Section 5])**.** *A good functor on* $\mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}$ *is determined up to weak homotopy equivalence by its restriction on* $\mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}}$.

Proof. This follows directly from the definition of good functor, cf. Definition 3.1.3.ii). □

Proposition 4.1.10. *The restriction of a good functor on* $\mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}$ *to* $\mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}}$ *is isotopy invariant in the sense of Definition 3.1.3.* □

Construction 4.1.11 ([GKW01, Section 5])**.** Let $\kappa \colon \mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}} \to \mathbf{\Delta}_+$ be the functor $V \mapsto \pi_0(\mathrm{I} \setminus V)$. Denote by $\mathfrak{E}mb_\bullet$ the homotopy right Kan extension of the functor $\operatorname{Emb}(-)$ along κ.

$$\begin{tikzcd}
\mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}} \arrow[r, "\kappa"] \arrow[d, "\operatorname{Emb}(-)"'] & \mathbf{\Delta}_+ \arrow[dl, dashed, "\mathfrak{E}mb_\bullet"] \\
\mathbf{CGH} \arrow[ur, phantom, "\Longrightarrow"] &
\end{tikzcd}$$

Thus $\mathfrak{E}mb_\bullet$ is an augmented cosimplicial space. Furthermore its associated cosimplicial space has the following properties:

i) For $V \in \mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}}$, we have that $\mathfrak{E}mb_{\pi_0(\mathrm{I}\setminus V)} \simeq \operatorname{Emb}(V)$.

ii) We have $\mathrm{Tot}^n \mathfrak{E}mb_\bullet \simeq \mathrm{T}_n \mathrm{Emb}(\mathrm{I})$ and $\mathrm{Tot}\, \mathfrak{E}mb_\bullet \simeq \mathrm{holim}_{n\geq 0} \mathrm{T}_n \mathrm{Emb}(\mathrm{I})$.

To look into the technical details of the construction, let us begin with introducing several categories.

Definition 4.1.12.

i) Define the category $\mathbf{Man}_{\mathrm{m}}$ of smooth oriented m-dimensional manifold. Objects of $\mathbf{Man}_{\mathrm{m}}$ are the smooth oriented manifolds of dimension m, and the morphisms are the orientation-preserving smooth embeddings.
ii) Define the topological category $\mathcal{M}\mathrm{an}_{\mathrm{m/M}}$ of smooth oriented m-dimensional manifold. This category has the same objects as $\mathbf{Man}_{\mathrm{m}}$, and the morphisms are spaces of orientation-preserving smooth embeddings.
iii) Define the full topological subcategory $\mathbf{Disc}_{\mathrm{m}}$ of $\mathbf{Man}_{\mathrm{m}}$ whose objects are finite disjoint unions of $\mathbb{R}^m$ and $\mathbb{R}_{\geq 0} \times \mathbb{R}^{m-1}$.
iv) Define the full subcategory $\mathcal{D}\mathrm{isc}_{\mathrm{m}}$ of $\mathcal{M}\mathrm{an}_{\mathrm{m}}$ that has the same object as $\mathbf{Disc}_{\mathrm{m}}$.
v) Let M be a smooth oriented manifold of dimension m. Define the category

$$\mathbf{Disc}_{\mathrm{m/M}} := \mathbf{Disc}_{\mathrm{m}} \underset{\mathbf{Man}_{\mathrm{m}}}{\times} \mathbf{Man}_{\mathrm{m/M}},$$

where $\mathbf{Man}_{\mathrm{m/M}}$ is the over category over M. Objects of $\mathbf{Disc}_{\mathrm{m/M}}$ are embeddings of finite disjoint unions of $\mathbb{R}^m$ and $\mathbb{R}_{\geq 0} \times \mathbb{R}^{m-1}$ into M.
vi) Let M be a smooth oriented manifold of dimension m. Define the topological category

$$\mathcal{D}\mathrm{isc}_{\mathrm{m/M}} := \mathcal{D}\mathrm{isc}_{\mathrm{m}} \underset{\mathcal{M}\mathrm{an}_{\mathrm{m}}}{\times} \mathcal{M}\mathrm{an}_{\mathrm{m/M}},$$

where $\mathcal{M}\mathrm{an}_{\mathrm{m/M}}$ is the over category over M.
vii) Let M be a smooth oriented manifold of dimension m. Define the subcategory $\mathbf{Isot}_{\mathrm{m/M}}$ of $\mathbf{Disc}_{\mathrm{m/M}}$ which has the same objects as $\mathbf{Disc}_{\mathrm{m/M}}$, but keeps only the morphisms that are isotopy equivalences.

Remark 4.1.13. Note that we can always view ordinary category as topological category by considering the morphism set as discrete morphism spaces.

Proposition 4.1.14 ([AF15, Proposition 2.19])**.** *The canonical functor*

$$\mathbf{Disc}_{\mathrm{m/M}} \hookrightarrow \mathcal{D}\mathrm{isc}_{\mathrm{m/M}}$$

induces an equivalence of topological categories

$$\mathbf{Disc}_{\mathrm{m/M}}[\mathbf{Isot}_{\mathrm{m/M}}^{-1}] \simeq \mathcal{D}\mathrm{isc}_{\mathrm{m/M}},$$

where $\mathbf{Disc}_{\mathrm{m/M}}[\mathbf{Isot}_{\mathrm{m/M}}^{-1}]$ *is the (Dwyer-Kan) localisation, cf. [DK80], of* $\mathbf{Disc}_{\mathrm{m/M}}$ *at* $\mathbf{Isot}_{\mathrm{m/M}}$.

Remark 4.1.15. In [AF15], the authors used the language of ∞-categories (quasi-categories). For a translation between topological categories, simplicial categories and ∞-categories, see [Lur09, Section 1.1.3, 1.1.4 and 1.1.5].

For our application, we consider $m = 1$ and $M = \mathrm{I}$.

Definition 4.1.16. Define the subcategory $\mathbf{Disc}_{1/\mathrm{I}}^{\partial}$ of $\mathbf{Disc}_{1/\mathrm{I}}^{\partial}$ whose objects are the embeddings such that the boundary $\partial\mathrm{I}$ of I is in the image.

In the same way, let $\mathbf{Isot}_{1/\mathrm{I}}^{\partial}$ be the subcategory of $\mathbf{Isot}_{1/\mathrm{I}}$ whose objects are the embeddings such that $\partial\mathrm{I}$ is in their images.

Explicitly, objects of $\mathbf{Disc}_{1/\mathrm{I}}^{\partial}$ are embeddings of the form

$$[0, \frac{1}{4}) \sqcup \bigsqcup_{k=1}^{n} \mathbb{R} \sqcup (\frac{3}{4}, 1] \hookrightarrow \mathrm{I}, \quad n \in \mathbb{N}.$$

Using the same proof strategy as Proposition 4.1.14, we have

Corollary 4.1.17. *The canonical functor* $\mathbf{Disc}^{\partial}_{1/\mathrm{I}} \to \mathcal{D}\mathrm{isc}^{\partial}_{1/\mathrm{I}}$ *induces an equivalence of topological categories*

$$\mathbf{Disc}^{\partial}_{1/\mathrm{I}}\left[\left(\mathbf{Isot}^{\partial}_{1/\mathrm{I}}\right)^{-1}\right] \simeq \mathcal{D}\mathrm{isc}^{\partial}_{1/\mathrm{I}}. \qquad \square$$

Proposition 4.1.18.

i) *([AF15, Lemma 3.11]) The functor*

$$S\colon \left(\mathcal{D}\mathrm{isc}^{\partial}_{1/\mathrm{I}}\right)^{\mathrm{op}} \to \mathbf{\Delta}_+$$
$$(i\colon V \hookrightarrow \mathrm{I}) \mapsto \pi_0(\mathrm{I} \setminus i(V))$$

is an equivalence of topological categories.

ii) *The functor* $Im\colon \mathbf{Disc}^{\partial}_{1/\mathrm{I}} \to \mathbf{Open}_{\partial}(\mathrm{I})_{\mathrm{fin}}$, $(i\colon V \hookrightarrow \mathrm{I}) \mapsto \mathrm{im}(i)$ *is an equivalence of ordinary categories. Furthermore* $Im|_{\mathbf{Isot}^{\partial}_{1/\mathrm{I}}}\colon \mathbf{Iso}^{\partial}_{1/\mathrm{I}} \to \mathbf{Isot}(\mathrm{I})$ *is an equivalence, where* $\mathbf{Isot}(\mathrm{I})$ *is the subcategory of* $\mathbf{Open}_{\partial}(\mathrm{I})_{\mathrm{fin}}$ *which has the same objects as* $\mathbf{Open}_{\partial}(\mathrm{I})_{\mathrm{fin}}$, *but keeps only the morphisms that are isotopy equivalences.*

Proof. i) See [AF15, Lemma 3.11].

ii) First Im is essentially surjective, because the boundary of I is in the image of i for any $i \in \mathbf{Isot}^{\partial}_{1/\mathrm{I}}$. For any two objects $i_1\colon V_1 \hookrightarrow \mathrm{I}$ and $i_2\colon V_2 \hookrightarrow \mathrm{I}$ in $\mathbf{Disc}^{\partial}_{1/\mathrm{I}}$, the morphism set $\mathrm{Mor}_{\mathbf{Disc}^{\partial}_{1/\mathrm{I}}}(i_1, i_2)$ is either empty or a one element set. Also the morphism set $\mathrm{Mor}_{\mathbf{Open}_{\partial}(\mathrm{I})}(\mathrm{im}(i_1), \mathrm{im}(i_2))$ is either empty or has one element. Thus Im is fully faithful. Similarly it follows that $Im|_{\mathbf{Isot}^{\partial}_{1/\mathrm{I}}}$ is an equivalence. $\square$

Corollary 4.1.19. *We have the following equivalences of topological categories*

$$\mathbf{Open}_{\partial}(\mathrm{I})_{\mathrm{fin}}[\mathbf{Iso}(\mathrm{I})^{-1}] \simeq \mathbf{Disc}^{\partial}_{1/\mathrm{I}}\left[\left(\mathbf{Isot}^{\partial}_{1/\mathrm{I}}\right)^{-1}\right] \simeq \mathcal{D}\mathrm{isc}^{\partial}_{1/\mathrm{I}} \overset{S}{\simeq} \mathbf{\Delta}_+^{\mathrm{op}}. \qquad \square$$

Construction 4.1.20. Let $F\colon \mathbf{Open}_{\partial}(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}} \to \mathbf{CGH}$ be an isotopy invariant functor. By the universal property of localisation, F factors through $\mathbf{Open}_{\partial}(\mathrm{I})_{\mathrm{fin}}[\mathbf{Iso}(\mathrm{I})^{-1}]$, say the factorisation is called F_{loc}. From the chain of equivalences in Corollary 4.1.19, we get the following diagram

$$\begin{tikzcd}
\mathbf{Open}_{\partial}(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}} \arrow[r, "L"] \arrow[d, "F"'] & \mathbf{Open}_{\partial}(\mathrm{I})_{\mathrm{fin}}[\mathbf{Iso}(\mathrm{I})^{-1}]^{\mathrm{op}} \arrow[r, "\simeq"] \arrow[dl, "F_{\mathrm{loc}}"] & (\mathcal{D}\mathrm{isc}^{\partial}_{1/\mathrm{I}})^{\mathrm{op}} \arrow[r, "S"] \arrow[dll, "\widetilde{F}"] & \mathbf{\Delta}_+ \arrow[dlll, dashed, "\mathfrak{F}_\bullet"] \\
\mathbf{CGH} & & &
\end{tikzcd},$$

where L is the localisation functor, and $\widetilde{F}$ and $\mathfrak{F}_\bullet$ are induced by the equivalences of categories. Note that $\mathfrak{F}_\bullet$ is an augmented cosimplicial space.

Denote by $\kappa\colon \mathbf{Open}_{\partial}(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}} \to \mathbf{\Delta}_+$ the composition of the horizontal functors in the diagram above. Using Proposition 4.1.18, we see that $\kappa(V) = \pi_0(\mathrm{I} \setminus V)$. Thus we get

$$\mathfrak{F}_{\pi_0(\mathrm{I}\setminus V)} \simeq F(V),$$

for any $V \in \mathbf{Open}_{\partial}(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}}$.

Conversely, given an augmented cosimplicial space $Y_\bullet$, we obtain an isotopy invariant functor by precomposition with κ.

Proposition 4.1.21 ([GKW01, Section 5])**.** *The n-th partial totalisation* $\mathrm{Tot}^n \mathfrak{F}_\bullet$ *of is weakly homotopy equivalent to* $\mathrm{T}_n F(\mathrm{I})$ *for* $n \geq 0$, *and thus* $\mathrm{Tot}\,\mathfrak{F}_\bullet$ *is weakly homotopy equivalent to* $\mathrm{holim}_{n\geq 0}\,\mathrm{T}_n F(\mathrm{I})$.

Proof. We have $\mathrm{Tot}^n \mathfrak{F}_\bullet = \mathrm{holim}_{\mathfrak{F}_\bullet|_{\Delta_{\leq n}}} \mathfrak{F}_\bullet \simeq \mathrm{holim}_{L(\mathbf{Open}^n_\partial(\mathrm{I}))} F_{\mathrm{loc}} \simeq \mathrm{holim}_{\mathbf{Open}^n_\partial(\mathrm{I})} F$. The first equality is by definition. The second weak homotopy equivalence comes from the equivalences of categories from Corollary 4.1.19, and one can validate the third weak homotopy equivalence by checking that L is homotopy initial. □

Corollary 4.1.22. *Apply Construction 4.1.20 and Proposition 4.1.21 to the embedding functor* $\mathrm{Emb}(-)$ *from Notation 3.2.1, we obtain a (augmented) cosimplicial space* $\mathfrak{Emb}_\bullet$ *satisfying Construction 4.1.11.i)–ii).*

4.2. **A integral homotopy spectral sequence for $\mathfrak{Emb}_\bullet$.** To the cosimplicial space $\mathfrak{Emb}_\bullet$, we can associate the Bousfield–Kan homotopy spectral sequence $\{E_{p,q}\}_{q\geq p\geq 0}$ with integral coefficients, cf. [BK72, Chapter X]. The diagonal and anti-diagonal of the first page, as well as the differential $d^1\colon E^1_{p-1,p} \to E^1_{p,p}$, have combinatorial interpretations in terms of unitrivalent graphs. The unitrivalent graphs we will encounter are closely related to Vassiliev invariants [Bar95; Con08; CT04b].

4.2.1. *A spectral sequence for cosimplicial spaces.* We begin by introducing techniques that we need for the computation of Bousfield–Kan spectral sequences.

Notation 4.2.1. Let $X_\bullet\colon \mathbf{\Delta} \to \mathbf{CGH}$ be a cosimplicial space. For $0 \leq i \leq n$, denote by $\delta^i\colon X^{[n-1]} \to X^{[n]}$ its coface maps and $s^i\colon X^{[n+1]} \to X^{[n]}$ its codegeneracy maps.

Given a cosimplicial space $X_\bullet\colon \mathbf{\Delta} \to \mathbf{CGH}$, there is a tower of fibrations (cf. [BK72, Chapter 6, Section 6.1])

$$\cdots \to \mathrm{Tot}^{n+1} X_\bullet \to \mathrm{Tot}^n X_\bullet \to \cdots \to \mathrm{Tot}^1 X_\bullet \to \mathrm{Tot}^0 X_\bullet. \tag{4.2.1}$$

Denote by $L^{n+1}X_\bullet$ the homotopy fibre of $\mathrm{Tot}^{n+1} X_\bullet \to \mathrm{Tot}^n X_\bullet$ and $L^0 X_\bullet \coloneqq \mathrm{Tot}^0 X_\bullet$.

Thus by applying Bousfield–Kan homotopy spectral sequence, [BK72, Section X.6], to the tower of fibrations (4.2.1), we obtain a spectral sequence calculating the homotopy groups of $\mathrm{Tot}\, X_\bullet$, whose first page is given by

$$E^1_{p,q} = \pi_{q-p}(L^p X_\bullet),$$

where $q \geq p \geq 0$.

With the help of the cosimplicial structure, we can calculate $\pi_\bullet(L^p X_\bullet)$ and the differential d^1 in terms of $\pi_\bullet(X^{[p]})$.

Proposition 4.2.2 ([BK72, Section X.6.2])**.** *Given a cosimplicial space $X_\bullet\colon \mathbf{\Delta} \to \mathbf{CGH}$, we have*

$$\pi_{q-p}(L^p X_\bullet) \cong \pi_q\left(X^{[p]} \cap \bigcap_{i=0}^{p-1} \ker(s^i)\right) \cong \pi_q(X^{[p]}) \cap \bigcap_{i=0}^{p-1} \ker(s^i_*),$$

where the push-forward $s^i_*\colon \pi_q(X^{[p]}) \to \pi_q(X^{[p-1]})$ is induced by the codegeneracy maps s^i on $X^{[p]}$.

Proposition 4.2.3 ([BK72, Chapter X, 7])**.** *Given a cosimplicial space $X_\bullet$, we obtain the Bousfield–Kan homotopy spectral sequence whose first page is given by*

$$E^1_{p,q} \cong \pi_q(X^{[p]}) \cap \bigcap_{i=0}^{p-1} \ker(s^i_*),$$

where $q \geq p \geq 0$, and the push-forward $s^i_*\colon \pi_q(X^{[p]}) \to \pi_q(X^{[p-1]})$ is induced by the codegeneracy maps s^i on $X^{[p]}$. The differential $d^1\colon E^1_{p,q} \to E^1_{p+1,q}$ on the first page is given by

$$x \mapsto \sum_{i=0}^{p+1} (-1)^i \delta^i_*(x),$$

where the push-forward $\delta^i_*\colon \pi_q(X^{[p]}) \to \pi_q(X^{[p+1]})$ is induced by the coface maps δ^i on $X^{[p]}$.

4.2.2. *Homotopy groups of* $\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)$. From Section 4.2.1 we see that we need to compute the homotopy groups of $\mathfrak{E}mb_n$ for $n \geq 0$, in order to do some computation of the Bousfield-Kan spectral sequence for the cosimplicial space $\mathfrak{E}mb_\bullet$. By Construction 4.1.11 we know that $\mathfrak{E}mb_n$ relates closely to configuration spaces of $\mathbb{R}^2 \times \mathrm{D}^1$. Therefore let us gather some information about the homotopy groups of the configurations space in this section.

Definition 4.2.4. Let M be a smooth manifold (possibly with boundary). Define the *configuration space* $\mathrm{Conf}_n(M)$ of $n \geq 1$ points on M as

$$\mathrm{Conf}_n(M) \coloneqq \{(x_1, \dots, x_n) \in (M \setminus \partial M)^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Now we focus on $\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)$ for $n \geq 0$.

Convention 4.2.5. We define[15] $\mathrm{Conf}_0(\mathbb{R}^2 \times \mathrm{D}^1) \coloneqq \{(0,0,-1),(0,0,1)\} \subseteq \partial(\mathbb{R}^2 \times \mathrm{D}^1)$.

Situation 4.2.6. Let us fix the following points of $\mathbb{R}^2 \times \mathrm{D}^1$. Define $e \coloneqq (1,0,0) \in \mathbb{R}^2 \times \mathrm{D}^1$, and $q_1 \coloneqq (0,0,0)$, and $q_k = q_1 + 4(k-1)e$ for $k \geq 1$ Also define the set of points $Q_0 \coloneqq \emptyset$ and $Q_k \coloneqq \{q_1, q_2, \dots, q_k\}$.

Theorem 4.2.7 ([FN62, Theorem 2])**.** *For $n \geq 2$ and $n \geq k \geq 0$, the map*

$$\begin{aligned} \mathrm{pr}_{k,n} \colon \mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_k) &\to \mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_k \\ (x_1, x_2, \cdots, x_n) &\mapsto x_1 \end{aligned}$$

is a fibre bundle whose fibre is homeomorphic to $\mathrm{Conf}_{n-1}(\mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_{k+1})$. For $k \geq 0$, the map $\mathrm{pr}_{k,n}$ admits a cross section[16]*.*

Thus we can compute $\pi_\bullet(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))$ inductively via the splitting long exact sequences for the fibre bundles $\mathrm{pr}_{k,n}$ for $0 \leq k \leq n$ and $n \geq 2$. And we can conclude the following corollary.

Corollary 4.2.8 ([FN62, Corollary 2.1])**.** *For $n \geq 2$, we have*

$$\pi_i(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)) \cong \bigoplus_{k=0}^{n-1} \pi_i(\mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_k) \cong \bigoplus_{k=1}^{n-1} \pi_i(\vee_{j=1}^k \mathrm{S}^2).$$

In particular, $\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)$ *is simply connected.*

Now we are going to introduce a set of generators for $\pi_2(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))$, which we will use in the computations in Section 4.2.3.

Definition 4.2.9. For $1 \leq i,j \leq n$ and $i \geq j$, define the map x_{ij} as the composition of the following two maps

$$\begin{aligned} \mathrm{S}^2 &\to \mathrm{Conf}_{n-j+1}(\mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_{j-1}) \\ x &\mapsto (q_i + x, q_j, \dots, q_{n+1}), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Conf}_{n-j+1}(\mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_{j-1}) &\hookrightarrow \mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1) \\ (x_1, \dots, x_{n-j+1}) &\mapsto (q_1, \dots, q_{j-1}, x_1, \dots, x_{n-j+1}). \end{aligned}$$

Proposition 4.2.10. *The maps $x_{ij} \colon \mathrm{S}^2 \to \mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)$ for $1 \leq i < j \leq n$ generate the group $\pi_2(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))$.*

[15] We define this conventions because we will use in the next section that $\mathfrak{E}mb_n \simeq \mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1) \times (\mathrm{S}^2)^n$ for $n \geq 1$.

[16] The case $k = 0$ works since we are looking at Euclidean spaces.

Proof. The image $S_{ij} \coloneqq \mathrm{im}(x_{ij})$ of x_{ij} is homeomorphic to a 2-sphere. For a fixed j with $1 \leq j \leq n$, the space $S_{1j} \vee S_{2j} \vee \cdots \vee S_{i-1,j} \subseteq \mathbb{R}^2 \times \mathrm{D}^1$ is homotopy equivalent to $\mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_{j-1}$. Note that for every i with $1 \leq i < j$, the map x_{ij} is the positive generator of $\pi_2(S_{ij})$. Thus by the Seifert–van Kampen theorem, the maps x_{ij} for $i = 1, \ldots, j-1$ generates the group $\pi_2(S_{1j} \vee S_{2j} \vee \cdots \vee S_{j-1,j}) \cong \pi_2(\mathbb{R}^2 \times \mathrm{D}^1 \setminus Q_{j-1})$. Now let j vary and apply Corollary 4.2.8, we have that the maps x_{ij} for $1 \leq i < j \leq n$ generate the group $\pi_2(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))$. □

Remark 4.2.11. The proof provides a decomposition of $\pi_\bullet(\mathrm{Conf}_n(\mathbb{R}^r \times \mathrm{D}^1))$ as

$$\pi_\bullet(\mathrm{Conf}_n(\mathbb{R}^r \times \mathrm{D}^1)) \cong \bigoplus_{j=2}^{n} \pi_\bullet(S_{1j} \vee \cdots \vee S_{ij} \vee \cdots \vee S_{j-1,j}),$$

where for $1 \leq i < j \leq n$ the positive generator of S_{ij} is x_{ij}. Thus by the following theorem of Hilton about homotopy groups of wedges of spheres, we can compute all the homotopy groups of $\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)$.

Definition 4.2.12 ([Hil55],[Whi78, Page 511–512])**.** Let $T \coloneqq \mathrm{S}^{r_1+1} \vee \mathrm{S}^{r_2+2} \vee \cdots \vee \mathrm{S}^{r_k+1}$ and denote by ι_i the positive generator of S^{r_i+1}. Note that ι_i can be considered as an element of $\pi_{r_i+1}(T)$ via the canonical embedding $\mathrm{S}^{r_i+1} \hookrightarrow T$.[17]

i) The *basic products* of weight 1 are the elements $\iota_1, \iota_2, \cdots, \iota_k$. We order the set of basic products of weight 1 by $\iota_1 < \iota_2 < \cdots < \iota_k$. We define basic products of weight bigger than 1 recursively. A basic product of weight ω is a Whitehead product $[a, b]$, where a and b are both basic products of weights $\alpha < \omega$ and $\beta < \omega$ respectively such that
   a) $\alpha + \beta = \omega$ and $a < b$, and
   b) if b is defined as the Whitehead product $[c, d]$ of basic products c and d, then we have $c \leq a$.

   We declare every basic product of weight w to be greater than any basic product of smaller weight. We order the set of basic products of weight w lexicographically, i.e. for two basic products $[a, b]$ and $[a', b']$ of weight w, we set $[a, b] < [a', b']$ if $a < a'$, or $a = a'$ and $b < b'$.
ii) Thus a basic product p of weight ω is a suitably bracketed word in the symbols ι_i for $i = 1, \ldots, k$. Assume ι_i appears ω_i times in p. We define the height $h(p)$ of p as $\sum_{i=1}^{k} r_i \omega_i$.

Theorem 4.2.13 ([Whi78, Theorem 8.1])**.** *Using the notation of Definition 4.2.12, let P be the set of (formal) basic products of $\iota_1, \ldots, \iota_k$. We have*

$$\pi_\bullet(T) \cong \bigoplus_{p \in P} \pi_\bullet(\mathrm{S}^{h(p)+1}).$$

where the direct summand $\pi_\bullet(\mathrm{S}^{h(p)+1})$ is embedded in $\pi_\bullet(T)$ by composition with the basic product $p \in \pi_{h(p)+1}(T)$.

The generator x_{ij} for $1 \leq i, j \leq n$ and $i \neq j$ of $\pi_2(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))$ satisfy some relations, which we will use in the computation in the Section 4.2.3.

Proposition 4.2.14 ([Hil55, Corollary 5.2, Theorem 5.3])**.** *Let X be a topological space. Then the Whitehead product $[,]$ on $\pi_\bullet(X)$ is bilinear, antisymmetric and satisfies the Jacobi identity, i.e. for $\alpha \in \pi_{a+1}(X)$, $\beta \in \pi_{b+1}(X)$ and $\gamma \in \pi_{c+1}(X)$ we have*

[17] We consider basic products eventually as homotopy classes, but to get a well-defined definition, one has to first define basic products as 'formal' products, cf. [Whi78, Page 511–512]. The index set P in Theorem 4.2.13 below is then the set of formal basic products, and this ensures a posteriori that we do not have to distinguish formal products and Whitehead products of homotopy classes.

i) $[\alpha, \beta+\gamma] = [\alpha,\beta] + [\alpha,\gamma]$ *and* $[\alpha+\beta,\gamma] = [\alpha,\gamma]+[\beta,\gamma]$,
ii) $[\alpha,\beta] = (-1)^{(a+1)(b+1)}[\beta,\alpha]$, *and*
iii) $(-1)^{c(a+1)}[\alpha,[\beta,\gamma]] + (-1)^{a(b+1)}[\beta,[\gamma,\alpha]] + (-1)^{b(c+1)}[\gamma,[\alpha,\beta]] = 0$.

Proposition 4.2.15 ([FH01])**.** *The elements* $x_{ij} \in\subseteq \pi_2(\mathrm{Conf}_2(\mathbb{R}^2 \times \mathrm{D}^1))$ *for* $1 \leq i,j \leq n$ *and* $i \neq j$ *satisfy the following relations:*
i) $x_{ij} = -x_{ji}$,
ii) $[x_{ij}, x_{jk}] = [x_{ji}, x_{ik}] = [x_{ik}, x_{kj}]$, *if* $n \geq 3$;
iii) $[x_{ij}, x_{kl}] = 0$, *if* $\{i,j\} \cap \{l,m\} = \emptyset$ *and* $n \geq 4$.

4.2.3. *A homotopy spectral sequence for the Taylor tower of* $\mathrm{Emb}(-)$. Recall from Construction 4.1.11 the cosimplicial space $\mathfrak{E}mb^\bullet$ which by Construction 4.1.20 corresponds to the embedding functor $\mathrm{Emb}(-)$ from Notation 3.2.1. We have by Construction 4.1.11 that

$$\mathfrak{E}mb_n \simeq \mathrm{Emb}(V) \simeq \mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1) \times \left(\mathrm{S}^2\right)^n, \tag{4.2.2}$$

for any $V \in \mathbf{Open}_\partial(\mathrm{I})^{\mathrm{op}}_{\mathrm{fin}}$ such that $\pi_0(\mathrm{I} \setminus V) \cong [n]$. The S^2 components in the product represent the tangent vectors at points of embeddings.

For the computation of the homotopy spectral sequence associated to $\mathfrak{E}mb^\bullet$ we need to compute the induced maps on homotopy groups of the face and codegeneracy maps. Recall that $\pi_*(\mathfrak{E}mb_n) \cong \pi_*(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)) \times (\pi_*(\mathrm{S}^2))^n$. By abuse of notation, we consider x_{ij} for $1 \leq i < j \leq n$ elements of $\pi_*(\mathfrak{E}mb_n)$ under the natural inclusion.

Let $l \in \mathbb{N}$ and $0 \leq l \leq n$. Recall the notations from Construction 4.1.11. Let $V_{n+1} \in \mathbf{Open}_\partial(\mathrm{I})_{\mathrm{fin}}$ be the object such that $\kappa(V_{n+1}) = [n+1]$. We obtain an object $V_n \subseteq V_{n+1}$ by removing the $(l+2)$-th subinterval of $V_{n+1} \setminus \partial\mathrm{I}$. Then the degeneracy map s^l for $\mathfrak{E}mb^\bullet$ is the induced restriction map $\mathrm{Emb}(V_{n+1}) \to \mathrm{Emb}(V_n)$, i.e. forgetting the embedding of the $(l+2)$-th interval With respect to the homotopy equivalence 4.2.2, we can write s^l with $0 \leq l \leq n$ concretely as

$$\begin{aligned} s^l \colon \mathrm{Conf}_{n+1}(\mathbb{R}^2 \times \mathrm{D}^1) \times \left(\mathrm{S}^2\right)^{n+1} &\to \mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1) \times \left(\mathrm{S}^2\right)^n \\ (x_1, \ldots, x_{n+1}) \times (v_1, \ldots, v_{n+1}) &\mapsto (x_1, \ldots, \widehat{x_{l+1}}, \ldots, x_{n+1}) \times (v_1, \ldots, \widehat{v_{l+1}}, \ldots, v_{n+1}), \end{aligned}$$

Therefore, precomposing with the map from S^2 representing the generators x_{ij}, we have

Proposition 4.2.16. *Let* $i,j,l,n \in \mathbb{N}$ *and* $1 \leq i < j \leq n+1$ *and* $0 \leq l \leq n$ *and* $n \geq 2$.
i) *We have*

$$s^l_*(x_{ij}) = \begin{cases} x_{i-1,j-1} & \text{if } l < i-1 \\ x_{i,j-1} & \text{if } i-1 < l < j-1 \\ x_{i,j} & \text{if } l > j-1 \\ 0 & \text{otherwise} \end{cases}$$

ii) *Denote by* $s^l_*(c) \colon \pi_r(\mathrm{Conf}_{[n+1]}(\mathbb{R}^2 \times \mathrm{D}^1)) \to \pi_r(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))$ *the restriction of the map* s^l_* *on to the* $\pi_r(\mathrm{Conf}_{[n+1]}(\mathbb{R}^2 \times \mathrm{D}^1))$ *component.*
Denote by Z *the set of basic products of the elements* $x_{i,j}$ *that contain* $x_{u,l+1}$ *or* $x_{l+1,v}$ *for* $1 \leq u \leq l$ *and* $l+2 \leq v \leq n+1$. *Via the isomorphism in Theorem 4.2.13, the kernel of the map* $s^l_*(c)$ *is isomorphic to* $\bigoplus_{p \in Z} \pi_r(\mathrm{S}^{h(p)}+1)$, *for* $r \geq 2$.
iii) *Denote by* $s^l_*(t) \colon \left(\pi_r(\mathrm{S}^2)\right)^{n+1} \to \left(\pi_r(\mathrm{S}^2)\right)^n$ *the restriction of the map* s^l_* *on to the* $\left(\pi_r(\mathrm{S}^2)\right)^{n+1}$ *component.*
For $r \geq 2$, *we have that* $s^l_*(t)$ *is the canonical projection where one forgets the* l*-th component. Thus the kernel of* $s^l_*(t)$ *is isomorphic to* $(0)^{l-1} \times \pi_r(\mathrm{S}^2) \times (0)^{n-l}$.

Proof. i) and iii) follows from the description of s^l right above the proposition.

For the proof of iii), let us abbreviate $s^l_*(c)$ by s^l_* in this part of the proof. Note that for $n \geq 2$, we have $s^l_*(x_{uv}) = 0$ if and only if $u = l+1$ or $v = l+1$. Thus for $n \geq 2$ and

$z \in Z$, we have $s^l_*(z) = 0$ because of the naturality of the Whitehead product. Thus s^l_* factors through $\pi_r(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))/\bigoplus_{p\in Z} \pi_r(\mathrm{S}^{h(p)} + 1)$

$$\begin{tikzcd}
\pi_r(\mathrm{Conf}_{n+1}(\mathbb{R}^2 \times \mathrm{D}^1)) \arrow[rr, "s^l_*"] \arrow[d, "p"'] & & \pi_r(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)) \\
\pi_r(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))/\sum_{p\in Z} \pi_r(\mathrm{S}^{h(p)} + 1) \arrow[urr, dashed, "\bar{s}^l_*"'] & &
\end{tikzcd}$$

By inspecting the value of s^l_* on x_{ij}, we conclude that for two basic products $w_1 \le w_2$ with $s^l_*(w_k) \neq 0$ for $k \in \{1, 2\}$, we have $0 \neq s^l_*(w_1) \le s^l_*(w_2)$. Also the height of w_k and $s^l_*(w_k)$ is the same.

Thus $\bar{s}^l_*$ sends a basis of $\pi_r(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))/\sum_{p\in Z} \pi_r(\mathrm{S}^{h(p)+1})$ (i.e. basic products that are not in Z) injectively to a basis of $\pi_r(\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1))$. Thus $\bar{s}^l_*$ is injective, which implies that the kernel of s^l_* is isomorphic to $\bigoplus_{p\in Z} \pi_r(\mathrm{S}^{h(p)+1})$, via the isomorphism from Theorem 4.2.13. □

Similar analysis of the definition of the face maps tells us that the face map δ^l of $\mathfrak{Emb}^\bullet$ corresponds to "break" the embeddings of the $(l + 1)$-th interval into the embedding of two subintervals. Therefore, one representative[18] for the map δ^l with $0 < l < n + 1$ can be the following:

$$\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1) \times \left(\mathrm{S}^2\right)^n \to \mathrm{Conf}_{n+1}(\mathbb{R}^2 \times \mathrm{D}^1) \times \left(\mathrm{S}^2\right)^{n+1}$$
$$(x_1, \dots, x_n) \times (v_1, \dots, v_n) \mapsto (x_1, \dots, x_l, x_l + \epsilon v_l, x_{l+1}, \dots, x_n) \times (v_1, \dots, v_l, v_l, v_{l+1}, \dots, v_n),$$

where the scalar $\epsilon \in \mathbb{R}$ is so chosen that $(x_1, \dots, x_l, x_l + \epsilon v_l, x_{l+1}, \dots, x_n)$ is a well-defined point in $\mathrm{Conf}_{n+1}(\mathbb{R}^2 \times \mathrm{D}^1)$. For $l = 0$ and $l = n + 1$, we have

$$\delta^0\left((x_1, \dots, x_n) \times (v_1, \dots, v_n)\right) = (x_{-1} + \epsilon e, x_1, \dots, x_n) \times (e, v_1, \dots, v_n)$$
$$\delta^n\left((x_1, \dots, x_n) \times (v_1, \dots, v_n)\right) = (x_1, \dots, x_n, x_{+1} + \epsilon' e,) \times (v_1, \dots, v_n, e),$$

where $x_{-1} = (0, 0, -1)$ and $x_{+1} = (0, 0, 1)$ and $e = (0, 0, 1)$.

Therefore, we can calculate explicitly that

Proposition 4.2.17. *Let $i, j, l, n \in \mathbb{N}$ and $n \ge 2$ and $1 \le i < j \le n$ and $0 \le l \le n + 1$.*

i) *For $n \in \mathbb{N}$ and $n \ge 2$, we have*

$$\delta^l_*(x_{ij}) = \begin{cases} x_{i+1,j+1} & \text{if } l < i \\ x_{i,j+1} + x_{i+1,j+1} & \text{if } l = i \\ x_{i,j+1} & \text{if } i < l < j \\ x_{i,j} + x_{i,j+1} & \text{if } l = j \\ x_{ij} & \text{otherwise} \end{cases}.$$

ii) *Denote by y_k a generator for the k-th component $\pi_2(\mathrm{S}^2)$ of $(\pi_2(\mathrm{S}^2))^n$. We have that*

$$\delta^l_*(y_k) = \begin{cases} y_{k+1} & \text{if } l < k \\ x_{k,k+1} + y_k + y_{k+1} & \text{if } l = k \\ y_k & \text{otherwise} \end{cases}.$$

□

Now we can compute $E^1_{p-1,p}$ and $E^1_{p,p}$ of the homotopy spectral sequence associated to the cosimplicial space $\mathfrak{Emb}^\bullet$.

[18]There is no natural homotopy equivalence between $\mathfrak{Emb}_n$ and $\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1) \times \left(\mathrm{S}^2\right)^n$

Corollary 4.2.18. *Let $l, n, r \in \mathbb{N}$ and $n \geq 2$ and $0 \leq l \leq n$ and $r \geq 2$, and recall the notations from Proposition 4.2.16. For the degeneracy map $\mathfrak{E}mb^{n+1} \xrightarrow{s^l} \mathfrak{E}mb^n$, we have*

$$\ker s^l_* = \ker s^l_*(c) \times \ker s^l_*(t) \text{ and}$$
$$\bigcap_{l=0}^{n-1} \ker s^l_* \cong \bigcap_{l=0}^{n-1} \ker s^l_*(c) \times (0)^n$$

Proof. We have that $s^l_* = s^l_*(c) \times s^l_*(t)$. □

Proposition 4.2.19.

i) *For $p \geq 3$ and $1 \leq i < p-1$, let T be the set of basic products of the elements $x_{i,p-1}$ of height $p-2$, such that each $x_{i,p-1}$ appears exactly once. Let F be the set of basic products of elements $x_{i,p-1}$ of height $p-1$, such that one $x_{k,p-1}$ appears exactly twice and all other $x_{i,p-1}$ appear exactly once. Then we have*

$$E^1_{p-1,p} \cong \bigoplus_T \pi_p(\mathrm{S}^{p-1}) \oplus \bigoplus_F \pi_p(\mathrm{S}^p) \tag{4.2.3}$$

where $\pi_p(\mathrm{S}^{p-1})$ and $\pi_p(\mathrm{S}^p)$ are embedded in $\pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1))$ by composition with the basic products in T and F respectively.

ii) *For $p \geq 2$, let H be the set of basic products of height $p-1$ of the elements in $x_{i,p}$ for $1 \leq i \leq p-1$ such that each $x_{i,p}$ appears exactly once. Then*

$$E^1_{p,p} \cong \bigoplus_H \pi_p(\mathrm{S}^p), \tag{4.2.4}$$

where the direct summands $\pi_p(\mathrm{S}^p)$ are embedded in $\pi_p(\mathrm{Conf}_p(\mathbb{R}^2 \times \mathrm{D}^1))$ by composition with the basic products in H.

Proof. i) Recall from Proposition 4.2.3 that $E^1_{p-1,p} \cong \pi_p(\mathfrak{E}mb_{p-1}) \cap \bigcap_{l=0}^{p-2} \ker(s^l_*)$. By Corollary 4.2.18 we only need to consider the $\pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1))$ component of $\pi_p(\mathfrak{E}mb_{p-1})$, i.e.

$$E^p_{p-1} \cong \pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1)) \cap \bigcap_{l=0}^{p-2} \ker(s^l_*(c)).$$

Recall from Corollary 4.2.8 that

$$\pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1)) \cong \bigoplus_{j=2}^{p-1} \pi_p(S_{1j} \vee S_{2j} \vee \cdots \vee S_{j-1,j}),$$

and x_{ij} is the positive generator of S_{ij}, $1 \leq i < j \leq p-1$. For a fixed j, let $\{b^{(j)}_k\}_{k\in\mathbb{N}}$ be the set of basic products of the elements x_{ij} for $i = 1, \ldots, j-1$. By Theorem 4.2.13 of Hilton, we have

$$\pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1)) \cong \bigoplus_{\substack{1<j<p-1 \\ k\in\mathbb{N},\ h(b^{(j)}_k)\leq p-1}} \pi_p(\mathrm{S}^{h(b^{(j)}_k)+1}).$$

Next we need to examine which elements of $\pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1))$ lie in $\bigcap_{l=1}^{p-2} \ker(s^l_*(c))$. By Proposition 4.2.17.iii), it is sufficient to see which basic products lie in $\bigcap_{i=1}^{p-2} \ker(s^l_*(c))$. Let us consider the following cases:

a) We have $j \neq p-1$. In this case, we have $s^{p-1}_*(b^{(j)}_k) \neq 0$.

b) We have $j = p-1$ and $h(b^{(j)}_k) \leq p-3$. In this case there exists at least one index $1 \leq i < p-1$ such that $x_{i,p-1}$ does not appear in $b^{(j)}_k$, and thus $s^{i-1}_*(b^{(j)}_k) \neq 0$.

c) We have $j = p-1$ and $h(b_k^{(j)}) = p-2$. In this case each $x_{i,p-1}$ with $1 \le i \le p-2$ appears in $b_k^{(j)}$ exactly once. Thus for all $0 \le l \le p-2$, we have $s_*^l(b_k^{(j)}) = 0$, since $s_*^l(x_{l+1,p-1}) = 0$ and $x_{l+1,p-1}$ appears in $b_k^{(j)}$.

d) We have $j = p-1$ and $h(b_k^{(j)}) = p-1$. In this case there exists an index i_k such that $x_{i_k,p-1}$ appears exactly twice in $b_k^{(j)}$, and all other $x_{i,p-1}$ with $1 \le i \le p-2$ and $i \ne i_k$ appear exactly once in $b_k^{(j)}$. As in c) we see $s_*^l(b_k^{(j)}) = 0$ for $0 \le l \le p-2$.

Thus $\bigcap_{i=1}^{p-2} \ker(s_*^i(c))$, or $E_{p-1,p}^1$, is generated by basic products of the form in c) and d), which yields Equation 4.2.3).

ii) Similar as in i), we recall that $E_{p,p}^1 \cong \pi_p(\mathrm{Conf}_p(\mathbb{R}^2 \times \mathrm{D}^1)) \cap \bigcap_{l=0}^{p-1} \ker(s_*^l(c))$. Furthermore $\bigcap_{l=0}^{p-1} \ker(s_*^l(c))$ is generated by the basic products $\{b_k^{(p)}\}_{k\in\mathbb{N}}$ of elements x_{ip} with $1 \le i \le p-1$ such that $h(b_k^{(p)}) = p-1$, and for each i with $1 \le i \le p-1$, the element x_{ip} appears exactly once in $b_k^{(p)}$. □

With the description of $E_{p-1,p}^1$ and $E_{p,p}^1$ in terms of elements x_{ij} with $1 \le i \le j$ and $j = p-1$ or $j = p$, we are going to give an explicit formula for the differential $d^1\colon E_{p-1,p}^1 \to E_{p,p}^1$.

Convention 4.2.20. Since the groups $E_{p-1,p}^1$ and $E_{p,p}^1$ are isomorphic to subgroups of $\pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1))$ and $\pi_p(\mathrm{Conf}_p(\mathbb{R}^2 \times \mathrm{D}^1))$ respectively, we only need to use $\delta_*^l(c)$ for the calculation of the differential d^1, according to Proposition 4.2.3. In the remaining text, abbreviate $\delta_*^l(c)$ by δ_*^l.

Proposition 4.2.21.

i) *The differential $d^1\colon E_{p-1,p}^1 \to E_{p,p}^1$ is trivial for $p = 1$ and $p = 3$.*

ii) *For $p = 2$, the differential $d^1\colon E_{1,2} \to E_{2,2}$ is an isomorphism.*

Proof. First we have

$$E_{0,1}^1 \cong \pi_1(\mathfrak{E}mb_0) = 0, \text{ and}$$
$$E_{1,1}^1 \cong \pi_1(\mathfrak{E}mb_1) \cong \pi_1(\mathrm{Conf}_1(\mathbb{R}^2 \times \mathrm{D}^1)) \times \pi_1(\mathrm{S}^2) = 0.$$

Thus $d^1\colon E_{0,1}^1 \to E_{1,1}^1$ is trivial.

For $d^1\colon E_{2,3}^1 \to E_{3,3}^1$, we apply Proposition 4.2.19 to see that

$$E_{2,3}^1 \cong \pi_3(S^2) \cong \mathbb{Z},$$

with generator x_{12}. Thus

$$\begin{aligned} d^1(x_{12}) &= \sum_{i=0}^{3} \delta_*^i(x_{12}) \\ &= x_{23} - (x_{13} + x_{23}) + (x_{12} + x_{13}) - x_{12} \\ &= 0. \end{aligned}$$

Therefore $d^1\colon E_{2,3}^1 \to E_{3,3}^1$ is trivial.

As for $d^1\colon E_{1,2}^1 \to E_{2,2}^1$, we have

$$E_{1,2}^1 \cong \pi_2(\mathfrak{E}mb_1) \cong \pi_2(\mathrm{Conf}_1(\mathbb{R}^2 \times \mathrm{D}^1)) \times \pi_2(\mathrm{S}^2) \cong \{0\} \times \pi_2(\mathrm{S}^2)$$

with generator y_1 for the component $\pi_2(\mathrm{S}^2)$, and

$$E_{2,2}^1 \cong \pi_2(\mathfrak{E}mb_2) \cap \bigcap_{l=0}^{1} s_*^l \cong \pi_2(\mathrm{Conf}_2(\mathbb{R}^2 \times \mathrm{D}^1)) \times \{e\} \times \{e\},$$

with generator x_{12} for the component $\pi_2(\mathrm{Conf}_2(\mathbb{R}^2\times \mathrm{D}^1))$. Apply the formula in Proposition 4.2.3, we have

$$\begin{aligned} d^1(y_1) &= \sum_{i=0}^{2} \delta_*^i(y_1) \\ &= y_2 - (x_{12} + y_1 + y_2) + y_1 \\ &= x_{12}. \end{aligned}$$

Therefore, $d^1\colon E^1_{1,2} \to E^1_{2,2}$ is an isomorphism. □

Now let us consider $d^1\colon E^1_{p-1,p} \to E^1_{p,p}$, for $p \geq 4$. According to Proposition 4.2.19 we have for $p \geq 4$,

$$\begin{aligned} E^1_{p-1,p} &\cong \bigoplus_T \pi_p(\mathrm{S}^{p-1}) \oplus \bigoplus_F \pi_p(\mathrm{S}^p) \\ &\cong \bigoplus_T \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_F \mathbb{Z} \\ E^1_{p,p} &\cong \bigoplus_H \pi_p(\mathrm{S}^p) \cong \bigoplus_H \mathbb{Z}. \end{aligned}$$

Since $E^1_{p,p}$ is torsion free, we see that d^1 is trivial on $\bigoplus_T \pi_p(\mathrm{S}^{p-1})$, and we conclude that we only need to consider the restriction of d^1 to $\bigoplus_F \pi_p(\mathrm{S}^p)$.

Notation 4.2.22. We denote the torsion-free part of $E^1_{p-1,p}$ by $E^1_{p-1,p}/\mathrm{tors}$, i.e. the summand $\bigoplus_F \pi_p(\mathrm{S}^p)$ in Equation 4.2.3.

Proposition 4.2.23. *For $p \geq 4$, denoted by D_p^{sep} the set of Whitehead products of the elements $x_{i,p-1}$ for $i = 1, \dots, p-2$ with the following properties:*

i) *For every $w \in D_p^{sep}$, there exists one $x_{k(w),p-1}$ that appears exactly twice and all other $x_{i,p-1}$ with $1 \leq i \leq p-2$ and $i \neq k(w)$ appear exactly once.*
ii) *Every $w \in D_p^{sep}$ is of the form $w = [c_1, c_2]$ where c_1 is an iterated Whitehead product of elements $x_{i,p-1}$ with $i \in I$ and c_2 is an iterated Whitehead product of elements $x_{i,p-1}$ with $j \in J$ such that $I, J \subseteq \{1, \dots, p-2\}$, $I \cap J = \{k(w)\}$ and $I \cup J = \{1, 2, \dots, p-2\}$.*

Then, $E^1_{p-1,p}/\mathrm{tors}$ is generated by elements of D_p^{sep}.

Proof. Denote by D_p the set of iterated Whitehead products of the elements $x_{i,p-1}$ with $i = 1, \dots, p-2$ satisfying only condition i). Using the same argument as in the proof of Proposition 4.2.19, we see that $D_p^{\mathrm{sep}} \subseteq E^1_{p-1,p}$ and $D_p \subseteq E^1_{p-1,p}$. In particular, the basic products in F, which are a basis of $E^1_{p-1,p}/\mathrm{tors}$, are contained in D_p. We have reduced the desired statement to the following claim which we prove by induction.

Claim. *For $p \geq 4$, any element of D_p can be written as a linear combination of elements of D_p^{sep} using only the Jacobi identity and antisymmetry relations (cf. Proposition 4.2.14).*

For $p = 4$, the claim follows by listing all the elements of D_4 and using the Jacobi identity of the Whitehead product.

Assume that the claim is true for all $p \leq n$ with $n \geq 4$. Let $p = n+1$ and consider $\widetilde{w} = [a_1, a_2] \in D_{n+1}$. Without loss of generality, we can assume that $x_{1,n}$ is the repeated element in $\widetilde{w}$. If the two copies of $x_{1,n}$ appear in a_1 and a_2 separately, then $\widetilde{w}$ is already an element of D_p^{sep}. Otherwise, both copies of $x_{1,n}$ appear in either a_1 or a_2, say they appear in a_1. By assumption a_1 is a Whitehead product of elements $x_{m,n}, m \in M$ with $1 \in M$, $\#M \leq n-2$, and for $m \neq 1$, the $x_{m,n}$ appears exactly once in a_1.

There is a bijection $r\colon \{x_{m,n} \mid m \in M\} \to \{x_{i,\#M+1} \mid i = 1, \dots, \#M\}$ such that $x_{1,n}$ is mapped to $x_{1,\#M+1}$. Define $a_1' \in D_{\#M+2}$ by replacing each occurrence of $x_{m,n}$ in a_1 by

$r(x_{m,n})$. By the induction assumption, we can write a'_1 as the finite sum $a'_1 = \sum_{i\in I}[c'_{i1}, c'_{i2}]$ such that $[c'_{i1}, c'_{i2}] \in D^{\text{sep}}_{\#M+2}$ and $x_{1,\#M+1}$ appears exactly twice in $[c_{i1}, c_{i2}]$. Thus, by replacing each $x_{i,\#M+1}$ by $r^{-1}(x_{i,\#M+1})$ we obtain $a_1 = \sum_{i\in I}[c_{i1}, c_{i2}]$ such that $[c_{i1}, c_{i2}]$ is a Whitehead product of the elements $x_{m,n}$ with $m \in M$, where $x_{1,n}$ appears exactly twice and $x_{m,n}$ appears exactly once for $m \neq 1$.

Therefore $\widetilde{w}$ can be written as

$$\begin{aligned}\widetilde{w} &= \sum_{i\in I} \big[[c_{i1}, c_{i2}], a_2\big] \\ &= \sum_{i\in I} (-1)^{\epsilon_1}\big[[c_{i1}, a_2], c_{i2}\big] + (-1)^{\epsilon_2}\big[[a_2, c_{i2}], c_{i1}\big],\end{aligned}$$

where ϵ_1 and ϵ_2 denote the signs which come from the Jacobi identity for the Whitehead product. For every $i \in I$, we have that $[[c_{i1}, a_2], c_{i2}], [[a_2, c_{i2}], c_{i1}] \in D^{\text{sep}}_{n+1}$. Thus $\widetilde{w}$ is a linear combination of elements of D^{sep}_p. □

The upshot is that it is sufficient, for the computation $d^1 \colon E^1_{p-1,p} \to E^1_{p,p}$, to compute $d^1(w)$ for every $w \in D^{\text{sep}}_p$.

Proposition 4.2.24. *Let $w = [c_1, c_2] \in D^{\text{sep}}_p$, say with repeated occurrence of $x_{k,p-1}$. We write*[19] *c_1 and c_2 as $c_1 = [\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots]$ and $c_2 = [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots]$. The index k will be fixed through out the proposition. Then we have*

$$d^1(w) = \partial^k(w) + \partial^{p-1}(w),$$

where

$$\begin{aligned}\partial^k(w) = (-1)^k &\big[[\ldots x_{i',p} \ldots x_{k,p} \ldots], [\ldots x_{k+1,p} \ldots x_{j',p} \ldots]\big] \\ &+ (-1)^k \big[[\ldots x_{i',p} \ldots x_{k+1,p} \ldots], [\ldots x_{k,p} \ldots x_{j',p} \ldots]\big]\end{aligned}$$

and

$$\begin{aligned}\partial^{p-1}(w) = (-1)^{p-1} &\big[[\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]\big] \\ &+ (-1)^{p-1} \big[[\ldots x_{i,p} \ldots x_{k,p} \ldots], [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots]\big],\end{aligned}$$

where $i' = i$ if $i < k$ and $i' = i+1$ if $i > k$ and $i' = i$ if $i < k$ and $i' = i+1$ if $i > k$.

Before we prove the proposition, let us take a look at an example of computation of d^1.

Example 4.2.25. Let $k = 2$ and $p = 8$ and $w \in D^{\text{sep}}_8$ of the form

$$w = \big[\,[[x_{37}, x_{27}], x_{57}], [[x_{17}, x_{27}], [x_{47}, x_{67}]]\,\big].$$

Now let us calculate $d^1(w)$ using the formulas in Proposition 4.2.24. We have

$$\partial^2(w) = \big[\,[[x_{48}, x_{28}], x_{68}], [[x_{18}, x_{38}], [x_{58}, x_{78}]]\,\big] + \big[\,[[x_{48}, x_{38}], x_{68}], [[x_{18}, x_{28}], [x_{58}, x_{78}]]\,\big],$$

and

$$\partial^7(w) = -\big[\,[[x_{37}, x_{27}], x_{57}], [[x_{18}, x_{28}], [x_{48}, x_{68}]]\,\big] - \big[\,[[x_{38}, x_{28}], x_{58}], [[x_{17}, x_{27}], [x_{47}, x_{67}]]\,\big].$$

Proof of Proposition 4.2.24. First we proof the proposition for $k \leq p-3$.

Claim 1. *For $l \neq k, p-1, p$, each of the elements $(-1)^{l-1}\delta^{l-1}_*(w)$, $(-1)^l\delta^l_*(w)$ and* $(-1)^{l+1}\delta^{l+1}_*(w)$ can be written canonically as a sum of two iterated Whitehead products of $x_{i,p}$ with $1 \leq i < p$ such that every summand of $(-1)^l\delta^l_*(w)$ appears in $(-1)^{l-1}\delta^{l-1}_*(w)$ or $(-1)^{l+1}\delta^{l+1}_*(w)$ with opposite sign.

[19]By abuse of notation we hide the inner brackets of iterated Whitehead products when we write an elements as done here.

We note first that $\delta_*^{l-1}(x_{in}) = \delta_*^l(x_{in}) = \delta_*^{l+1}(x_{in})$ for $i \neq l-1, l$ and $l+1$. Moreover,

$$\delta_*^r(x_{in}) = \begin{cases} x_{i+1,p}, & \text{if } i > l+1 \\ x_{i,p}, & \text{if } i < l-1 \end{cases}$$

for $r = l-1, l$ and $l+1$.

Without loss of generality, we write $w = [\dots x_{l,p-1} \dots x_{l-1,p-1} \dots x_{l+1,p-1} \dots]$, where we show only the elements that are interesting for us. Note that in this presentation of w, the order in which the elements $x_{l,p-1}, x_{l-1,p-1}$ and $x_{l+1,p-1}$ appear does not play a role. We calculate $\delta_r^\bullet(w)$ for $r = l-1, l$ and $l+1$:

$$\begin{aligned} \delta_*^{l-1}(w) &= [\dots x_{l+1,p} \dots x_{l-1,p} + x_{l,p} \dots x_{l+2,p} \dots] \\ \delta_*^{l}(w) &= [\dots x_{l,p} + x_{l+1,p} \dots x_{l-1,p} \dots x_{l+2,p} \dots] \\ \delta_*^{l+1}(w) &= [\dots x_{l,p} \dots x_{l-1,p} \dots x_{l+1,p} + x_{l+2,p} \dots] \end{aligned}$$

Thus we see that $[\dots x_{l+1,p} \dots x_{l-1,p} \dots x_{l+2,p} \dots]$ of $\delta_*^l(w)$ appears in $\delta_*^{l-1}(w)$ and the term $[\dots x_{l,p} \dots x_{l-1,p} \dots x_{l+2,p} \dots]$ of $\delta_*^l(w)$ appears in $\delta_*^{l+1}(w)$. Since the signs in front of $\delta_*^r, r = l-1, l$ and $l+1$ in d^1 are alternating, we see that in $d^1(w)$ the terms of $(-1)^l\delta_*^l(c)$ are cancelled by terms of $(-1)^{l-1}\delta_*^{l-1}(w)$ and $(-1)^{l+1}\delta_*^{l+1}(w)$ as desired.

Claim 2. *For $l = k$, after cancelling terms of $(-1)^k\delta_*^k(w)$ by terms of $(-1)^{k-1}\delta_*^{k-1}(w)$ and $(-1)^{k+1}\delta_*^{k+1}(w)$ as in **Claim 1.**, the remaining terms of $(-1)^k\delta_*^k(w)$ is $\partial^k(w)$.*

For convenience of the proof, we write without loss of generality

$$w = [\dots x_{k,p-1} \dots x_{k-1,p-1} \dots x_{k,p-1} \dots x_{k+1,p-1} \dots].$$

Again note that for $i \neq k-1, k, k+1$, we have $\delta_*^{k-1}(x_{i,p-1}) = \delta_*^k(x_{i,p-1}) = \delta_*^{k+1}(x_{i,p-1})$, and note

$$\begin{aligned} \delta_*^{k-1}(w) &= [\dots x_{k+1,p} \dots x_{k-1,p} + x_{k,p} \dots x_{k+1,p} \dots x_{k+2,p} \dots] \\ \delta_*^{k}(c) &= [\dots x_{k,p} + x_{k+1,p} \dots x_{k-1,p} \dots x_{k,p} + x_{k+1,p} \dots x_{k+2,p} \dots] \\ \delta_*^{k+1}(w) &= [\dots x_{k,p} \dots x_{k-1,p} \dots x_{k,p} \dots x_{k+1,p} + x_{k+2,p} \dots]. \end{aligned}$$

Thus after cancelling with terms of $\delta_*^{k-1}(w)$ and $\delta_*^{k+1}(w)$, the remaining term of $\delta_*^k(w)$ is

$$(-1)^k[\dots x_{k,n+1} \dots x_{k-1,p} \dots x_{k+1,p} \dots x_{k+2,p} \dots] \\ + (-1)^k[\dots x_{k+1,p} \dots x_{k-1,p} \dots x_{k,p} \dots x_{k+2,p} \dots].$$

Writing w as $w = [c_1, c_2] = [[\dots x_{i'n} \dots x_{kn} \dots], [\dots x_{kn} \dots x_{j'n} \dots]]$, the remaining terms of $\delta_*^k(w)$ look like

$$(-1)^k \big[[\dots x_{i',p} \dots x_{k,p} \dots], [\dots x_{k+1,p} \dots x_{j',p} \dots]\big] \\ + (-1)^k \big[[\dots x_{i',p} \dots x_{k+1,p} \dots], [\dots x_{k,p} \dots x_{j',p} \dots]\big],$$

which we recognise as $\partial^k(w)$ as desired.

Claim 3. After cancelling with terms of $(-1)^{p-2}\delta_*^{p-2}(w)$ and $(-1)^p\delta_*^p(w)$, the remaining term of $(-1)^{p-1}\delta_*^{p-1}(w)$ is $\partial^{p-1}(w)$.

In order to prove this claim, we write $w = [c_1, c_2]$ as in Proposition 4.2.24. Note that $\delta_*^{p-1}(x_{i,p-1}) = x_{i,p-1} + x_{i,p}$, so we have

$$\begin{aligned} \delta_*^{p-1}(w) &= [\delta_*^{p-1}(c_1), \delta_*^{p-1}(c_2)] \\ &= \big[[\dots x_{i,p-1} + x_{i,p} \dots x_{k,p-1} + x_{k,p} \dots], [\dots x_{k,p-1} + x_{k,p} \dots x_{j,p-1} + x_{j,p} \dots]\big]. \end{aligned}$$

Recall that by assumption c_1 and c_2 are iterated Whitehead products of the elements $x_{i,p-1}$ with $i \in I$ and $x_{j,p-1}$ with $j \in J$ where $I, J \subseteq \{1, \dots, p-2\}$ and $I \cap J = \{k\}$ and

$I \cup J = \{1, \ldots, p-2\}$. Furthermore, each $x_{i,p-1}$ with $i \in I$ appears exactly once in c_1 and similarly each $x_{j,p-1}$ with $j \in J$ appears exactly once in c_2.

Recall that $E^1_{p-1,p}/\text{tors}$ is a subgroup of $\pi_p(\mathfrak{E}mb_{p-1}) \cong \pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1))$. For $w \in \pi_p(\mathrm{Conf}_{p-1}(\mathbb{R}^2 \times \mathrm{D}^1))$, which has the form $[\ldots[x_{i,p-1} + x_{i,p}, x_{j,p-1} + x_{j,p}]\ldots]$ with $i \neq j$, we have by Proposition 4.2.15 the following equality

$$\big[\ldots[x_{i,p-1} + x_{i,p}, x_{j,p-1} + x_{j,p}]\ldots\big] = \big[\ldots[x_{i,p-1}, x_{j,p-1}]\ldots\big] + \big[\ldots[x_{i,p}, x_{j,p}]\ldots\big].$$

Thus by induction on the number of brackets the brackets, we have

$$\begin{aligned}
\delta^{p-1}_*(c_1) &= [\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots] + [\ldots x_{i,p} \ldots x_{k,p} \ldots] \\
\delta^{p-1}_*(c_2) &= [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots] + [\ldots x_{k,p} \ldots x_{j,p} \ldots].
\end{aligned}$$

and thus

$$\begin{aligned}
\delta^{p-1}_*(w) = &\big[[\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots], [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots]\big] \\
&+ \big[[\ldots x_{i,p} \ldots x_{k,p} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]\big] \\
&+ \big[[\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]\big] \\
&+ \big[[\ldots x_{i,p} \ldots x_{k,p} \ldots], [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots]\big].
\end{aligned}$$

Let us now look at $\delta^{p-2}_*(w)$ and $\delta^p_*(w)$. We have

$$\delta^p_*(w) = \big[[\ldots x_{ik} \ldots x_{k,p-1} \ldots], [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots]\big].$$

Assume without loss of generality that $x_{p-2,p-1}$ appears in c_1, and write c_1 of the form $c_1 = [\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots x_{p-2,p-1} \ldots]$. We get

$$\begin{aligned}
\delta^{p-2}_*(w) &= \big[[\ldots x_{i,p} \ldots x_{k,p} \ldots x_{p-2,p} + x_{p-1,p} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]\big] \\
&= \big[[\ldots x_{i,p} \ldots x_{k,p} \ldots x_{p-2,p} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]\big] \\
&+ \big[[\ldots x_{i,p} \ldots x_{k,p} \ldots x_{p-1,p} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]\big].
\end{aligned}$$

Thus after cancelling with the terms of $(-1)^{p-2}\delta^{p-2}_*(w)$ and $(-1)^p\delta^p_*(w)$, the remaining term of $(-1)^{p-1}\delta^{p-1}_*(w)$ is

$$\begin{aligned}
(-1)^{p-1}&\big[[\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]\big] \\
&\qquad + (-1)^{p-1}\big[[\ldots x_{i,p} \ldots x_{k,p} \ldots], [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots]\big],
\end{aligned}$$

which is exactly $\partial^{p-1}(w)$ as desired.

Finally one can proof Claim 1 for $k = p-2$ analogously. Then we can explicitly write down $\delta^r_*(w)$ for $r = p-3, p-2, p-1$ and p and obtain the desired formula in the proposition. □

Remark 4.2.26. Note that in the calculations of the proof we only changed the indices of $x_{i,p-1}$ for $i = 1, \ldots, p-2$, but the bracketing of c_1 and c_2 was not changed at all. More explicitly, the expressions $[[\ldots x_{i',p} \ldots x_{k,p} \ldots], [\ldots x_{k+1,p} \ldots x_{j',p} \ldots]]$ in the formula of $\partial^k(w)$ and $[[\ldots x_{i,p-1} \ldots x_{k,p-1} \ldots], [\ldots x_{k,p} \ldots x_{j,p} \ldots]]$ in the formula of $\partial^{p-1}(w)$ have the same bracketing as the one of c_1. The expressions $[[\ldots x_{i',p} \ldots x_{k+1,p} \ldots], [\ldots x_{k,p} \ldots x_{j',p} \ldots]]$ in $\partial^k(w)$ and $[[\ldots x_{i,p} \ldots x_{k,p} \ldots], [\ldots x_{k,p-1} \ldots x_{j,p-1} \ldots]]$ in $\partial^{p-1}(w)$ have the same bracketing as c_2. Note that the bracketing determines the shape of the unitrivalent graphs in the combinatorial interpretation in the next section.

4.2.4. *Combinatorial intepretation.* In this section, we will give a combinatorial interpretation of Proposition 4.2.19 and Proposition 4.2.24.

Definition 4.2.27. A *unitrivalent graph* Γ is a graph whose nodes have only degree 1 or 3, together with a cyclic order on the edges at each node. We call the nodes of degree 1 *leaves* and nodes of degree 3 *trivalent node*. When Γ has n leaves, we define a *labelling* (or total ordering) on Γ to be a bijection of the set $\{1, 2, \ldots, n\}$ to the set of leaves. Denote by UTG the set of labelled unitrivalent graphs. We define the degree Γ as the number of nodes divided by 2.

When we draw a labelled unitrivalent graph, we place the leaves on a oriented line, ordered according to the labelling. Unless explicitly mentioned, the cyclic orders of the trivalent nodes are counterclockwise. See Figure 20 for an example of labelled unitrivalent graphs.

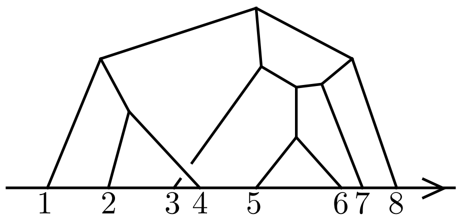


FIGURE 20. A unitrivalent tree of degree 4.

Definition 4.2.28. We define the following relations on $\mathbb{Z}[\mathrm{UTG}]$:

i) Two labelled unitrivalent graphs Γ_1 and $-\Gamma_2$ are *AS-related* if Γ_1 and Γ_2 are the same up to the cyclic order at one node. This is depicted in Figure 21.

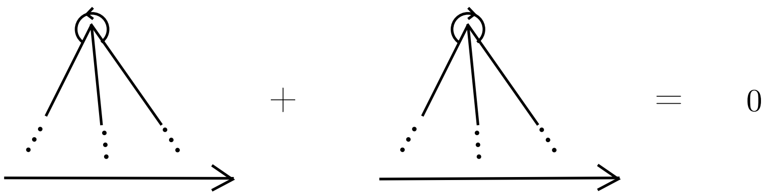


FIGURE 21. A visualisation of the AS-relation.

ii) Let Γ be a labelled unitrivalent graph. Let e be an edge in Γ between two trivalent nodes v and w. Then Γ is *IHX-related* to the difference $\Gamma' - \Gamma''$ of the following two labelled unitrivalent graphs Γ' and Γ''. Let $\{e, e'_v e''_v\}$ be the ordered set of edges at the node v, i.e. we have $e < e'_v < e''_v$ (cyclic order). In the same way, let $\{e, e'_w, e''_w\}$ be the ordered set of edges at the node w. The graph Γ' arise from Γ by deleting the edge e and the nodes v and w of Γ, and adding an edge e' and two trivalent nodes v' and w' such that the ordered set of edges at v' are $\{e', e''_v, e'_w\}$ and the ordered set of edges at w' are $\{e', e''_w, e'_v\}$. The unitrivalent graph Γ'' is constructed similar to Γ'. For Γ'' the ordered set of edges at the nodes v' is $\{e' < e'_v < e'_w\}$ and at the nodes w' is $\{e' < e''_w < e''_v\}$. This is depicted in Figure 22.

Definition 4.2.29. Denote by UTT_p the set of labelled unitrivalent trees of degree p. Define $\mathcal{T}_p$ as the abelian group generated by UTT_p modulo AS- and IHX-relations, i.e.

$$\mathcal{T}_p := \mathbb{Z}\left[\mathrm{UTT}_p\right] / \sim_{\mathrm{AS}}, \sim_{\mathrm{IHX}} .$$

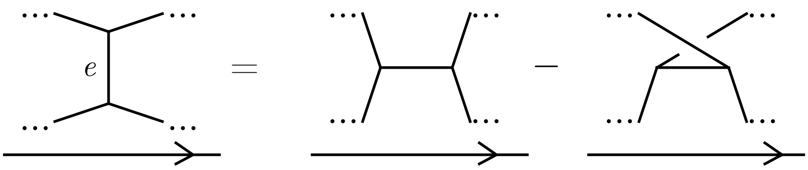


FIGURE 22. A visualisation of the IHX-relation.

Construction 4.2.30. For $p \geq 2$, denote by T_p the set of iterated Whitehead products of the elements $x_{i,p}$ with $1 \leq i < p$ such that $x_{i,p}$ appears at most once in a given iterated Whitehead product. We are going to construct a one-to-one correspondence

$$T_p \underset{\Phi_T}{\overset{\Psi_T}{\rightleftarrows}} \left\{ \begin{array}{l} \text{labelled unitrivalent tree of degree at most } p-1 \\ \text{together with a monotone bijection of their labelling} \\ \text{with a subset of } \{1,\dots,p\} \text{ containing } p \end{array} \right\}.$$

Define the length of $\tau \in T_p$ to be the total number of occurrences of $x_{i,p}$, $i = 1, \dots, p$ in τ. We will define Ψ_T inductively on the length n of τ. Define $\Psi_T(x_{ip})$ to be the degree 1 labelled unitrivalent tree consisting of two nodes labelled by i and p and an edge connecting them. Assume that for all $\tau_k \in T_p$ of length k at most $n-1 < p$, we have that $\Psi_T(\tau_k)$ is a degree $k-1$ unitrivalent tree with labelling $L_{\tau_k} := \{i \in \mathbb{N} \mid x_{ip} \text{ appears in } \tau_k\} \cup \{p\}$. For $\tau_n = [\tau', \tau''] \in T_p$ of length n, we know that τ' and τ'' are elements of T_p and of length smaller than n. By induction hypothesis, both $\Gamma' := \Psi_T(\tau')$ and $\Gamma'' := \Psi_T(\tau'')$ have a leaf with label p. We define the labelled unitrivalent tree $\Psi_T(\tau_n)$ to be the tree that arises by joining the tree Γ' and Γ'' at the respective leaves labelled by p, and joining to this joint point a new leaf labelled by p. The set of labels of $\Psi_T(\tau_n)$ is $L_{\tau'} \cup L_{\tau''}$. This construction is depicted in Figure 23, where also the cyclic order at the joint node is indicated.

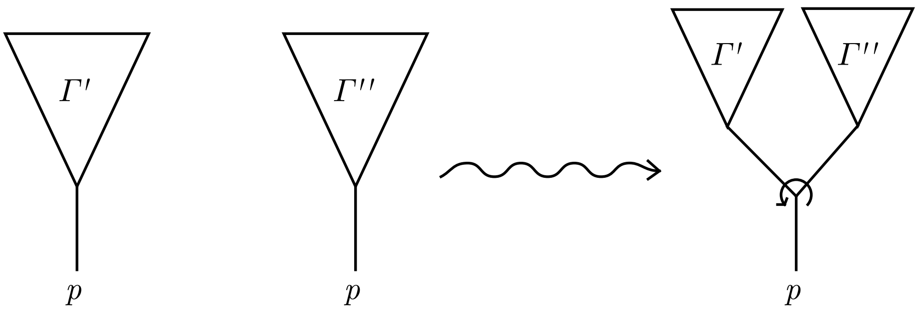


FIGURE 23. Unitrivalent trees Γ', Γ'', and the unitrivalent tree obtained from Γ' and Γ'' by joining their leaves labelled by p.

Now we proceed to define the inverse map Φ_T. For a labelled unitrivalent tree Γ_1 of degree 1 with the set of labellings $\{i, p\}$, set $\Phi_T(\Gamma_1) = x_{ip}$. Assume that we have already defined Φ_T for unitrivalent trees of degree smaller than $n < p - 1$. Every unitrivalent tree Γ_n of degree n can be depicted as in Figure 23. Then define $\Phi_T(\Gamma_n) := [\Phi_T(\Gamma'), \Phi_T(\Gamma'')]$, where Γ' and Γ'' are the trees depicted in Figure 23. By construction, the two maps Ψ_T and Φ_T are inverse to each other.

Proposition 4.2.31. *Recall the group $E^1_{p,p} \cong \bigoplus_H \mathbb{Z}$ from Proposition 4.2.19. For $p \geq 2$, the construction above induces an isomorphism $E^1_{p,p} \cong \mathcal{T}_{p-1}$ of groups.*

Lemma 4.2.32. *Let $J_p \subseteq T_p$ be the set of iterated Whitehead products of the elements $x_{i,p}$, $i = 1, \dots, p-1$ such that $x_{i,p}$ appears exactly once in the iterated Whitehead product.*

Then we have

$$E^1_{p,p} \cong \mathbb{Z}[J_p]/\sim,$$

where $\sim$ *denotes the antisymmetry and Jacobi identity relations from Proposition 4.2.14.*

Proof. Recall H from Proposition 4.2.19, and note that $H \subseteq J_p \subseteq E^1_{p,p}$. Thus any element of $J_p \setminus H$ can be written as linear combination of elements of H. Furthermore, this linear combination is produced by applying the Jacobi identity and antisymmetry relation to the element, cf. [Hal50, Theorem 3.1]. Thus we obtain the desired a group isomorphism $E^1_{p,p} \cong \mathbb{Z}[J_p]/\sim$. □

Proof of Proposition 4.2.31. We are going to define group homomorphisms

$$\mathbb{Z}[J_p] \underset{\phi_T}{\overset{\psi_T}{\rightleftarrows}} \mathbb{Z}[\mathrm{UTT}_{p-1}],$$

which become inverses to each other once we pass to the quotients $E^1_{p,p}$ and $\mathcal{T}_{p-1}$. We will define our morphisms on generators and extend linearly to the whole group.

For $p = 2$, define $\psi_T(x_{12}) = \Gamma$ and $\phi_T(\Gamma) = x_{12}$, where Γ is the labelled unitrivalent tree of degree 1 and with labelling set $\{1, 2\}$. There is no relation to consider when passing to the quotients $E^1_{2,2}$ and $\mathcal{T}_1$. Thus $\bar{\psi}_T$ and $\bar{\phi}_T$ are inverse to each other by definition.

For $p \geq 3$ and $v = [v_1, v_2] \in J_p$, define

$$\psi_T([v_1, v_2]) = (-1)^{\#L_1 + \#(L_1 \times_> L_2)} \Psi_T([v_1, v_2])$$

where[20] $L_i := \{j \in \mathbb{N} \mid x_{jp} \text{ appears in } v_i\} \cup \{p\}$ and

$$L_1 \times_> L_2 := \{(a, b) \in L_1 \times L_2 \mid a > b, \text{ and } a, b \notin L_1 \cap L_2\}.$$

To see that the anti-symmetry of the Whitehead product corresponds to the AS-relation, we look at $\psi_T([v_1, v_2] + (-1)^{\#(L_1 \times L_2)}[v_2, v_1])$ which equals

$$(-1)^{\#L_1 + \#(L_1 \times_> L_2)} \Psi_T([v_1, v_2]) + (-1)^{\#(L_1 \times L_2) + \#(L_2 \times_> L_1) + \#L_2} \Psi_T([v_2, v_1]). \tag{4.2.5}$$

Recall from Construction 4.2.30 that the only difference between the trees $\Psi_T([v_1, v_2])$ and $\Psi_T([v_2, v_1])$ is the cyclic order at the trivalent node which is adjacent to the leaf with label p, i.e. $\Psi_T([v_1, v_2]) \sim_{\mathrm{AS}} -\Psi_T([v_2, v_1])$. Note that the sum of signs in Equation 4.2.5 is

$$\begin{aligned}\#L_1 + \#\left(L_1 \times_> L_2\right) + \#\left(L_1 \times L_2\right) + \#\left(L_2 \times_> L_1\right) + \#L_2 \\ = \#L_1 + \#\left(L_1 \times L_2\right) + \#L_2 + (\#L_1 - 1)(\#L_2 - 1),\end{aligned}$$

because $\#(L_1 \times_> L_2) + \#\left(L_2 \times_> L_1\right) = (\#L_1 - 1)(\#L_2 - 1)$. Thus we obtain that the element in Formula 4.2.5 AS-related to 0.

Now let us consider the Jacobi identity. For this, take $v = [v_1, [v_2, v_3]] \in J_p$ with $v_i \in \pi_{l_i}(\mathrm{Conf}_p(\mathbb{R}^2 \times \mathrm{D}^1))$, where $l_i = \#L_i$. Then

$$\begin{aligned}&\psi_T\left((-1)^{(l_3-1)l_1}[v_1, [v_2, v_3]] + (-1)^{(l_1-1)l_2}[v_2, [v_3, v_1]] + (-1)^{(l_2-1)l_3}[v_3, [v_1, v_2]]\right) \\ &\quad = (-1)^{\epsilon_1}\Psi_T([v_1, [v_2, v_3]]) + (-1)^{\epsilon_2}\Psi_T([v_2, [v_3, v_1]]) + (-1)^{\epsilon_3}\Psi_T([v_3, [v_1, v_2]]),\end{aligned} \tag{4.2.6}$$

where ϵ_1, ϵ_2 and ϵ_3 are the suitable signs. Again by Construction 4.2.30, we have that

$$(-1)^{\epsilon_1}[\Psi_T([v_1, [v_2, v_3]])] + (-1)^{\epsilon_2}[\Psi_T([v_2, [v_3, v_1]])] + (-1)^{\epsilon}[\Psi_T([v_3, [v_1, v_2]])]$$

is IHX-related to 0, and thus the element in Equation 4.2.6 is IHX-related to 0. Therefore, $\bar{\psi}_T$ is well-defined.

Let Γ_{p-1} be a labelled unitrivalent tree of degree $p-1$, drawn as in Figure 23, define

$$\phi_T(\Gamma_n) = (-1)^{\#L_1 + \#(L_1 \times_> L_2)} \Phi_T(\Gamma_n),$$

[20]Note that L_i is the set of labels of the tree $\Psi_T(v_i)$, in particular $\#L_i$ is the number of leaves of this tree.

where L_1 and L_2 is the set of labels of Γ' and Γ'' respectively.

Similar to the discussion of $\overline{\psi}_T$, one can show that $\overline{\phi}_T$ is well-defined. Finally, the maps $\overline{\psi}_T$ and $\overline{\phi}_T$ are inverses to each other by construction. □

Remark 4.2.33. Construction 4.2.30 and Proposition 4.2.31 are inspired by [Con08, Section 4]. Our proof adds missing details of [Con08, Proposition 4.7].

Definition 4.2.34. Let $i, j \in \mathbb{N}$ with $i \neq j$. An (i, j)-*marked unitrivalent graph* Γ_{ij} is a unitrivalent graph of degree j together with two nodes v_i and v_j that satisfy the following properties:

i) The underlying graph of Γ is connected and has exactly one simple cycle[21].
ii) The two marked nodes v_i and v_j are adjacent to the leaf with label i and j, respectively, and lie on simple cycle.

Denote by $\mathrm{UTG}_{i,j}$ the set of (i, j)-marked unitrivalent graphs.

Definition 4.2.35. For $p \geq 3$, define $\mathcal{D}_p$ to be the abelian group generated by the set of (i, p)-marked unitrivalent graphs with $1 \leq i < p$, modulo AS- and IHX$^{\mathrm{sep}}$-relations, i.e.

$$\mathcal{D}_p := \mathbb{Z}[\cup_{i=1}^{p-1} \mathrm{UTG}_{i,p}]/\sim_{\mathrm{AS}}, \sim_{\mathrm{IHX}^{\mathrm{sep}}},$$

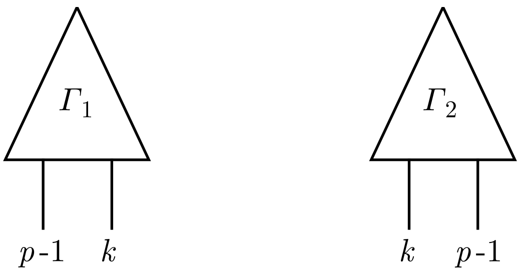
where the IHX$^{\mathrm{sep}}$-relation is the usual IHX-relation, except that the edge e, which appears in Definition 4.2.28,T is not allowed to be an edge that adjacent to the marked nodes.

Construction 4.2.36. Recall from Proposition 4.2.23 the definition of D_p^{sep} for $p \geq 4$. We are going to construct a one-to-one correspondence

$$D_p^{\mathrm{sep}} \underset{\Phi_D}{\overset{\Psi_D}{\rightleftarrows}} \bigcup_{i=1}^{p-2} D_{i,p-1}$$

We use Ψ_T and Φ_T from Construction 4.2.30 to define Ψ_D and Φ_D.

For $w = [c_1, c_2] \in D_p^{\mathrm{sep}}$, say with repeated occurrence of $x_{k,p-1}$, apply Ψ_T to c_1 and to c_2. We obtain the following two labelled unitrivalent trees $\Gamma_1 := \Psi_T(c_1)$ and $\Gamma_2 := \Psi_T(c_2)$.



Define $\Psi_D(w)$ to be the following $(i, p-1)$-marked unitrivalent graphs.

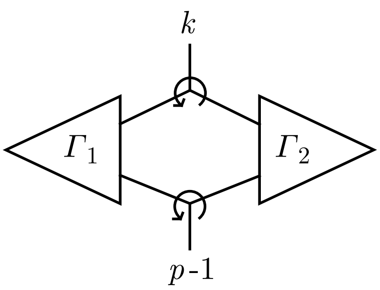


FIGURE 24. A visualisation of the labelled unitrivalent graph $\Psi_D(w)$.

[21]A simple cycle is defined as a loop in the graph with no repetitions of nodes and edges allowed, other than the repetition of the starting and ending nodes.

For a $(i, p-1)$-marked unitrivalent diagram $\Gamma_{i,p-1}$, we can draw $\Gamma_{i,p-1}$ as in Figure 24. Then define $\Phi_D(\Gamma_{i,p-1}) \coloneqq [\Phi_T(\Gamma_1), \Phi_T(\Gamma_2)]$. By construction the maps Ψ_D and Φ_D are inverse to each other.

Proposition 4.2.37. *For every $w = [[c_1, [c_2, c_3]], c_4] \in D_p^{\mathrm{sep}}$, say with $c_i \in \pi_{l_i}(\mathbb{R}^2 \times \mathrm{D}^1)$, we have in $\mathbb{Z}[D_p^{\mathrm{sep}}]$ the* separated Jacobi identity

$$(-1)^{(l_3-1)l_1}\big[[c_1,[c_2,c_3]],c_4\big] + (-1)^{(l_1-1)l_2}\big[[c_2,[c_3,c_1]],c_4\big] + (-1)^{(l_2-1)l_3}\big[[c_3,[c_1,c_2]],c_4\big] = 0.$$

Proof. Apply the Jacobi identity to $[c_1, [c_2, c_3]]$. □

Remark 4.2.38. Note that the usual Jacobi identity is not well-defined in $\mathbb{Z}[D_p^{\mathrm{sep}}]$.

Proposition 4.2.39. *Recall the group $E^1_{p-1,p}/\mathrm{tors}$ from Proposition 4.2.23. For $p \geq 4$, the construction above induces an isomorphism $E^1_{p-1,p}/\mathrm{tors} \cong \mathcal{D}_{p-1}$ of groups.*

Similar to Lemma 4.2.32, we obtain the following presentation of $E^1_{p-1,p}/\mathrm{tors}$.

Lemma 4.2.40. *We have*

$$E^1_{p-1,p}/\mathrm{tors} \cong \mathbb{Z}[D_p^{\mathrm{sep}}]/\sim,$$

where $\sim$ denotes relation induced by antisymmetry and the separated Jacobi Identity. □

Proof of Proposition 4.2.39. Similar as in the proof of Proposition 4.2.31, we can define group homomorphisms

$$\mathbb{Z}[D_p^{\mathrm{sep}}] \underset{\phi_D}{\overset{\psi_D}{\rightleftarrows}} \mathbb{Z}[\bigcup_{i=1}^{p-1} D_{i,p-1}].$$

For $w = [c_1, c_2] \in D_p^{\mathrm{sep}}$, say with repeated occurrence of $x_{k,p-1}$, define the set of labels $L_i \coloneqq \{j \in \mathbb{N} \mid x_{j,p-1} \text{ appears in } c_i\} \cup \{p-1\}$. Define $\psi_D(w) \coloneqq (-1)^{\#(L_1 \times_> L_2)}\Psi_D(w)$.

For $\Gamma_{k,p-1} \in D_{k,p-1}$, drawn as in Figure 24, denote by L_{Γ_i} the labelling of Γ_i. Define $\phi_D(\Gamma_{k,p-1}) \coloneqq (-1)^{\#(L_{\Gamma_1} \times_> L_{\Gamma_2})}\Phi_D(\Gamma_{k,p-1})$.

Similar as in the proof of Proposition 4.2.31 we can check by explicit computation that the induced maps $\bar{\psi}_D$ and $\bar{\phi}_D$ on the quotients $E^1_{p-1,p}/\mathrm{tors}$ and $\mathcal{D}_{p-1}$ are well-defined. More precisely, antisymmetry corresponds to AS-relation and the separated Jacobi identity corresponds to IHX$^{\mathrm{sep}}$-relation. By construction the maps $\bar{\psi}_D$ and $\bar{\phi}_D$ are inverse to each other. □

Definition 4.2.41. Define the following relations on $\mathbb{Z}[\mathrm{UTG}]$:

i) Let Γ be a labelled unitrivalent graph. Let e be the edge connecting the leaf labelled by n and the adjacent trivalent node v. Then Γ is *STU-related* to the difference $\Gamma' - \Gamma''$ of the following two labelled unitrivalent graphs Γ' and Γ'' Denote by $\{e, e_1, e_2\}$ the ordered set of edges at the node v, i.e. $e < e_1 < e_2$ (cyclic order). The graph Γ' arise from Γ the following steps: First, delete the edge e, the node v and the leaf labelled by n. Second, add two leaves, labelled by n and $n+1$ with adjacent edges e_2 and e_1 respectively. Third, relabel the leaves labelled by m with $m+1$ if $m > n$. The graph Γ'' is constructed similar to Γ'. For Γ'' the adjacent edges of the leaves labelled by n and $n+1$ are e_1 and e_2 respectively. The STU-relation is depicted in Figure 25.

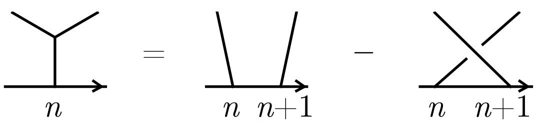


FIGURE 25. A visualisation of the STU-relation.

ii) Let $\varGamma$ be a labelled unitrivalent graph. Then $\varGamma_1' - \varGamma_1''$ is *STU²-related* to $\varGamma_2' - \varGamma_2''$, where $\varGamma_1' - \varGamma_1''$ is obtained by performing the STU-relation at the leaf of $\varGamma$ labelled by n, and $\varGamma_2' - \varGamma_2''$ is obtained by performing the STU-relation at the leaf of $\varGamma$ labelled by m. The STU^2-relation is depicted in Figure 26.

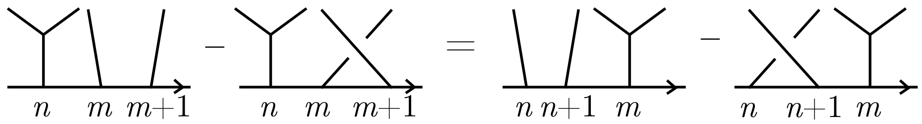


FIGURE 26. A visualisation of the STU^2-relation.

Remark 4.2.42. Note that the STU^2-relation is finer than the STU-relation.

Proposition 4.2.43. *The STU²-relation is well-defined on $\mathbb{Z}[UTT_p]$ for $p \geq 1$.* □

Proof. This is clear because the STU^2-relation does not change the connectivity and degree of the unitrivalent graphs, and it also does not add simple cycles to the graphs. □

Recall the computation of the differential $d^1 \colon E^1_{p-1,p} \to E^1_{p,p}$ from Proposition 4.2.24. Note that in the computation only $d^1|_{E^1_{p-1,p}/\mathrm{tors}}$ is relevant, and by abuse of notation we will write d^1 instead of $d^1|_{E^1_{p-1,p}/\mathrm{tors}}$ in the following. By Proposition 4.2.31 and Proposition 4.2.39 we can consider d^1 as a map between two groups of unitrivalent graphs as follows:

$$\varPsi_T \circ d^1 \circ \varPhi_D \colon \mathcal{D}_{p-1} \to \mathcal{T}_{p-1}.$$

By abuse of notation we will also denote this map between $\mathcal{D}_{p-1}$ and $\mathcal{T}_{p-1}$ by d^1.

The following proposition describes what d^1 means on the level of unitrivalent graphs.

Proposition 4.2.44 ([Con08, Proposition 4.8])**.** *Let $\tau \in \mathcal{T}_{p-1}$, then $\tau \in \operatorname{im}(d^1)$ if and only if τ is STU²-related to 0.*

Proof. "$\Leftarrow$" It is sufficient to show that for $w = [c_1, c_2] \in D_p^{\mathrm{sep}}$, say with repeated occurrence of $x_{k,p-1}$, we have $\overline{\psi}_T(d^1(w)) = \overline{\psi}_T(\partial^k(w)) + \overline{\psi}_T(\partial^{p-1}(w))$ is STU^2-equivalent to 0. As in Proposition 4.2.24, we write (without loss of generality) $c_1 = [\dots x_{i,p-1} \dots x_{k,p-1} \dots]$ and $c_2 = [\dots x_{k,p-1} \dots x_{j,p-1} \dots]$. Thus

$$\begin{aligned}
\overline{\psi}_T(c_1) &= (-1)^{\epsilon_1} \varGamma_1 \\
\overline{\psi}_T(c_2) &= (-1)^{\epsilon_2} \varGamma_2 \\
\overline{\psi}_D(w) &= (-1)^{\epsilon_w} \varGamma_w,
\end{aligned}$$

where $\varGamma_1$, $\varGamma_2$ and $\varGamma_w$ are depicted in Figure 27. We will discuss the signs ϵ_1, ϵ_2 and ϵ_w in the end.

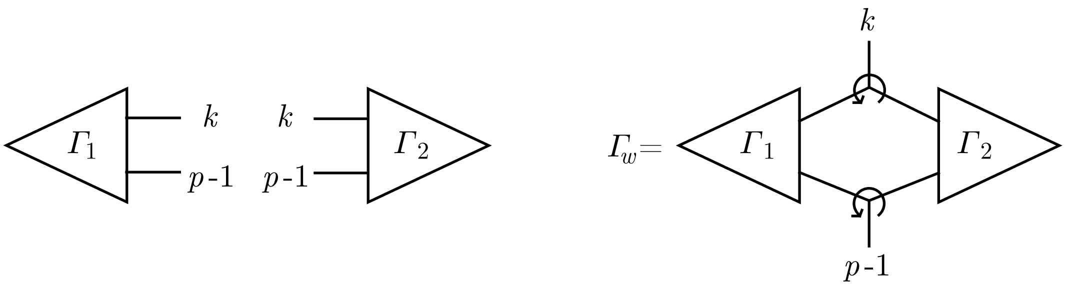


FIGURE 27. Visualisations of $\varGamma_1$, $\varGamma_2$ and $\varGamma_w$.

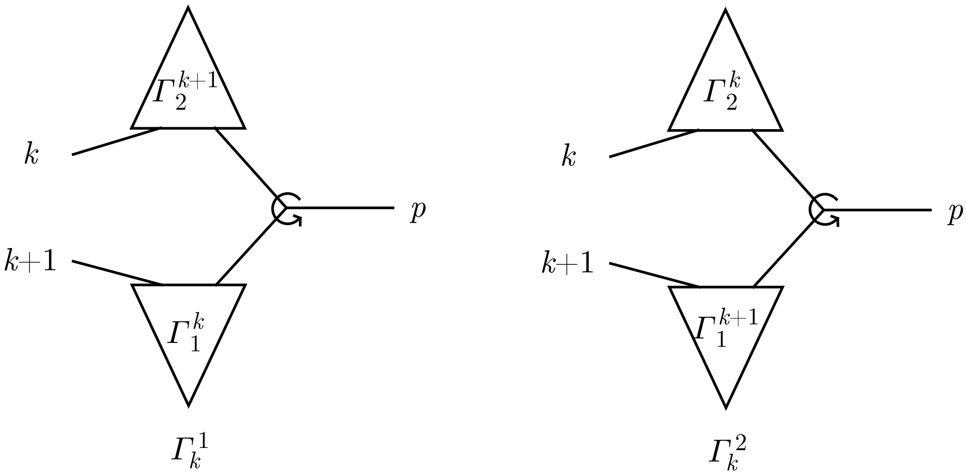


FIGURE 28. Descriptions of the labelled unitrivalent trees Γ_k^1 and Γ_k^2.

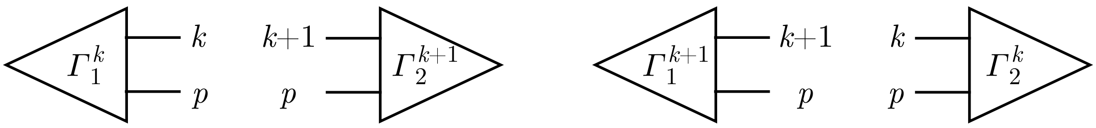
Recall the formula for $\partial^k(w)$ from Proposition 4.2.24. We write

$$\partial^k(w) = (-1)^k[c_1^k, c_2^{k+1}] + (-1)^k[c_1^{k+1}, c_2^k],$$

where c_1^k, c_2^{k+1}, c_1^{k+1} and c_2^k correspond exactly to the four Lie brackets in the formula of $\partial^k(w)$. Thus we have

$$\begin{aligned}\overline{\psi}_T(c_1^k) &= (-1)^{\epsilon_1}\Gamma_1^k\\ \overline{\psi}_T(c_2^{k+1}) &= (-1)^{\epsilon_2}\Gamma_2^{k+1}\\ \overline{\psi}_T(c_1^{k+1}) &= (-1)^{\epsilon_1}\Gamma_1^{k+1}\\ \overline{\psi}_T(c_2^k) &= (-1)^{\epsilon_2}\Gamma_2^k,\end{aligned}$$

where Γ_1^k, Γ_2^{k+1}, Γ_1^{k+1} and Γ_2^k are the labelled unitrivalent trees depicted below



Note that the underlying unlabelled trees of Γ_i^k and Γ_i^{k+1} for $i = 1, 2$ are the same as the ones for Γ_i respectively. Furthermore, we have

$$\begin{aligned}\overline{\psi}_T([c_1^k, c_2^{k+1}]) &= (-1)^{\epsilon_k^1}\Gamma_k^1\\ \overline{\psi}_T([c_1^{k+1}, c_2^k]) &= (-1)^{\epsilon_k^2}\Gamma_k^2\end{aligned}$$

where Γ_k^1 and Γ_k^2 are the labelled unitrivalent trees depicted in Figure 28. We perform the same steps for $\partial^{p-1}(w) = (-1)^{p-1}[c_1, c_2^p] + (-1)^{p-1}[c_1^p, c_2]$, thus obtain

$$\begin{aligned}\overline{\psi}_T(c_1^p) &= (-1)^{\epsilon_1}\Gamma_1^p\\ \overline{\psi}_T(c_2^p) &= (-1)^{\epsilon_2}\Gamma_2^p,\end{aligned}$$

where Γ_1^p and Γ_2^p are the labelled unitrivalent trees depicted in Figure 29.

Furthermore, $\overline{\psi}_T([c_1, c_2^{k+1}]) = (-1)^{\epsilon_p^1}\Gamma_p^1$ and $\overline{\psi}_T([c_1^{k+1}, c_2]) = (-1)^{\epsilon_p^2}\Gamma_p^2$, where Γ_p^1 and Γ_p^2 are depicted[22] in Figure 29. Reviewing the leaves of Γ_k^i and Γ_p^i as placed on the oriented

[22] Here we join the trees at the leaves labelled by k. Note that Construction 4.2.30 only considers the case where we join at the leaves labelled by p. The construction for k instead of p is analog. But, one has to check that both constructions apply to the same tree yield the same iterated Whitehead product.

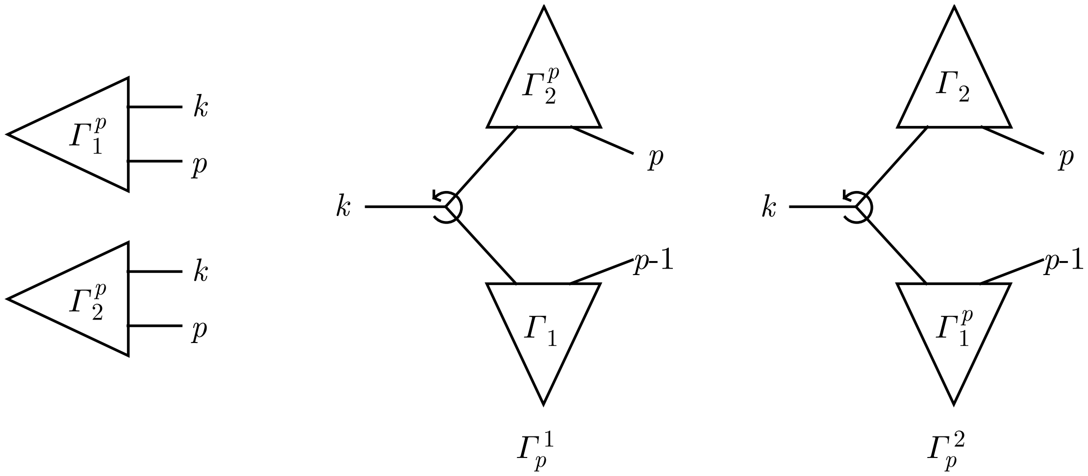
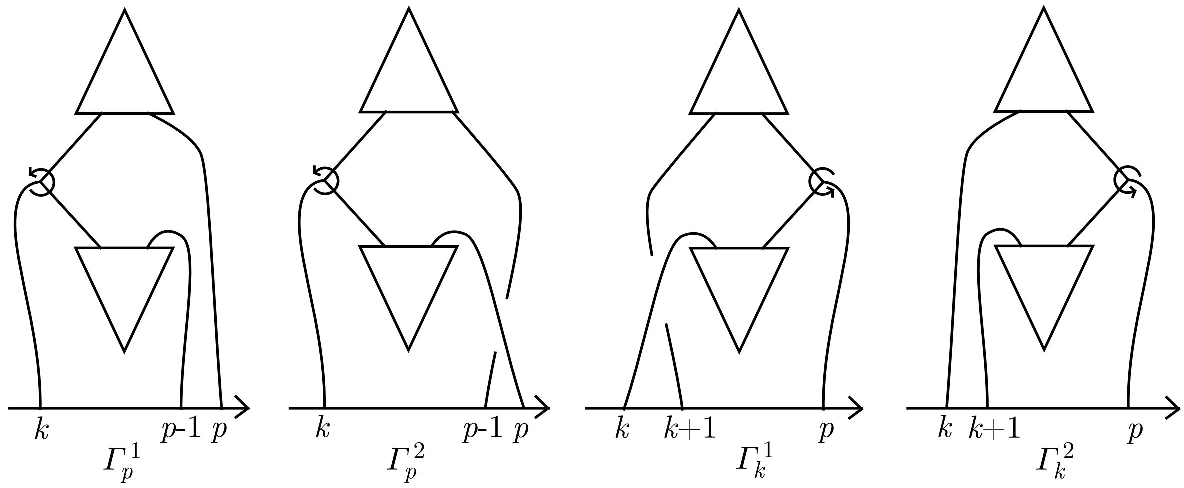


FIGURE 29. Descriptions of the labelled unitrivalent trees $\Gamma_1^p, \Gamma_2^p, \Gamma_p^1$ and Γ_p^2

line, we get



Thus up to signs, we have that

$$\overline{\psi}_T(d^1(w)) = (-1)^k\big((-1)^{\epsilon_k^1}\Gamma_k^1 + (-1)^{\epsilon_k^2}\Gamma_k^2\big) + (-1)^{p-1}\big((-1)^{\epsilon_p^1}\Gamma_p^1 + (-1)^{\epsilon_p^2}\Gamma_p^2\big)$$

is STU^2-related to 0. Now let us check the signs. For $i = 1, 2$, let L_i be the set of labels Γ_i. Thus we have

$$\epsilon_w = \epsilon_1 + \epsilon_2 + \#L_1 + \#(L_1 \times_> L_2).$$

where ϵ_1 and ϵ_2 appeared right at the beginning of the proof. For $i \in \{1, 2\}$ and $j \in \{k, k+1\}$ let L_i^k be the set of labels of Γ_i^j, and L_i^p be the set of labels of Γ_i^p. Then we have

$$\begin{aligned} \epsilon_k^1 &= \epsilon_1 + \epsilon_2 + \#L_1^k + \#(L_1^k \times_> L_2^{k+1}), \\ \epsilon_k^2 &= \epsilon_1 + \epsilon_2 + \#L_1^{k+1} + \#(L_1^{k+1} \times_> L_2^k), \\ \epsilon_p^1 &= \epsilon_1 + \epsilon_2 + \#L_1 + \#(L_1 \times_> L_2^p), \\ \epsilon_p^2 &= \epsilon_1 + \epsilon_2 + \#L_1^p + \#(L_1^p \times_> L_2). \end{aligned}$$

Recall Remark 4.2.26 and thus observe that $\#L_1 = \#L_1^k = \#L_1^{k+1} = \#L_1^p$, Furthermore, $\#(L_1^{k+1} \times_> L_2^k) = \#(L_1^k \times_> L_2^{k+1}) + 1$, where the $+1$ arises because of the pair $(k+1, k)$, and $\#(L_1^p \times_> L_2) = \#(L_1 \times_> L_2^p) + 1$, where the $+1$ arises because of the pair $(p, p-1)$. Thus we can write $\overline{\psi}_T(d^1(w))$ as

$$\overline{\psi}_T(d^1(w)) = (-1)^{k+\epsilon_k^2}(\Gamma_k^2 - \Gamma_k^1) + (-1)^{p-1+\epsilon_p^1}(\Gamma_p^1 - \Gamma_p^2). \tag{4.2.7}$$

For the signs we use

$$\begin{aligned}
\#\left(L_1^{k+1} \times_> L_2^k\right) &= \#\left((L_1 \setminus \{k\}) \times_> (L_2 \setminus \{k\})\right) \\
&\quad + \#\left(\{k+1\} \times_> (L_2 \setminus \{k\})\right) \\
&\quad + \#\left((L_1 \setminus \{k\}) \times_> \{k\}\right), \\
\#\left(L_1 \times_> L_2^p\right) &= \#\left((L_1 \setminus \{k\}) \times_> (L_2 \setminus \{k\})\right) + \#(L_2 \setminus \{k, p-1\}), \\
\#(L_2 \setminus \{k, p-1\}) &= \#(\{k+1\} \times_> (L_2 \setminus \{k\})) + \#((L_2 \setminus \{k\}) \times_> \{k+1\}) \\
p-k-2 &= \#\left((L_2 \setminus \{k\}) \times_> \{k+1\}\right) + \#\left((L_1 \setminus \{k\}) \times_> \{k\}\right).
\end{aligned}$$

Thus we we have

$$k + \epsilon_k^2 + p - 1 + \epsilon_p^1 = 2p - 3,$$

i.e. the sign before $(\mathit{\Gamma}_k^2 - \mathit{\Gamma}_k^1)$ and $(\mathit{\Gamma}_p^1 - \mathit{\Gamma}_p^2)$ in Equation 4.2.7 are different. We conclude that$\bar{\psi}_T(d^1(w))$ is STU^2 related to 0.

"$\Rightarrow$" It is sufficient to prove that any linear combination of the form $\mathit{\Gamma}_p^1 - \mathit{\Gamma}_p^2 - \mathit{\Gamma}_k^2 + \mathit{\Gamma}_k^1$ with $1 \leq k \leq p-2$, lies in the image of d^1. Note that $\mathit{\Gamma}_k^2 - \mathit{\Gamma}_k^1$ is the result of performing a STU-relation at the trivalent node adjacent to the leaf with label k in $\mathit{\Gamma}_w = \bar{\psi}_D(w)$, cf. Figure 27. Similarly, $\mathit{\Gamma}_p^1 - \mathit{\Gamma}_p^2$ is the result of performing a STU-relation at the trivalent node adjacent to the leaf with label $p-1$ in $\mathit{\Gamma}_w$. Thus we can obtain the linear combination $\mathit{\Gamma}_p^1 - \mathit{\Gamma}_p^2 - \mathit{\Gamma}_k^2 + \mathit{\Gamma}_k^1$ via performing STU-relations on a $(k, p-1)$-marked unitrivalent graphs at its two marked trivalent nodes. By Proposition 4.2.39, the domain of d^1 is exactly the set of $(k, p-1)$-marked unitrivalent graphs.

In conclusion, the map $d^1 \colon \mathcal{D}_{p-1} \to \mathcal{T}_{p-1}$ is of the form depicted below

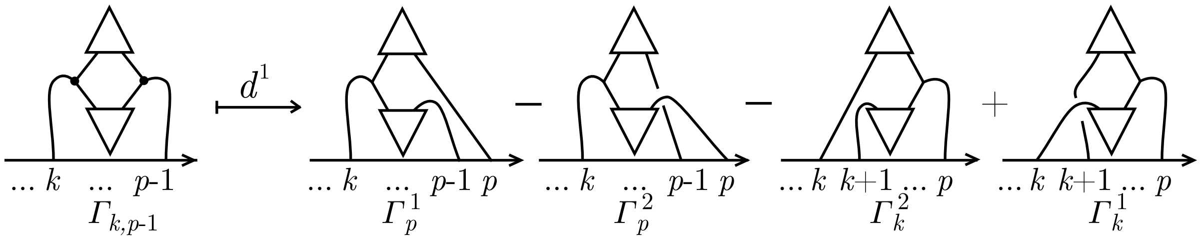


FIGURE 30. An example of d^1 applied to a $(k, p-1)$-marked unitrivalent graph.

Consider the sign in front of $\mathit{\Gamma}_{k,p-1}$ and $\mathit{\Gamma}_p^1$ in the equations $\overline{\psi}_D([c_1, c_2]) = (-1)^{\epsilon_w} \mathit{\Gamma}_{k,p-1}$ and $\overline{\psi}_T([c_1, c_2^p]) = (-1)^{\epsilon_p^1} \mathit{\Gamma}_p^1$. First recall that

$$\begin{aligned}
\epsilon_w &= \epsilon_1 + \epsilon_2 + \#\left(L_1 \times_> L_2\right), \\
\epsilon_p^1 &= \epsilon_1 + \epsilon_2 + \#\left(L_1 \times_> L_2^p\right)
\end{aligned}$$

Observe that we have $\#\left(L_1 \times_> L_2^p\right) = \#\left(L_1 \times_> L_2\right) + \#\left(L_2 \setminus \{k, p\}\right)$, where the term $\#\left(L_2 \setminus \{k, p\}\right)$ arises because of the set of pairs $\{(p-1, x) \in L_1 \times L_2^p \mid x \neq k, p\}$. Moreover $\# L_1 + \#\left(L_2 \setminus \{k, p\}\right) = p-1$. Thus the sign difference between $\overline{\psi}_D([c_1, c_2])$ and $\overline{\psi}_T([c_1, c_2^p])$ is $(-1)^{p-1}$, which cancels with the sign in front of $[c_1, c_2^p]$ in the expression of $\partial^{p-1}(w)$. Therefore, $\mathit{\Gamma}_{k,p-1}$ and $\mathit{\Gamma}_p^1$ have the same sign, i.e. ϵ_w and ϵ_p^1 have the same parity. $\square$

Remark 4.2.45. Viewed on generators, d^1 maps a $(i, p-1)$-marked unitrivalent graph to linear combination corresponding to a STU^2-relation at two trivalent nodes which are adjacent to the leaves with labels k and $p-1$ respectively, cf. Figure 30.

The proof presented here adds missing details to the article [Con08], especially to the proof of Proposition 4.8. Furthermore, [Con08, Proposition 4.8] considers rational coefficients, whereas our proof verifies the version of the proposition with integral coefficients.

Corollary 4.2.46.

i) *For $p \geq 4$, the group $E^2_{p,p}$ is isomorphic to the abelian group generated by unitrivalent trees of degree $p-1$, modulo AS-, IHX-, and STU^2-relations.*

ii) *For $p = 3$, we have $E^2_{3,3} \cong \mathcal{T}_2 \cong \mathbb{Z}$.*

iii) *For $p = 0, 1, 2$, we have $E^2_{p,p} = 0$.*

Proof. Statement i) follows from Proposition 4.2.44. Statement ii) follows from the fact that $d^1 \colon E^1_{2,3} \to E^1_{3,3}$ is trivial, cf. Proposition 4.2.21, and $E^1_{3,3} \cong \mathcal{T}_2$, cf. Proposition 4.2.31. Statement iii) follows from the fact that $E^1_{p,p} = 0$ for $p = 0, 1, 2$, cf. Proposition 4.2.21. $\square$

5. Conclusion and further works

To conclude our presentation of the connection between Vassiliev invariant and the Taylor tower of the embedding functor $\mathrm{Emb}(-)$, let us reflect on our approach encountered in the text and mention some further work.

First, let us mention briefly the methods used in [BCKS17], where our inspiration comes from. In [BCKS17], the authors also construct a cosimplicial space $C_\bullet$ for the tower of fibrations $\cdots \to \mathrm{T}_n \mathrm{Emb}(\mathrm{I}) \to \mathrm{T}_{n-1} \mathrm{Emb}(\mathrm{I}) \to \cdots \to \mathrm{T}_0 \mathrm{Emb}(\mathrm{I})$, via a compactification of $\mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)$, such that $C_{[n]} \simeq \mathrm{Conf}_n(\mathbb{R}^2 \times \mathrm{D}^1)$ and $\mathrm{Tot}^n C_\bullet \simeq \mathrm{T}_n \mathrm{Emb}(\mathrm{I})$. Compared with the cosimplicial space $\mathfrak{Emb}_\bullet$ in Construction 4.1.11, the cosimplicial space $C_\bullet$ has the advantage that it is geometric and various versions of $C_\bullet$ have been used in context concerning Vassiliev invariants or (finite type) invariants of 3-manifold, cf. [AS94] and [BT94]. Using this cosimplicial model, the space $\mathcal{K}$ and $\mathrm{T}_n \mathrm{Emb}(\mathrm{I})$ become E_1-spaces (i.e. spaces with an action of the little intervals operad) and this E_1-action on $\mathrm{T}_n \mathrm{Emb}(\mathrm{I})$ is used to define an abelian group structure on $\pi_0(\mathrm{T}_n \mathrm{Emb}(\mathrm{I}))$, cf. [BCKS17, Corollary 4.13]. Furthermore, the authors of [BCKS17] proves that this group multiplication is compatible with connected sum of equivalent classes of knots, cf. [BCKS17, Section 5.7], which is an important ingredient of our proof of Theorem 3.2.6. We are working on achieving similar results using the cosimplicial space $\mathfrak{Emb}_\bullet$, for example applying [MS04, Theorem 3.1].

The connection between Vassiliev invariant and the embedding functor $\mathrm{Emb}(-)$ can be expressed in the following diagram:

$$\begin{tikzcd}
 & \pi_0(\mathrm{T}_n \mathrm{Emb}(\mathrm{I})) & E^\infty_{n,n} \arrow[l, hook'] & E^2_{n,n} \arrow[l, two heads] & E^1_{n,n} \arrow[l, two heads] \\
\pi_0(\mathcal{K}) \arrow[ur, "\eta_n(\mathrm{I})"] \arrow[dr] & & & & \\
 & \pi_0(\mathcal{K})/{\sim_{C_n}} \arrow[uu, "\bar\eta_n(\mathrm{I})"'] & \mathcal{G}_{n-1} \arrow[l, hook'] \arrow[uu, "\bar\eta_n(\mathrm{I})|_{\mathcal{G}_{n-1}}"'] \arrow[ur, phantom, "?"] & \frac{\mathbb{Z}[\mathrm{UTT}_{n-1}]}{\mathrm{AS,\ IHX,\ STU}^2} \arrow[l, two heads, "\text{[CT04a]}"] \arrow[uu, "\cong"'] & \frac{\mathbb{Z}[\mathrm{UTT}_{n-1}]}{\mathrm{AS,\ IHX}} \arrow[l, two heads] \arrow[uu, "\cong"']
\end{tikzcd},$$

where $n \geq 3$, and[23]

$$E^\infty_{n,n} \cong \ker\big(\pi_0(\mathrm{T}_n \mathrm{Emb}(\mathrm{I})) \to \pi_0(\mathrm{T}_{n-1} \mathrm{Emb}(\mathrm{I}))\big),$$
$$\mathcal{G}_{n-1} \cong \ker\big(\pi_0(\mathcal{K})/{\sim_{C_n}} \to \pi_0(\mathcal{K})/{\sim_{C_{n-1}}}\big).$$

Except for the diagram marked with "?", we know that the diagrams commute. We would like to know whether the diagram "?" also commute, which could involve calculation of higher differentials of the spectral sequence $E^r_{p,q}, q \geq p \geq 0$ and their combinatorial interpretations. Thus, as step one, we wonder whether we can generalise the combinatorial interpretation from Section 4.2.4 to the whole first page of the spectral sequence.

[23] See [CT04a] for the definition of $\mathcal{G}_{n-1}$.

Last but not the least, in the proof of Theorem 3.2.6, we actually proved that a genus one capped grope of degree n gives a path in $\mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$. However, grope cobordism of knots in general does not restrict to genus one gropes, cf. [CT04b]. Therefore, we would like to find out whether higher genus grope also has an interpretation in the Taylor tower of $\mathrm{Emb}(-)$. In [BCKS17] the authors conjectured that the map $\eta_n(\mathrm{I})_*\colon \pi_0(\mathcal{K}) \to \mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$ is a universal additive Vassiliev invariant of degree at most $n-1$. Since grope cobordism characterises universal additive Vassiliev invariant, one of our goals is to construct a tower of fibrations $\cdots \to \mathscr{K}_n \to \mathscr{K}_{n-1} \to \ldots$ using grope cobordism such that $\mathscr{K}_n \simeq \mathrm{T}_n\,\mathrm{Emb}(\mathrm{I})$.

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